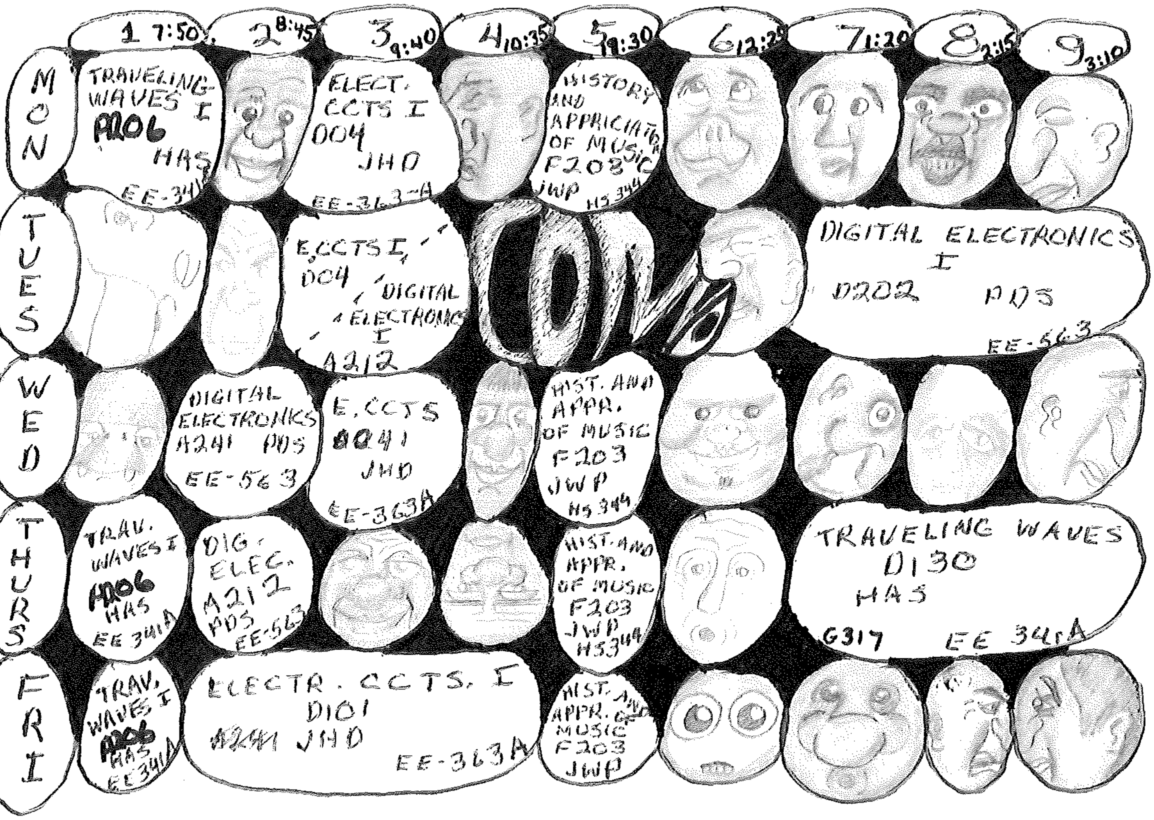
Traveling Waves

R.J. Marks II Class Notes
Rose-Hulman Institute of Technology (1972)

TRAVELING WAVES



9-1?-71 MON

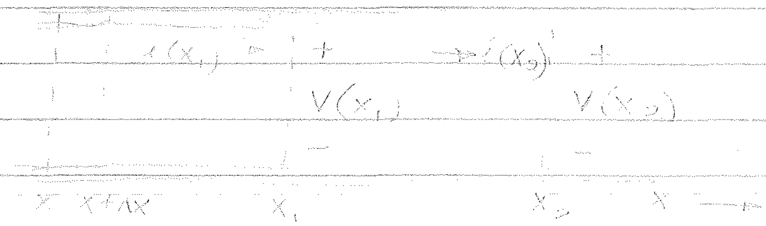
READ CHAP. ? pg ?? prob 10, 12, ?, ?, 20 [illegible] FRIDAY

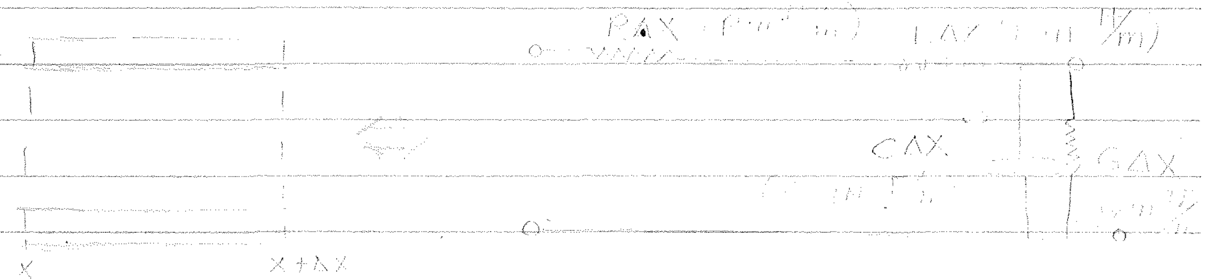
NOTE

DISTRIBUTED SYSTEM [illegible]

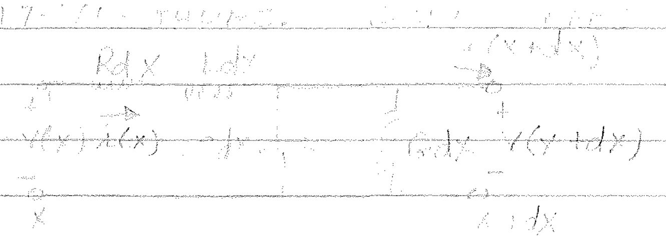
INDEPENDENT VARIABLES

TRANSMISSION LINES





9-17-71 [illegible]



KIRCHOFF'S VOLTAGE LAW

$\Rightarrow v(x) = i(x) R\,dx + L\,dx \frac{\delta i(x,t)}{\delta t} + v(x+dx)$

$\therefore [v(x+dx) - v(x)]/dx = i(x)R - L$ [illegible]

TAKE LIMIT AS $dx \to 0$

$\Rightarrow \frac{\delta v(x)}{\delta x}$ [illegible]

$\frac{\delta v}{\delta x} = -iR - L$ [illegible]

HIGHER FREQ $\Rightarrow$ HIGH $\frac{\delta i}{\delta t}$ $\Rightarrow$ [illegible]

USING KIRCHHOFF'S CURRENT LAW [illegible]

$i(x) = C\,dx\,\frac{\delta V(x+dx,t)}{\delta t} + G\,dx\,V(x+dx) + i(x+dx)$

$\therefore \frac{i(x+dx) - i(x)}{dx} = -C\,\frac{\delta V(x+dx,t)}{\delta t} - G\,V(x+dx)$

$$\boxed{\frac{\delta i(x)}{\delta x} = -C\,\frac{\delta V(x)}{\delta t} - G\,V(x)}$$

IN SUMMARY [illegible]

(TRANSMISSION [illegible])

$\frac{\delta V}{\delta x} = -iR - L\,\frac{\delta i}{\delta t}$

$\frac{\delta i}{\delta x} = -GV - C\,\frac{\delta V}{\delta t}$

LAB

$\frac{\delta V}{\delta x} = -iR - L\,\frac{\delta i}{\delta t}$

$\frac{\delta i}{\delta x} = -GV - C\,\frac{\delta V}{\delta t}$

CASE A: LOSSLESS LINE, R = G = 0

$\frac{\delta V}{\delta x} = -L\,\frac{\delta i}{\delta t}\ ;\ \frac{\delta i}{\delta x} = -C\,\frac{\delta V}{\delta t}$

$\frac{\delta^2 V}{\delta x^2} = -L\,\frac{\delta^2 i}{\delta x\,\delta t}\ ;\ \frac{\delta^2 i}{\delta x\,\delta t} = -C\,\frac{\delta^2 V}{\delta t^2}$

$\therefore \boxed{\frac{\delta^2 V}{\delta^2 x} = LC\,\frac{\delta^2 V}{\delta t^2}}$ ← CLASSICAL WAVE EQUATION

([illegible])

$V_p = \frac{1}{\sqrt{LC}}$; [illegible] FOR PHASE, V_p [illegible] VELOCITY

ALSO [illegible]

CASE B: RC LINE, L = G = 0

$\frac{\delta V}{\delta x} = -iR$; [illegible]

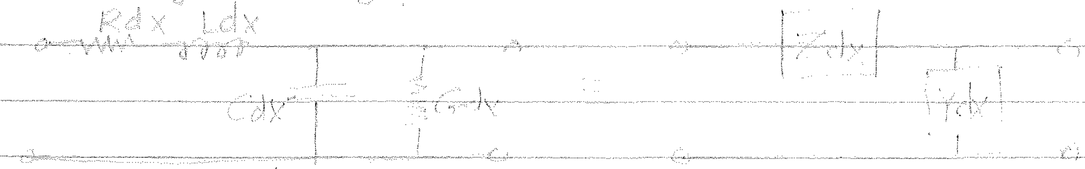
$\frac{\delta^2 i}{\delta^2 x} = -C\,\frac{\delta^2 V}{\delta x\,\delta t}$, $\frac{\delta^2 V}{\delta x\,\delta t} = -R\,\frac{\delta i}{\delta t}$

$\therefore \boxed{\frac{\delta^2 i}{\delta^2 x} = RC\,\frac{\delta i}{\delta t}}$ ← DIFFUSION EQUATION

(CURRENT [illegible])

9-17-71

$\gamma = \left[(R + j\omega L)(G + j\omega C)\right]^{\frac{1}{2}}$



2.14) $T_{d_{300}} = \dfrac{l}{V_p(300)}$, $T_{d_{400}}$ [illegible]

$\Delta T_d = l \left[\dfrac{1}{V_p(300)} - [illegible] \right]$

$= 10^{-8} L \left[\dfrac{1}{2.72} - \dfrac{1}{2.89} \right] <$ [illegible] $\times 10$ [illegible]

$L < 2.5 \times 10^{3} \left[[illegible] - 2.89 \right]^{-1}$

ANS [illegible]

9-20-71

WAVES ON [illegible] LINE

LOSSLESS LINE: [illegible]

$\frac{\delta^2 v}{\delta x^2} = \frac{1}{V_p^2}$ [illegible] $\frac{\delta^2 i}{\delta x^2} = \frac{1}{V_p^2} \frac{\delta^2 i}{\delta t^2}$ [illegible]

GENERAL SOLUTION:

$v(x,t) = f(x - V_p t) + g(x + V_p t)$ [illegible]

$i(x,t) = \frac{1}{Z_0}\left[[illegible] \right]$

INITIAL CONDITIONS:

$v(x,0) = V_0(x)$, [illegible]

THEM (1-10) $v(x,t) = \frac{1}{2}\left[V_0(x - V_p t)\right.$ [illegible]

$+ V_0(x + V_p t)$ [illegible]

EXAMPLE: (i @ 0) (EXAMPLE 1 [illegible])

9-24-71

$$\frac{\delta}{\delta t} f(x - V_p t) = -V_p f'(x - V_p t)$$

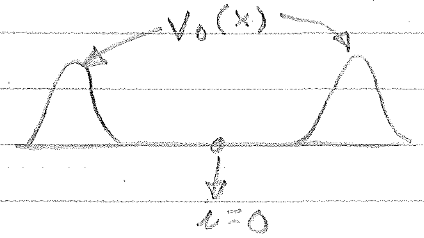
$$\frac{\delta}{\delta x} f(x - V_p t) = f'(x - V_p t)$$

$\leftarrow V(x)$ $\qquad$ $V(x) \rightarrow$

$i(x) = \frac{V(x)}{-Z_0} \qquad i(x) = \frac{V(x)}{Z_0}$

1-4) ON INFINITE LINE: $i(x,t) = \frac{1}{2Z_0}\left[V_0(x - V_p t) - V_0(x + V_p t)\right]$

1-2)



$i(x,t) = \frac{1}{2Z_0}\left[V_0(x - V_p t) - V_0(x + V_p t)\right]$

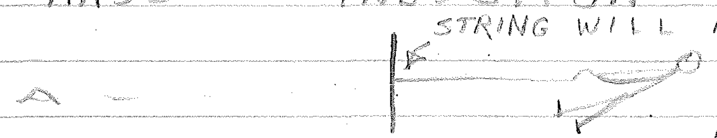
$V_0(x) = V_0(-x) \Rightarrow V_0(x)$ EVEN FUNC

$\Rightarrow i(0,t) = \frac{1}{2Z_0}\left[V_0(-V_p t) - V_0(V_p t)\right]$

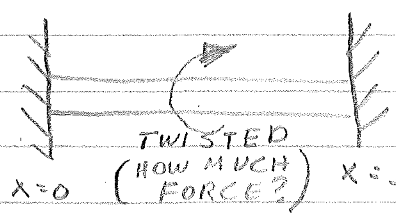
(STR) TENSION — VOLTAGE

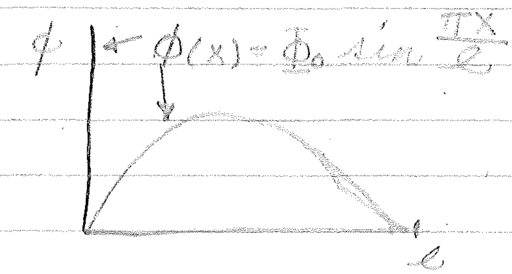
VELOCITY — CURRENT

MASS — INDUCTION



1-5)





$\phi(x,t) = 0 \;@\; x = 0, l$

$\Rightarrow \Phi(s,t) = 0 \;@\; x = 0, l$

d) $\phi(x,s) = \cos\left(\frac{\pi V_p}{l}\right) t \, \Phi_0 \sin \frac{\pi x}{l}$

(CHECK INITIAL CONDITIONS)

1-6) $\phi_0(x) = \Phi \sin \frac{n\pi v}{l}$

1-7)

MONDAY: 1-5, 1-6, 1-9

THURS: 1-8, 1-10, 1-11

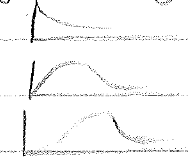
9-24-71

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$e^{-t/\tau} = \frac{1}{s+1/\tau}$$

$$t\,e^{-t/\tau} = \frac{1}{(s+1/\tau)^2}$$

$$t^2 e^{-t/\tau} =$$



FOR INDUCTOR SHUNT

$$Z_L = sL$$

$$\Gamma_L = \frac{sL - Z_0}{sL + Z_0} = 1 - \frac{2Z_0}{sL + Z_0}$$

9-27-71

1-5) $J = \rho I = \frac{MASS}{VOLUME} \times LENGTH^2 =$ MASS

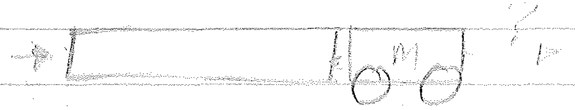
$\tau = C \frac{\delta \theta}{\delta x}$ $C = \infty \Rightarrow$ RIGID BODY

$\frac{1}{C_R} \frac{\delta \tau}{\delta t} = \frac{\delta \omega}{\delta x}$; $J \frac{\delta \omega}{\delta t} = \frac{\delta \tau}{\delta x}$ $\omega \sim I$; $\tau \sim V$

C   L

$Z_0 = \sqrt{\frac{L}{C}} \Rightarrow Z = \sqrt{J C_R}$

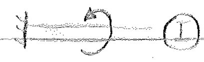
NORMAL OR PRINCIPLE MODES AS OSCILLATE IN SAME, OR HAPPY BELLY

1-10)    M ~ INDUCTANCE

9-30-71

$V \sim T$; $I \sim$ Veloc. ; $Q \sim$ POSITION

TAKE HOME TEST HINTS

1) 

2)

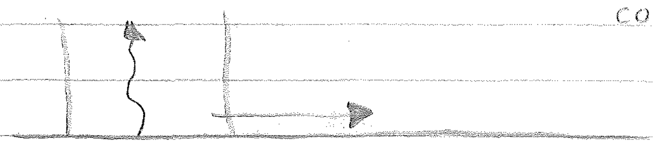
$$\rho \frac{\delta v}{\delta t} = \frac{\delta \sigma}{\delta x} \sim L \frac{\delta i}{\delta t} = \frac{\delta V}{\delta x}$$

$$\frac{1}{Y} \frac{\delta \sigma}{\delta t} = \frac{\delta v}{\delta x} \sim C \frac{\delta V}{\delta t} = -\frac{\delta i}{\delta x}$$

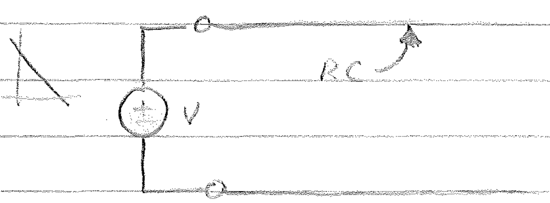
LASOR = LIGHT AMPLIFICATION BY STIMULATING . . .

10-10-71

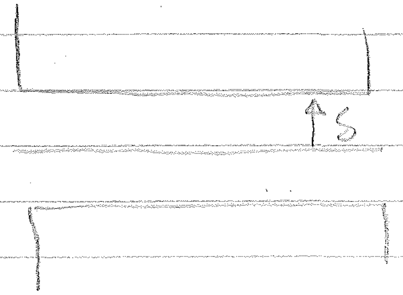
PROBLEM 4 (MIDTERM



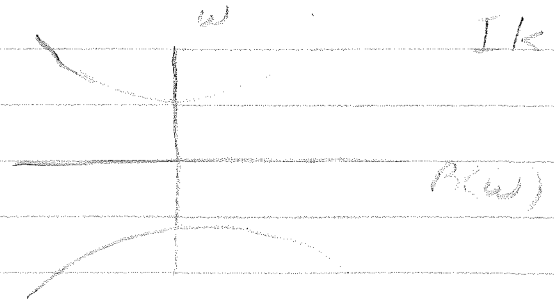
COMPUTE $E(t)$



1.48



$$\frac{\partial^2 \delta}{\partial t^2} = C^2 \frac{\partial^2 \delta}{\partial x^2} - Ik\,\delta$$

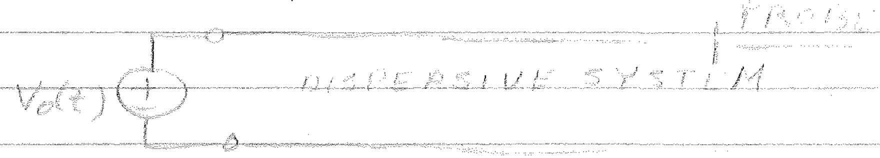


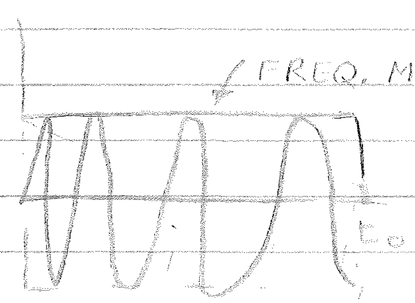
$$Ik = \omega_c^2$$

APPLICATION OF SPACE-TIME RAYS TO PULSE COMPRESSION

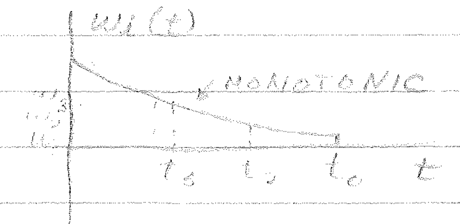
TRANSMIT (COMPLEX) SIGNALS

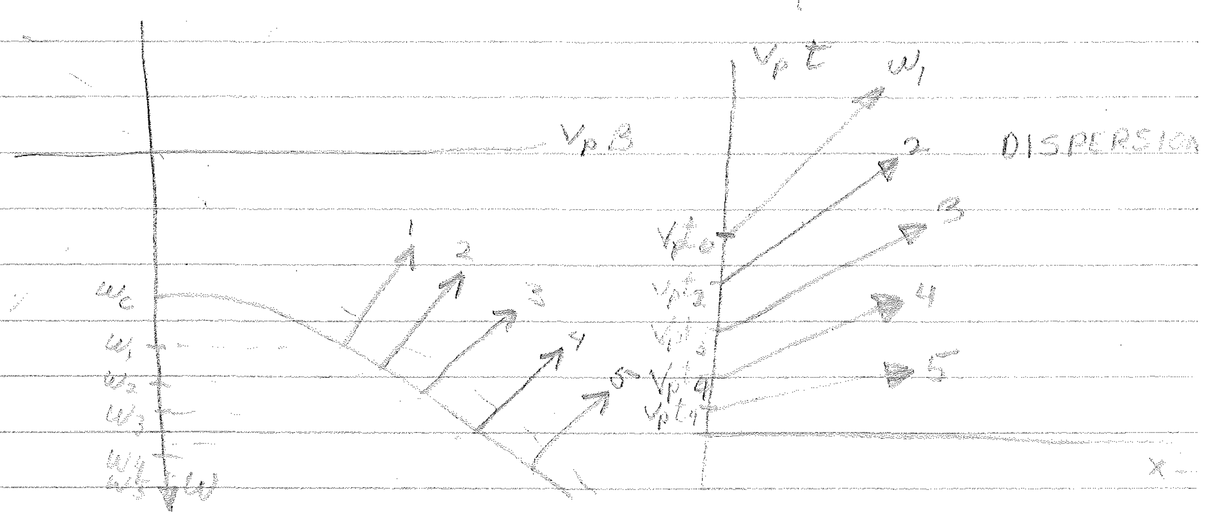
$$V_0(t) = V_0 \left[u(t) - u(t - t_0) \right] e^{j\theta(t)} \quad \text{(PHASE MOD.)}$$

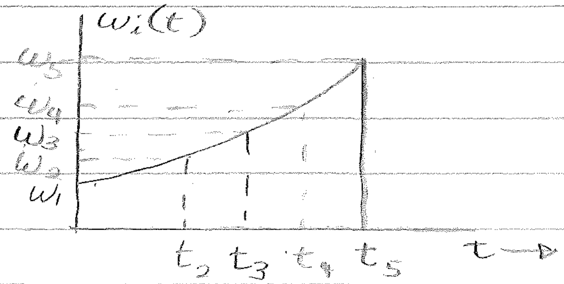


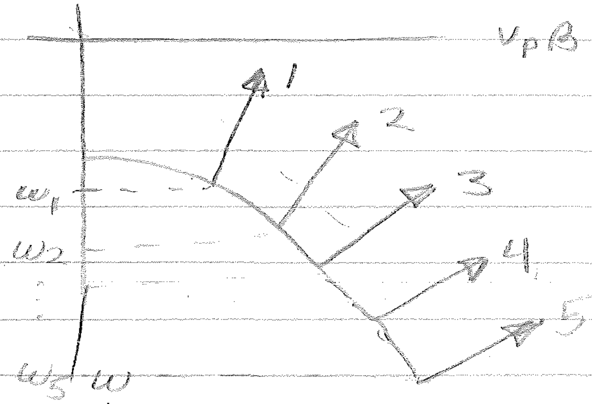


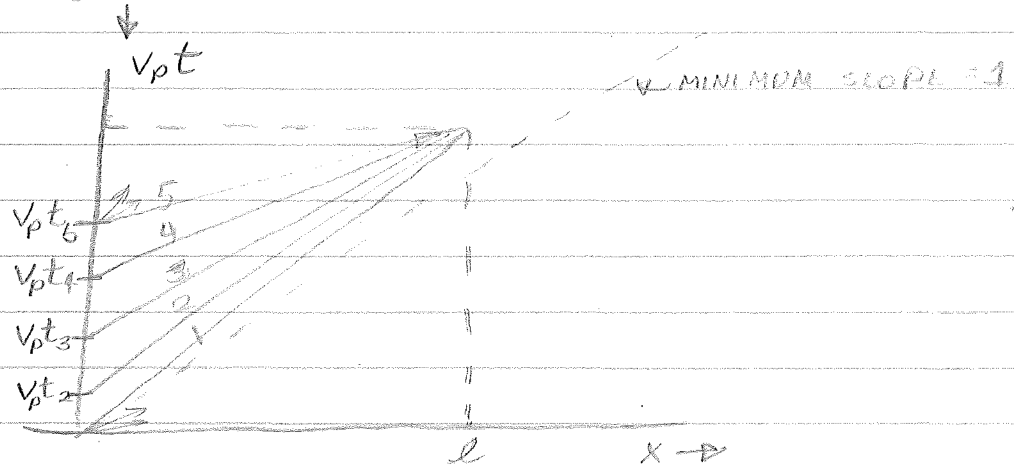
$$\omega_i(t) = \frac{d\theta}{dt}$$











WILL HAVE "JUNK" RAYS AT PULSE BEGINNING
AND END CORRESPONDING TO HIGH FREQUENCY
DUE TO PULSE INTRODUCTION

11-1-71

11, 13, 14, 15 IN CHAPT. 5

ON FINAL (1-9)

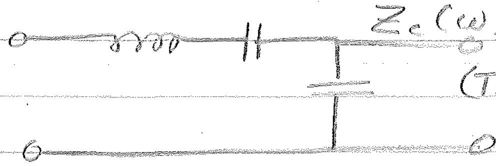
$$Z_c = \frac{Z_0}{\left[1-\left(\frac{\omega_c}{\omega_0}\right)^2\right]^{\frac{1}{2}}}$$

FOR $\omega_c > \omega_0$

$$Z_c = \frac{Z_0}{-j\left[\left(\frac{\omega_c}{\omega_0}\right)^2-1\right]^{\frac{1}{2}}}$$

FACTOR OUT $-j$ INSTEAD OF j SO WAVE WILL TRAVEL TO RIGHT.

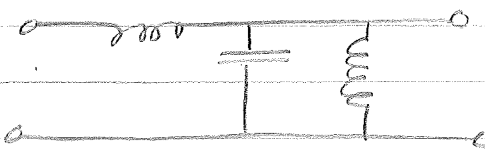
WAVEGUIDES



$$Z_c(\omega) = Z_0\left[1-\left(\frac{\omega_c}{\omega_0}\right)^2\right]^{1/2}$$

(T.M.) SAME β, V_{GP}, V_P, V_G BUT DIFFERENT Z

(CAPACITIVE @ $\omega < \omega_c$)

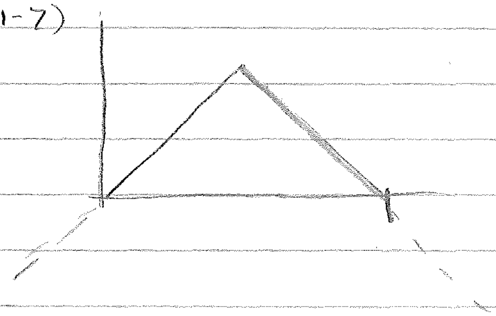


(T.E.) $Z_c(\omega) = \frac{Z_0}{\left[1-\left(\frac{\omega_c}{\omega_0}\right)^2\right]^{\frac{1}{2}}}$

(INDUCTIVE @ $\omega < \omega_c$)

11-71

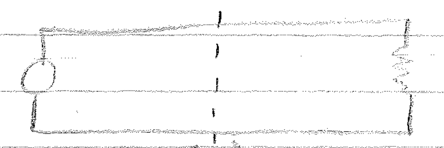
1-7)



EXPAND IN SIN SERIES

(5-11)

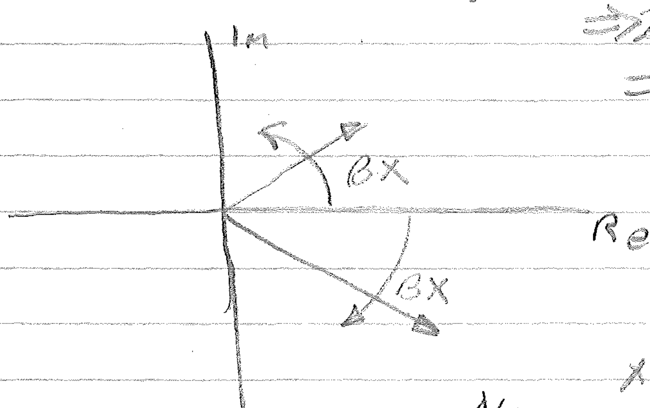
IDEAL LOSSLESS WAVEGUIDE IN SIN STEADY STATE



$v(t) = R_e\{V e^{j\omega t}\} = R_e\{|V| e^{j\theta} e^{j\omega t}\}$

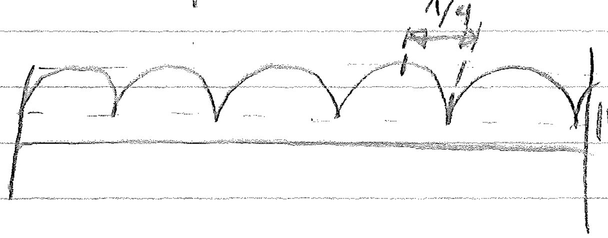
$$V(x) = V_+ e^{-j\beta x} + V_- e^{j\beta x}, \quad \beta = \frac{\omega}{V_p} = \omega\sqrt{LC}$$
$$= V_+ [e^{-j\beta x} + \Gamma e^{j\beta x}]$$
$$= V_+ [e^{-j\beta x} + |\Gamma| e^{j(\beta x + \phi)}]$$

IN PHASE: $-\beta X = \beta X + \phi + 2n\pi$



$$\Rightarrow 2\beta X = -\phi - 2n\pi$$
$$\Rightarrow X_{MAX} = -\left(\frac{\phi + 2n\pi}{2\beta}\right)$$
$$X_{MIN} = -\left(\frac{\phi + 2(n+\frac{1}{2})\pi}{2\beta}\right)$$

$$X_{MAX} - X_{MIN} = \frac{\pi}{2\beta} = \frac{\lambda}{4}$$



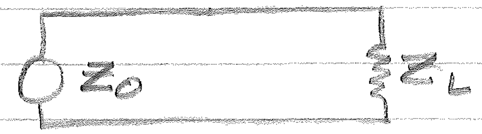
$$V(x) = V_+ \left[e^{-j\beta x} + |\Gamma| e^{j(\beta x + \phi)} \right]$$

$$|V_{MAX}| = V_+ \left[1 + |\Gamma| \right]$$

$$|V_{MIN}| = V_+ \left[1 - |\Gamma| \right]$$

VSWR = VOLTAGE STANDING WAVE RATIO

$$VSWR = \frac{|V|_{MAX}}{|V|_{MIN}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

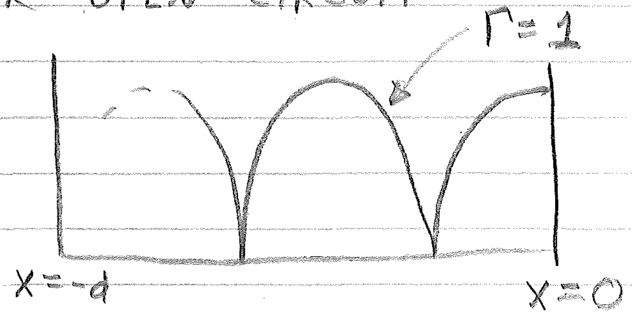


$$\Rightarrow \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

IF $Z_L > Z_0 \Rightarrow \varphi = 0$ $\quad |\Gamma| = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma$

$$\therefore VSWR = \frac{1 + \frac{Z_L - Z_0}{Z_L + Z_0}}{1 - \frac{Z_L - Z_0}{Z_L + Z_0}} = \frac{Z_L}{Z_0} \quad \text{(FOR REAL Z)}$$

FOR OPEN CIRCUIT

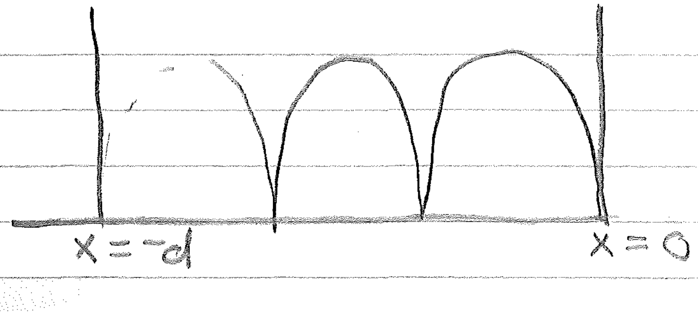


IF $Z_L < Z_0$ AND REAL

$\varphi = 180^\circ$ $\quad |\Gamma| = -\Gamma = \frac{Z_0 - Z_L}{Z_0 + Z_L}$

$$\Rightarrow VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{Z_0}{Z_L}$$

FOR SHORT CIRCUIT



$VSWR_{MIN} = 1$

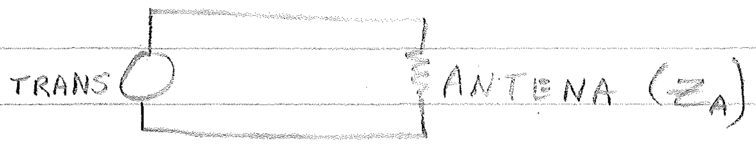
VSWR = +∞ FOR $Z_L = 0, \infty$, OR Z_L IS PURE REACTANCE

PROOF:

$Z_L = jX$

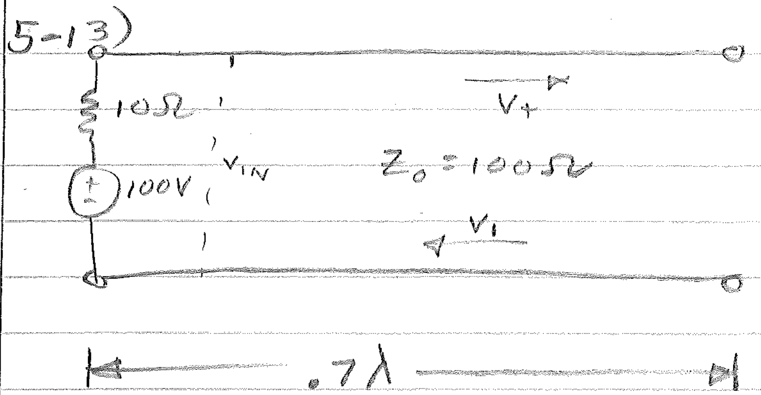
$\Gamma = \frac{jX - Z_0}{jX + Z_0} \Rightarrow |\Gamma| = 1$

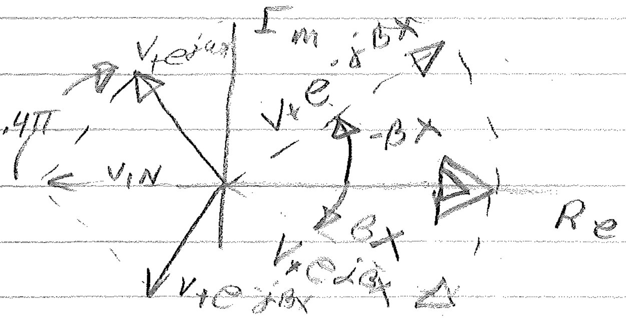
$SWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \infty$ FOR PURE REACTANCE



WISH TO HAVE SMALL SWR FOR MAXIMUM PWR. @ Z_A

5-13)





$V(x) = V_+(e^{-j\beta x} + \Gamma e^{j\beta x})$

$= V_+(e^{-j\beta x} + e^{j\beta x})$

INCIDENT VOLTAGE PHASOR = $V_+ e^{-j\beta x}$

$\Rightarrow V_{OUT} = 2V_+$

$-2V_+ \cos(0.4\pi) = V_{IN}$

IF TRANY LINE IS OPEN CIRCUITED AND $\frac{(2n+1)}{4}\lambda$, WILL HAVE SUPER BIG VOLTAGE OUTPUT

DERIVING DRIVING POINT IMPEDANCE

$$V(x) = V_+ \left[e^{-j\beta x} + \Gamma e^{+j\beta x} \right]$$

$$I(x) = \frac{V_+}{Z_0} \left[e^{-j\beta x} - \Gamma e^{+j\beta x} \right]$$

DRIVING POINT IMPEDANCE $= \left(V(x) / I(x) \right) \Big|_{x=-d}$

$$\therefore Z(x) = \frac{V(x)}{I(x)} = Z_0 \left[\frac{e^{-j\beta x} + \Gamma e^{j\beta x}}{e^{-j\beta x} - \Gamma e^{j\beta x}} \right] = Z_0 \left[\frac{1 + \Gamma e^{j2\beta x}}{1 - \Gamma e^{j2\beta x}} \right]$$

$$\boxed{Z(-d) = Z_0 \left[\frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}} \right]}$$

FOR THIS PROBLEM, $\Gamma = 1$, $\beta d = \left(\frac{2\pi}{\lambda} \right)(.7\lambda) = 1.4\pi$

$$\Rightarrow Z(-d) = 100 \left[\frac{1 + e^{-j2.8\pi}}{1 - e^{-j2.8\pi}} \right] = Z_{INPUT}$$

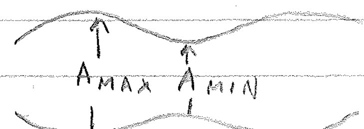
TO CHAPT. 9 (PP 250 - 270) 1, 2, 7, 13

ANS. 5-15

11-71

$\frac{\text{CAUSE}}{\text{EFFECT}} = \frac{V}{I} = Z$

$= \frac{\text{TORQUE}}{\angle \text{VELOCITY}}$



$\frac{A_{MAX}}{A_{MIN}} = SWR$

$\% \text{ REFL} = \frac{(SWR - 1)^2}{(SWR + 1)^2} \times 100\%$

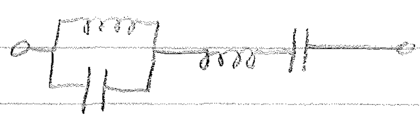
9-1; 9, 13

11-71 FINAL

2) CHECK CHAPT 1; TRANY LINE EQUIVALENT

(DIFFUSION)

c)



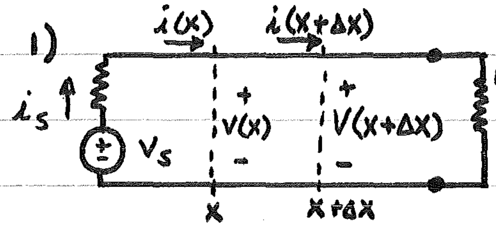
?

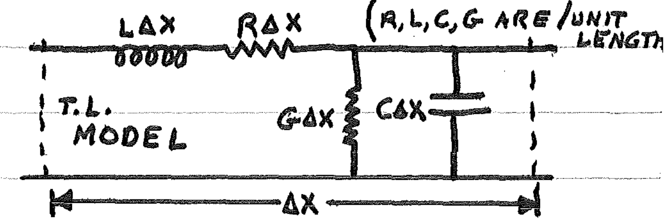
TRAVELING WAVE ENGINEERING

R.K. MOORE

I) TRAVELING ELECTROMAGNETIC WAVES

A) DERIVATION OF GENERAL TRANSMISSION LINE EQUATIONS

1)




$$V(x+\Delta x) = V(x) - R\Delta x\, i(x) - L\Delta x \frac{d i(x)}{dt}$$

$$\frac{V(x+\Delta x) - V(x)}{\Delta x} = -R\, i(x) - L\frac{d i(x)}{dt}$$

$$\frac{\delta V}{\delta x} = \lim_{\Delta x \to 0} \frac{V(x+\Delta x) - V(x)}{\Delta x} = -\left(R + L\frac{\delta}{\delta t}\right) i(x)$$

FIELD EQUIV: $\nabla \times E = -\mu \frac{\delta H}{\delta t}$

$\nabla \times E$ = CURL OF E

E = ELECTRIC FIELD (V/m)

μ = PERMEABILITY (H/m)

H = MAGNETIC FIELD (A/m)

$$i(x+\Delta x) = i(x) - G\Delta x\, V(x+\Delta x) - C\Delta x \frac{\delta V(x+\Delta x)}{\delta t}$$

$$V(x+\Delta x) = V(x) + \frac{\delta V(x)}{\delta x}\Delta x + \ldots \quad \text{(TAYLOR EXPANSION)}$$

$$\Rightarrow i(x+\Delta x) = i(x) - G\Delta x\, V(x) - C\Delta x \frac{\delta V(x)}{\delta t} - G(\Delta x)^2 \frac{\delta V(x)}{\delta x} - C(\Delta x)^2 \frac{\delta^2 V(x)}{\delta x\, \delta t} + \ldots$$

$$\frac{i(x+\Delta x) - i(x)}{\Delta x} = -G V(x) - C\frac{\delta V(x)}{\delta t} - \Delta x\left[G\frac{\delta V(x)}{\delta x} + C\frac{\delta^2 V(x)}{\delta x\, \delta t}\right] + \ldots$$

$$\lim_{\Delta x \to 0} \frac{i(x+\Delta x) - i(x)}{\Delta x} = \frac{\delta i}{\delta x} = -\left(G + C\frac{\delta}{\delta t}\right) V(x)$$

FIELD EQUIV: $\nabla \times H = \left(\sigma + \epsilon\frac{\delta}{\delta t}\right)E$

$\nabla \times H$ = CURL OF H

H = MAGNETIC FIELD (A/m)

σ = CONDUCTIVITY (℧/m)

ϵ = PERMITTIVITY (F/m)

E = ELECTRIC FIELD (V/m)

2) TELEGRAPHER'S EQUATIONS (FROM DERIVATION)

$$\begin{cases} \frac{\delta v}{\delta x} = -\left(R + L\frac{\delta}{\delta t}\right) i \Leftrightarrow \nabla \times E = -\mu \frac{\delta H}{\delta t} \\ \frac{\delta i}{\delta x} = -\left(G + C\frac{\delta}{\delta t}\right) v \Leftrightarrow \nabla \times H = \left(\sigma + \epsilon \frac{\delta}{\delta t}\right) E \end{cases}$$

DIFFERENTIATING BOTH (DISTANCE) AND COMBINING:

$$\frac{\delta^2 V}{\delta x^2} = -\left(R + L\frac{\delta}{\delta t}\right)\frac{\delta i}{\delta x} \; ; \; \frac{\delta^2 i}{\delta x^2} = -\left(G + C\frac{\delta}{\delta t}\right)\frac{\delta V}{\delta x}$$

$$\Rightarrow \frac{\delta^2 V}{\delta x^2} = \left(R + L\frac{\delta}{\delta t}\right)\left(G + C\frac{\delta}{\delta t}\right) V \Leftrightarrow \nabla^2 E = \mu \frac{\delta}{\delta t}\left(\sigma + \epsilon\frac{\delta}{\delta t}\right) E$$

$$\text{OR} \quad \frac{\delta^2 V}{\delta x^2} = RGV + (RC + LG)\frac{\delta V}{\delta t} + LC\frac{\delta^2 V}{\delta t^2} \Leftrightarrow \nabla^2 E = \mu\sigma\frac{\delta E}{\delta t} + \mu\epsilon\frac{\delta^2 E}{\delta t^2}$$

B) SOLUTION OF THE GENERAL TRANSMISSION-LINE EQUATIONS - STEADY STATE

$$\text{INSTANTANEOUS VALUES} \begin{cases} v = R_e V e^{j\omega t} = V\cos\omega t \\ i = R_e I e^{j\omega t} = I\cos\omega t \end{cases}$$

$$\Rightarrow \frac{\delta^2 V}{\delta x^2} = RGV + (LG + RC) j\omega V + (j\omega)^2 LCV$$

$$= (R + j\omega L)(G + j\omega C) V = \gamma^2 V \quad \text{HARMONIC EQUATION}$$

$$\therefore \gamma^2 = (R + j\omega L)(G + j\omega C) \Leftrightarrow \gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$$

SOLVING HARMONIC EQUATION $\left(\frac{\delta^2 V}{\delta x^2} = \gamma^2 V\right)$ YIELDS

$$V = Ae^{-\gamma x} + Be^{\gamma x} \quad \text{(A AND B UNDETERMINED CONSTANTS)}$$

SIMILARLY: $(R + j\omega L) I = -\frac{dV}{dx}$

$$I = \frac{-1}{R + j\omega L}\frac{dV}{dx} = \frac{1}{R + j\omega L}\left(A\gamma e^{-\gamma x} - B\gamma e^{\gamma x}\right)$$

$$= \sqrt{\frac{G + j\omega C}{R + j\omega L}}\left(Ae^{-\gamma x} - Be^{\gamma x}\right)$$

~~$Z_0 = \sqrt{R + j\omega L}$~~

THEN LET $Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \Leftrightarrow \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$

Z_0 = CHARACTERISTIC IMPEDANCE

η = INTRINSIC IMPEDANCE

$$\therefore \quad V = Ae^{-\gamma x} + Be^{\gamma x}$$

$$I = \frac{1}{Z_0}\left(Ae^{-\gamma x} - Be^{\gamma x}\right)$$

FOR PLANE WAVES (PROPOGATION IN X DIRECTION $\ni \frac{\delta}{\delta x} = \frac{\delta}{\delta y} = 0$)

$$\Leftrightarrow E_y = Ae^{-\gamma x} + Be^{\gamma x}$$

$$\Leftrightarrow H_z = \frac{1}{\eta}\left[Ae^{-\gamma x} - Be^{\gamma x}\right]$$

c) WAVE-PARAMETERS AND THE INFINITE LINE

ON AN INFINITE LINE, $B = 0 \Rightarrow$

$$V = Ae^{-\gamma x} \Leftrightarrow E_y = Ae^{-\gamma x}$$

$$I = \frac{A}{Z_0}e^{-\gamma x} \Leftrightarrow H_z = \frac{A}{\eta}e^{-\gamma x}$$

AND $\frac{V}{I} = Z_0 \Leftrightarrow \frac{E_y}{H_z} = \eta$

INFINATE AND
PROPERLY TERMINATED LINES

$$\underline{\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta}$$

α = ATTENUATION CONSTANT $\left(\frac{\text{NEPERS}}{m}\right)$

β = PHASE CONSTANT $\left(\frac{\text{RADIANS}}{m}\right)$

$$\therefore \quad V = Ae^{-\alpha x}e^{-j\beta x} \Leftrightarrow E_y = Ae^{-\alpha x}e^{-j\beta x}$$

$$I = \frac{A}{Z_0}e^{-\alpha x}e^{-j\beta x} \Leftrightarrow H_z = \frac{A}{\eta}e^{-\alpha x}e^{-j\beta x}$$

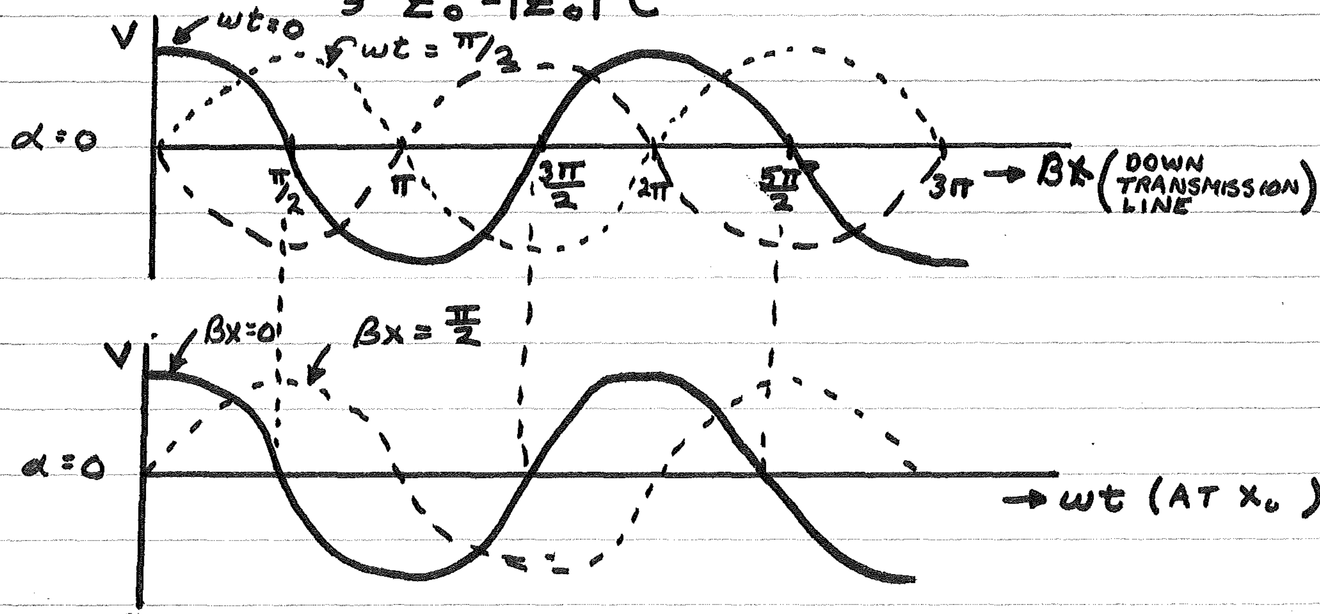
OR MORE APPROPRIATLY, $Ve^{j\omega t} = Ae^{-\alpha x}e^{j(\omega t - \beta x)} \Leftrightarrow E_y e^{j\omega t} = Ae^{-\alpha x}e^{j(\omega t - \beta x)}$

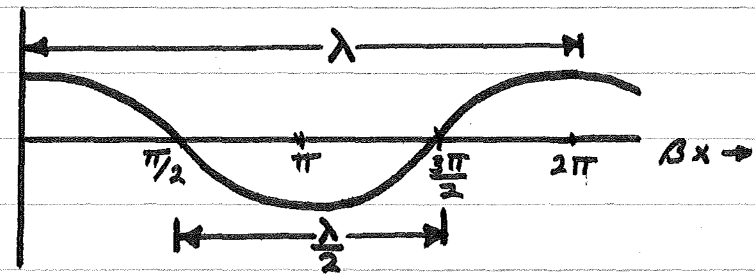
$$Ie^{j\omega t} = \frac{A}{Z_0}e^{-\alpha x}e^{j(\omega t - \beta x)} \Leftrightarrow H_z e^{j\omega t} = \frac{A}{\eta}e^{-\alpha x}e^{j(\omega t - \beta x)}$$

INSTANTANEOUS VALUES;

$$\begin{cases} v = Re\, Ae^{-\alpha x}e^{j(\omega t - \beta x)} = Ae^{-\alpha x}\cos(\omega t - \beta x) \\ i = Re\, \frac{A}{Z_0}e^{-\alpha x}e^{j(\omega t - \beta x)} = \frac{A}{|Z_0|}e^{-\alpha x}\cos(\omega t - \beta x - \theta) \end{cases}$$

$$\ni Z_0 = |Z_0|e^{j\theta}$$





ANALOGOUS SPACE AND TIME RELATIONS:

$$\beta\lambda = 2\pi \Leftrightarrow \omega T = 2\pi$$

$$\lambda = 2\pi/\beta \Leftrightarrow T = \frac{2\pi}{\omega} \quad ; \quad V_p = f\lambda$$

$$\beta x = \frac{2\pi}{\lambda}x \Leftrightarrow \omega t = \frac{2\pi}{T}t$$

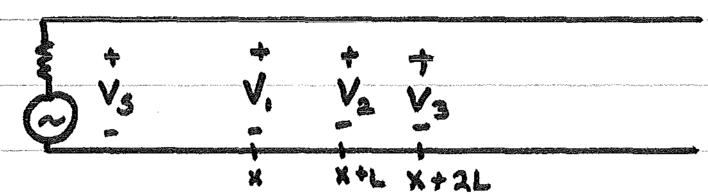
PHASE VELOCITY (SPEED WHICH A MOVING OBSERVER MUST TRAVEL TO OBSERVE STATIONARY VOLTAGE)

$\phi = \omega t + \beta x$ MUST BE CONSTANT

IF $V = A\cos\phi$ IS TO BE CONSTANT

$$\therefore x = \frac{1}{\beta}(\omega t + \phi)$$

$$\Rightarrow V_p = \frac{dx}{dt} = \frac{\omega}{\beta}$$



$$|V_1| = |V_S|e^{-\alpha x} ; \ |V_2| = |V_S|e^{-\alpha(x+L)} ; |V_3| = |V_S|e^{-\alpha(x+2L)}$$

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{V_3}{V_2}\right| = e^{-\alpha L}$$

$\alpha L = \log_e \frac{V_1}{V_2}$ NEPERS (MEASURES ATTENUATION = $^-$GAIN)

$$db = 10\log_{10}\frac{P_1}{P_2}$$

$= 20\log_{10}\left|\frac{V_1}{V_2}\right|$ FOR MATCHED IMPEDANCE

1 db = 8.686 NEPER

FOR HIGH RF FREQUENCIES (OR LOSSLESS LINE)

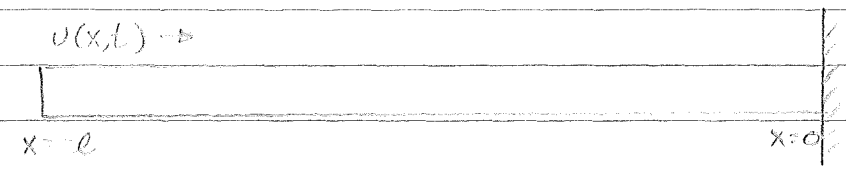
$$Z_0 = (L/C)^{1/2} \Leftrightarrow \eta = (\mu/\epsilon)^{1/2}$$

$$\alpha = \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right) \Leftrightarrow \alpha = \frac{1}{2}\sigma\eta$$

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}}$$

$$V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \Leftrightarrow V_p = \frac{1}{\sqrt{\mu\epsilon}}$$

1-8)



$$\frac{\delta v_x}{\delta x} = \frac{1}{Y}\frac{\delta \sigma(x,t)}{\delta t}$$

$$\Rightarrow v_x(x,t) = \frac{\delta u(x,t)}{\delta t}$$

Y = BULK MODULUS

σ = STRESS

ρ = MASS DENSITY

$$\frac{\delta \sigma}{\delta x} = \rho \frac{\delta^2 v_x(x,t)}{\delta t^2}$$

$$\frac{\delta^2 v_x(x,t)}{\delta x^2} = \frac{1}{Y}\frac{\delta^2 \sigma(x,t)}{\delta x \delta t}, \quad \frac{\delta^2 \sigma(x,t)}{\delta x \delta t} = \rho \frac{\delta^2 v_x(x,t)}{\delta t^2}$$

$$\Rightarrow \frac{\delta^2 v_x(x,t)}{\delta x^2} = \frac{\rho}{Y}\frac{\delta^2 v_x(x,t)}{\delta t^2} \; ; \; \frac{\rho}{Y} = \frac{1}{v_p^2}$$

$$\left(\frac{\delta^2 E(x,t)}{\delta x^2} = LC \frac{\delta^2 E(x,t)}{\delta t^2}\right)$$

$$\frac{d^2 \bar{V}_x(x,s)}{dx^2} = \left(\frac{s}{v_p}\right)^2 \bar{V}_x(x,s)$$

$$\text{SINCE} \quad v_x(x,0) = \frac{\delta v_x(x,0)}{\delta t} = 0$$

$$\Rightarrow \bar{V}_x(x,s) = \bar{V}_x^{+} e^{-\frac{s}{v_p}x} + \bar{V}_x^{-} e^{\frac{s}{v_p}x}$$

$\ni \bar{V}_x^{+}$ AND $\bar{V}_x^{-}$ ARE ARBITRARY CONSTANTS

ANALOGIES:

$v(x,t)$	$E(x,t)$
$\sigma(x,t)$	$-i(x,t)$
ρ	C
$\frac{1}{Y}$	L
$\sigma(-\ell,t) = 0$	$i(-\ell,t) = 0$ (sc)

$$Z_G = \infty$$

$v(0,t) = 0$	$E(0,t) = 0$ (oc)

$$Z_L = 0$$

$$Z_0 = (Y\rho)^{\frac{1}{2}} \qquad Z_0 = \sqrt{\frac{L}{C}}$$

BOB MARKS

LET $\Gamma_L = V_x^- / V_x^+ \Rightarrow V(0,s) = V_x^+ + V_x^- = V_x^+(1+\Gamma_L)$

$I(0,s) = \frac{1}{Z_0}(V_x^+ - V_x^-) = \frac{V_x^+}{Z_0}[1-\Gamma_L]$

$\Rightarrow \frac{V_x^+}{Z_L}(1+\Gamma_L) = \frac{V_x^+}{Z_0}(1-\Gamma_L)$

$\therefore \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = -1$ (SINCE $Z_L = 0$)

$\Rightarrow V(x,s) = V_x^+\left[e^{-\frac{s}{v_p}x} - e^{\frac{s}{v_p}x}\right]$

$I(x,s) = V_x^+/Z_0\left[e^{-\frac{s}{v_p}x} + e^{\frac{sx}{v_p}}\right]$

$Z(x,s) = Z_0\left[\frac{e^{-\frac{sx}{v_p}} - e^{\frac{sx}{v_p}}}{e^{-\frac{sx}{v_p}} + e^{sx/v_p}}\right]$

$Z_{IN} = Z(-\ell,s) = Z_0\left[\frac{1-e^{-2s\ell/v_p}}{1+e^{-2s\ell/v_p}}\right]$

$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} = 1$ (SINCE $Z_G = \infty$)

$V(-\ell,s) = \frac{Z_0\, V_G(s)}{Z_0 + Z_G}\left(\frac{1+e^{-\frac{2s\ell}{v_p}}}{1+e^{-2s\ell/v_p}}\right) V_g(s)$

$= V_G(s)\left(\frac{Z_0}{Z_0+Z_G}\right)\left(1+\sum_{n=1}^{\infty} 2(-1)^n e^{-\frac{2s\ell n}{v_p}}\right)$

FOR $T_G = 1 + \Gamma_G = 2$

$(\Gamma_G)^n = (1)^n = 1$

$(\Gamma_L)^n = (-1)^n$

DIVIDING BY S IN LAPLACE DOMAIN

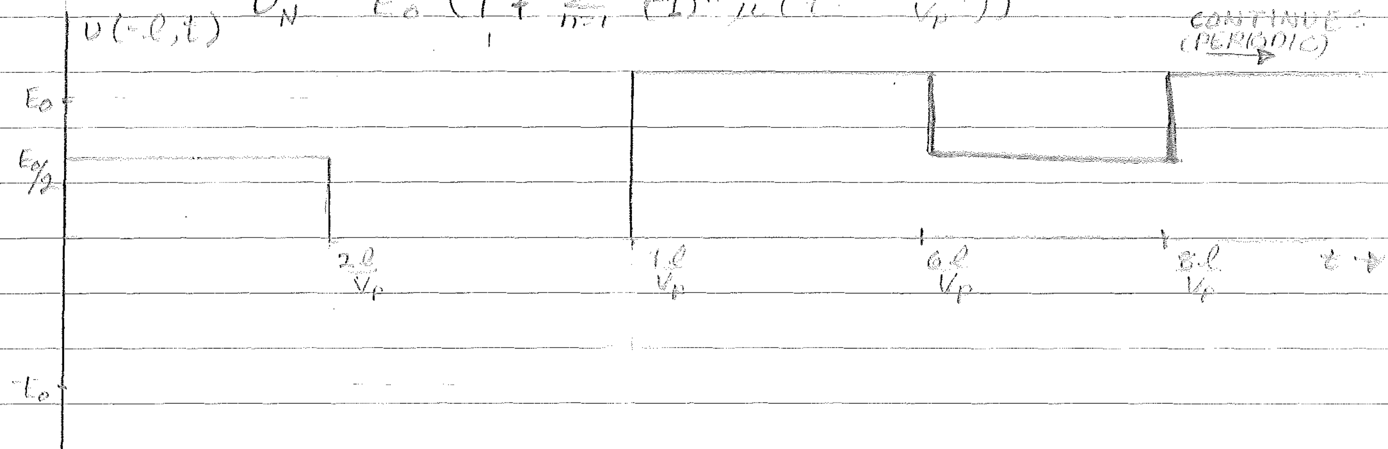
$\Rightarrow$ INTEGRATING IN TIME DOMAIN

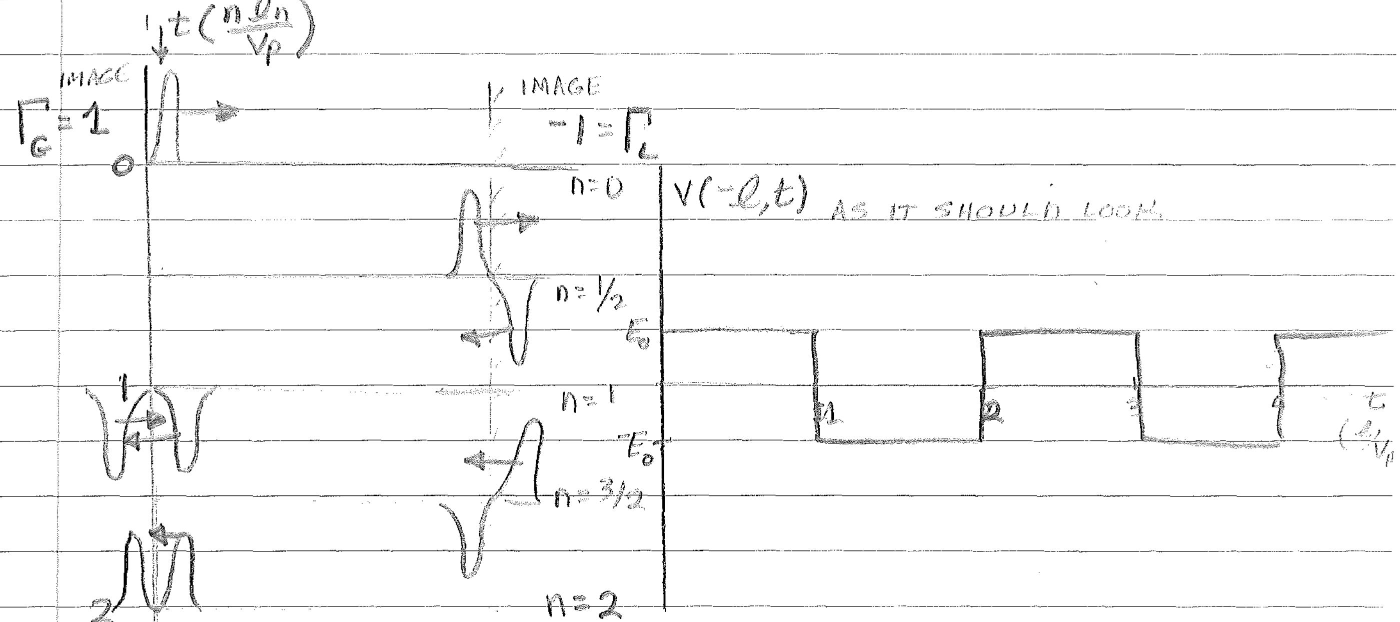
BOB MARKS

$$\frac{1}{s}V(-\ell,s) = V_G(s)\left(\frac{Z_0}{Z_0+Z_G}\right)\left(\frac{1}{s}+\frac{1}{s}\sum_{n=1}^{\infty}2(-1)^n e^{-\frac{2s\ell n}{v_p}}\right)$$

$$\therefore\ v(-\ell,t) = V_0\left(\frac{Z_0}{Z_0+Z_G}\right)\left(1+2\sum_{n=1}^{\infty}(-1)^n\mu\left(t-\frac{2\ell n}{v_p}\right)\right)$$

$$v_N = E_0\left(1+\sum_{n=1}^{\infty}(-1)^n\mu\left(t-\frac{2\ell n}{v_p}\right)\right)$$





$$V\left(-\ell,\frac{n\ell_n}{v_p}\right) = V\left(-\ell,\frac{(n+2)\ell}{v_p}\right)$$

x = -ℓ   x = 0

BOB MARKS

$V(x,s) = V_+\left[e^{-\frac{sx}{v_p}} + \Gamma_L e^{\frac{sx}{v_p}}\right] \ni \Gamma_L = \frac{j\omega M - Z_0}{j\omega M + Z_0}$

MASS IS ANALAGOUS TO INDUCTION

TORQUE TO VOLTAGE, VELOCITY TO CURRENT,

AND BULK MODULUS TO $1/C$

$I(x,s) = \frac{V^+}{Z_0}\left[e^{-\frac{sx}{v_p}} - \Gamma_L e^{-\frac{s\ell}{v_p}}\right]$

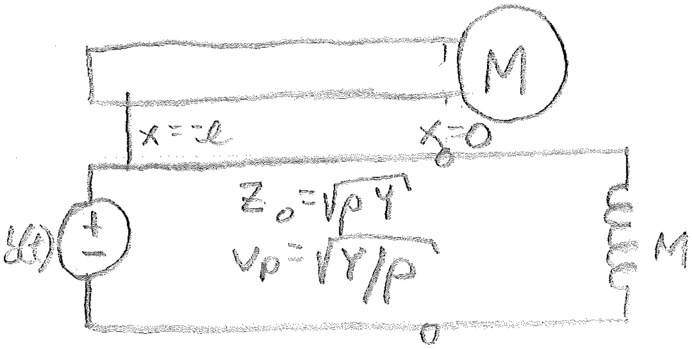
$v_p = \sqrt{Y/\rho}$; $Z_0 = \sqrt{Y\rho}$

AT $x = 0$; $I(0,s) = \left[(1-\Gamma_L)(1+\Gamma_L e^{-2s\ell/v_p})\right]\frac{1}{Z_0}$

$= \frac{1}{Z_0}\sum_{n=1}^{\infty}(-1)^{n+1} e^{-\frac{(2n-1)\ell}{v_p}}\Gamma_L^{n-1}(1-\Gamma_L)$

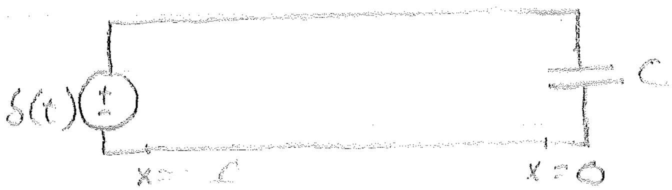
$\Rightarrow i(0,t) = \frac{1}{Z_0}\sum_{n=1}^{\infty}(-1)^{n+1}\Gamma_L^{n-1}(1-\Gamma_L)\,\delta\!\left(t - \frac{(2n-1)\ell}{v_p}\right)$

= VELOCITY OF MASS



BOB MARKS

1-11)



τ ANALAGOUS TO VOLTAGE

ω " TO CURRENT

NOW $\Gamma_L = \frac{1 - j\omega C}{1 + j\omega C}$

$$\Rightarrow I(0,s) = \frac{1}{Z_0} \sum_{n=1}^{\infty} \Gamma_L^{n-1}(1-\Gamma_L)(-1)^{n+1} e^{-\frac{(2n-1)\ell}{v_p}}$$

$$i(0,t) = \frac{1}{Z_0} \sum_{n=1}^{\infty} \Gamma_L^{n-1}(1-\Gamma_L)(-1)^{n+1} \delta\left(t - \frac{(2n-1)\ell}{v_p}\right)$$

(1-8). The diff. eqn. and boundary and initial conditions are:

(a) $\rho \dfrac{\partial^2 u}{\partial t^2} = Y \dfrac{\partial^2 u}{\partial x^2}$

(b) $\sigma(-L,t) = Y \dfrac{\partial u}{\partial x}\Big|_{x=-L} = -\sigma\delta(t)$

(the minus sign implies a compression)

$u(0,t) = 0 \quad ; \quad u(x,0) = \dot{u}(x,0) = 0.$

Taking $\mathcal{L}$-form: $\dfrac{d^2U}{dx^2} = \dfrac{\rho}{Y}\left[s^2U - su(x,0) - \dot{u}(x,0)\right] = \dfrac{\rho}{Y}s^2U = \dfrac{s^2}{v_p^2}U.$

$\therefore\ U(x,s) = A(s)e^{(s/v_p)x} + B(s)e^{-(s/v_p)x}$

$U(0,s) = \mathcal{L}[u(0,t)] = 0 = A(s) + B(s). \Rightarrow A = -B.$

$\dfrac{dU}{dx}\Big|_{x=-L} = \mathcal{L}\left[\dfrac{\partial u}{\partial x}\Big|_{x=-L}\right] = -\dfrac{\sigma}{Y} = \dfrac{s}{v_p}\left[Ae^{-sL/v_p} - Be^{sL/v_p}\right].$

$= \dfrac{sA}{v_p}\left[e^{-sL/v_p} + e^{sL/v_p}\right].$

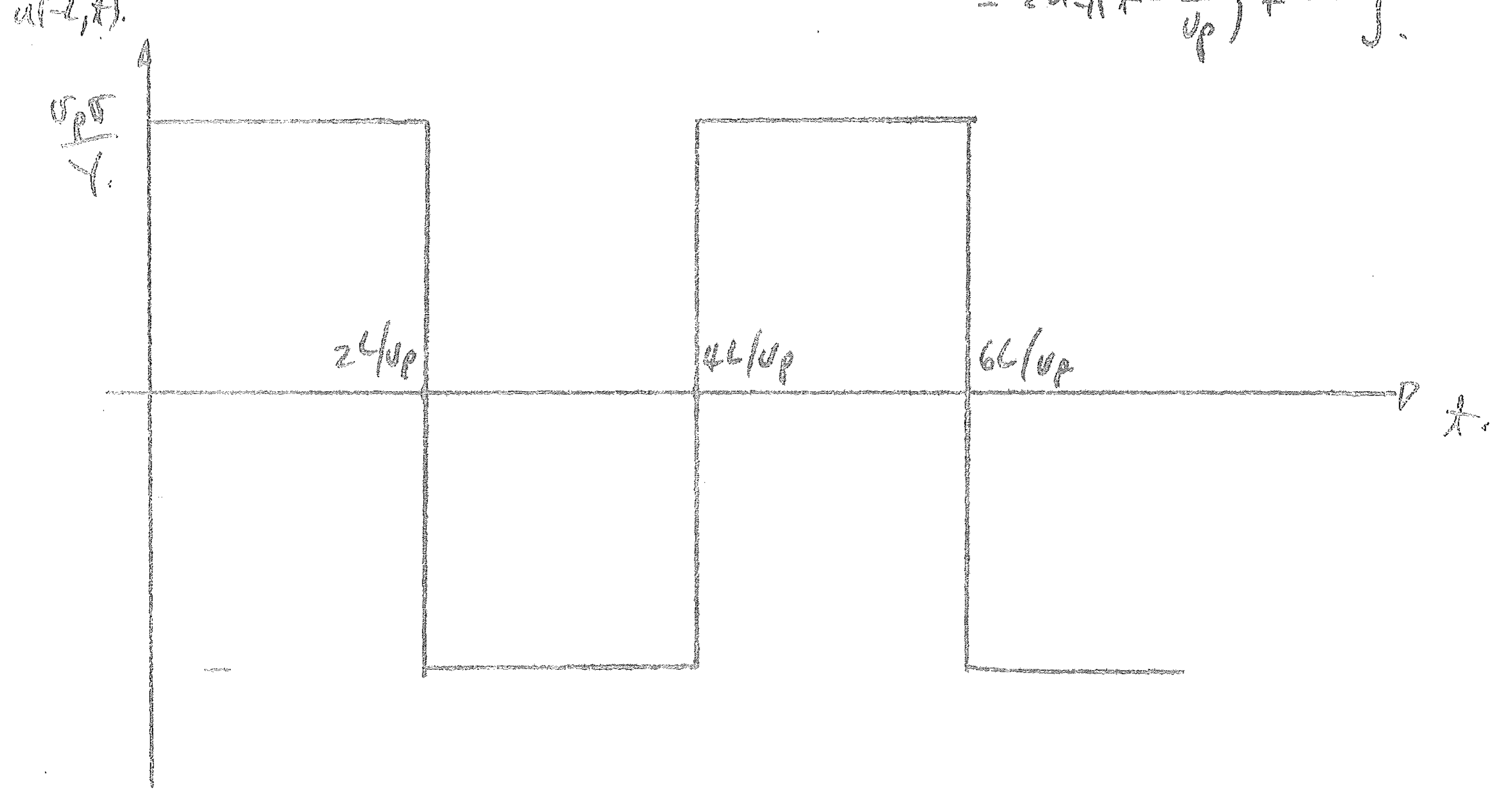
$\therefore\ A(s) = -\dfrac{\sigma v_p}{sY} \cdot \dfrac{1}{e^{sL/v_p} + e^{-sL/v_p}} = -B(s).$

$\therefore\ U(-L,s) = \dfrac{\sigma v_p}{sY} \cdot \dfrac{e^{sL/v_p} - e^{-sL/v_p}}{e^{sL/v_p} + e^{-sL/v_p}} = \dfrac{\sigma v_p}{sY}\left\{1 - 2e^{-2sL/v_p} + 2e^{-4sL/v_p} - 2e^{-6sL/v_p} + \dots\right\}$

Now $\mathcal{L}^{-1}\left[\dfrac{\alpha e^{-s\tau}}{s}\right] = \alpha u_{-1}(t-\tau).$

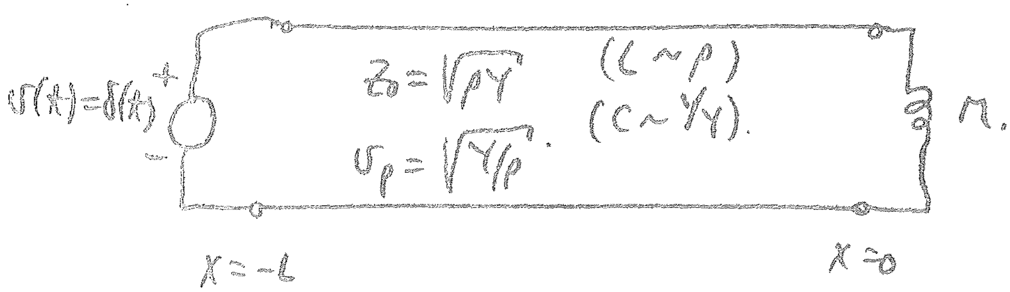
(e)

Hence, $$u(-L,t) = \mathcal{L}^{-1}\left[U(-L,s)\right] = \frac{v_p\sigma}{Y}\left[u_{-1}(t) - 2u_{-1}\left(t-\frac{2L}{v_p}\right) + 2u_{-1}\left(t-\frac{4L}{v_p}\right) - 2u_{-1}\left(t-\frac{6L}{v_p}\right) + \cdots\right].$$



Note that the initial <u>displacement</u> of the left face is <u>positive</u> (to the right) because the impulsive stress was <u>compressive</u> (negative), and then changes sign every $2L/v_p$ seconds (the round-trip time).

(1-10). If torque is analogous to voltage and velocity to current, then the equiv. ckt. is ①



Starting with (1-35) & (1-36);

$$V(x,s)=V_+\left[e^{-sx/v_p}+\Gamma_L e^{sx/v_p}\right],\qquad \Gamma_L=\frac{sM-Z_0}{sM+Z_0}.$$

$$I(x,s)=\frac{V_+}{Z_0}\left[e^{-sx/v_p}-\Gamma_L e^{sx/v_p}\right].$$

$$V(-\ell,s)=1=V_+\left[e^{+s\ell/v_p}+\Gamma_L e^{-s\ell/v_p}\right]\Rightarrow V_+=\frac{1}{\left[e^{s\ell/v_p}+\Gamma_L e^{-s\ell/v_p}\right]}$$

$$\therefore\ I(x,s)=\frac{1}{Z_0}\,\frac{\left[e^{-sx/v_p}-\Gamma_L e^{sx/v_p}\right]}{\left[e^{s\ell/v_p}+\Gamma_L e^{-s\ell/v_p}\right]}.$$

At $x=0$ (load); $$I(0,s)=\frac{1}{Z_0}\,\frac{(1-\Gamma_L)\cdot e^{-s\ell/v_p}}{\left[1+\Gamma_L e^{-2s\ell/v_p}\right]}.$$

$$=\frac{1}{Z_0}(1-\Gamma_L)e^{-s\ell/v_p}\left[1-\Gamma_L e^{-2s\ell/v_p}+\Gamma_L^2 e^{-4s\ell/v_p}-\Gamma_L^3 e^{-6s\ell/v_p}+\cdots\right].$$

$$=\frac{1}{Z_0}\left(e^{-s\ell/v_p}\cdot(1-\Gamma_L)-(\Gamma_L-\Gamma_L^2)e^{-3s\ell/v_p}+\Gamma_L^2(1-\Gamma_L)e^{-5s\ell/v_p}+\cdots\right].$$

= velocity of mass.

Now $1-\Gamma_L = 1-\dfrac{sM-Z_0}{sM+Z_0} = \dfrac{sM+Z_0-sM+Z_0}{sM+Z_0} = \dfrac{2Z_0}{sM+Z_0} = \dfrac{2Z_0/M}{s+Z_0/M}.$

$$\therefore \quad \mathcal{L}^{-1}\{(1-\Gamma_L)\} = (2Z_0/M)\cdot e^{-t\cdot Z_0/M}.$$

$\mathcal{L}^{-1}\{\Gamma_L\}$ as derived on p. 23b.

$$\mathcal{L}^{-1}\{\Gamma_L(1-\Gamma_L)\} = \int_0^t \left(\delta(\tau) - \frac{2Z_0}{M}e^{-\tau\cdot Z_0/M}\right)\cdot\frac{2Z_0}{M}e^{-(t-\tau)\cdot Z_0/M}\cdot d\tau$$

$$= \frac{2Z_0}{M}e^{-t\cdot Z_0/M} - \left(\frac{2Z_0}{M}\right)^2 t\, e^{-t\,Z_0/M}$$

$$= \frac{2Z_0}{M}e^{-t\cdot Z_0/M}\left(1-\frac{2Z_0}{M}t\right).$$

Etc.

$$\therefore \quad i(0,t) = i_{mass}(t) = \frac{1}{Z_0}\left[\frac{2Z_0}{M}e^{-(t-\ell/v_p)\cdot Z_0/M}\cdot u_{-1}(t-\ell/v_p)\right.$$

$$- \frac{2Z_0}{M}e^{-(t-3\ell/v_p)\cdot Z_0/M}\cdot\left(1-\frac{2Z_0}{M}(t-3\ell/v_p)\right)\cdot u_{-1}(t-3\ell/v_p)$$

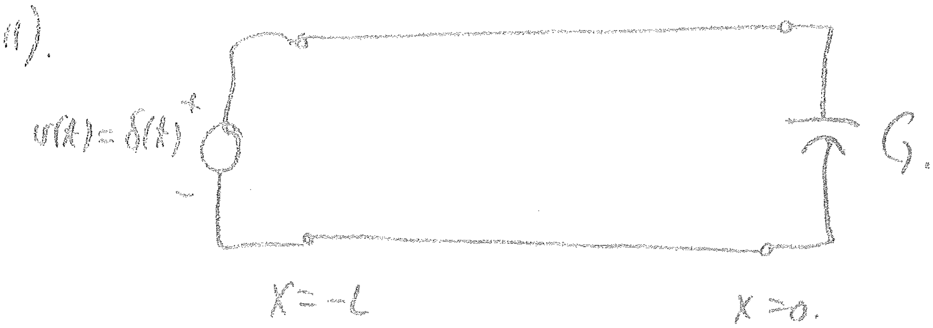
$$\left. + \cdots \right].$$

$$= \frac{2}{M}\left[e^{-(t-\ell/v_p)\cdot Z_0/M}\cdot u_{-1}(t-\ell/v_p) - e^{-(t-3\ell/v_p)\cdot Z_0/M}\cdot\left(1-\frac{2Z_0}{M}\left(t-\frac{3\ell}{v_p}\right)\right)u_{-1}(t-3\ell/v_p)\right.$$

$$\left. + \cdots \right].$$

Note that the first term is what one would obtain for the current flow through an inductor M in series with Z_0 and an impulsive voltage source of $\underline{2\,\delta(t)}$. The factor 2 is due to the impulsive voltage reflecting off of an <u>open circuit</u>.

(1-11).



$v \sim$ voltage
$\omega \sim$ current.

Everything proceeds as in Prob. (1-10) except that $\Gamma_L = \dfrac{1 - s Z_0 C_L}{1 + s Z_0 C_L}$.

$$\therefore \quad I(0,s) = \frac{1}{Z_0}\left[(1-\Gamma_L)e^{-sl/u_p} + \Gamma_L(1-\Gamma_L)e^{-3sl/u_p} + \cdots\right].$$

$$1 - \Gamma_L = 1 - \frac{1 - s Z_0 C_L}{1 + s Z_0 C_L} = \frac{2 Z_0 C_L s}{1 + Z_0 C_L s} = 2 - \frac{2}{1 + s Z_0 C_L}.$$

$$\mathcal{L}^{-1}\{(1-\Gamma_L)\} = 2\delta(t) - \frac{2}{Z_0 C_L} e^{-t/Z_0 C_L}.$$

From p. 236.

$$\mathcal{L}^{-1}\{\Gamma_L(1-\Gamma_L)\} = 2\int_0^t \left(-\delta(\tau) + \frac{2}{Z_0 C_L}e^{-\tau/Z_0 C_L}\right)\left(\delta(t-\tau) - \frac{1}{Z_0 C_L}e^{-(t-\tau)/Z_0 C_L}\right) d\tau$$

$$= 2\left[-\delta(t) + \frac{3}{Z_0 C_L}e^{-t/Z_0 C_L} - \frac{2}{(Z_0 C_L)^2}e^{-t/Z_0 C_L}\cdot t\right].$$

$$= -2\delta(t) + \frac{2e^{-t/Z_0 C_L}}{Z_0 C_L}\left(3 - \frac{2t}{Z_0 C_L}\right).$$

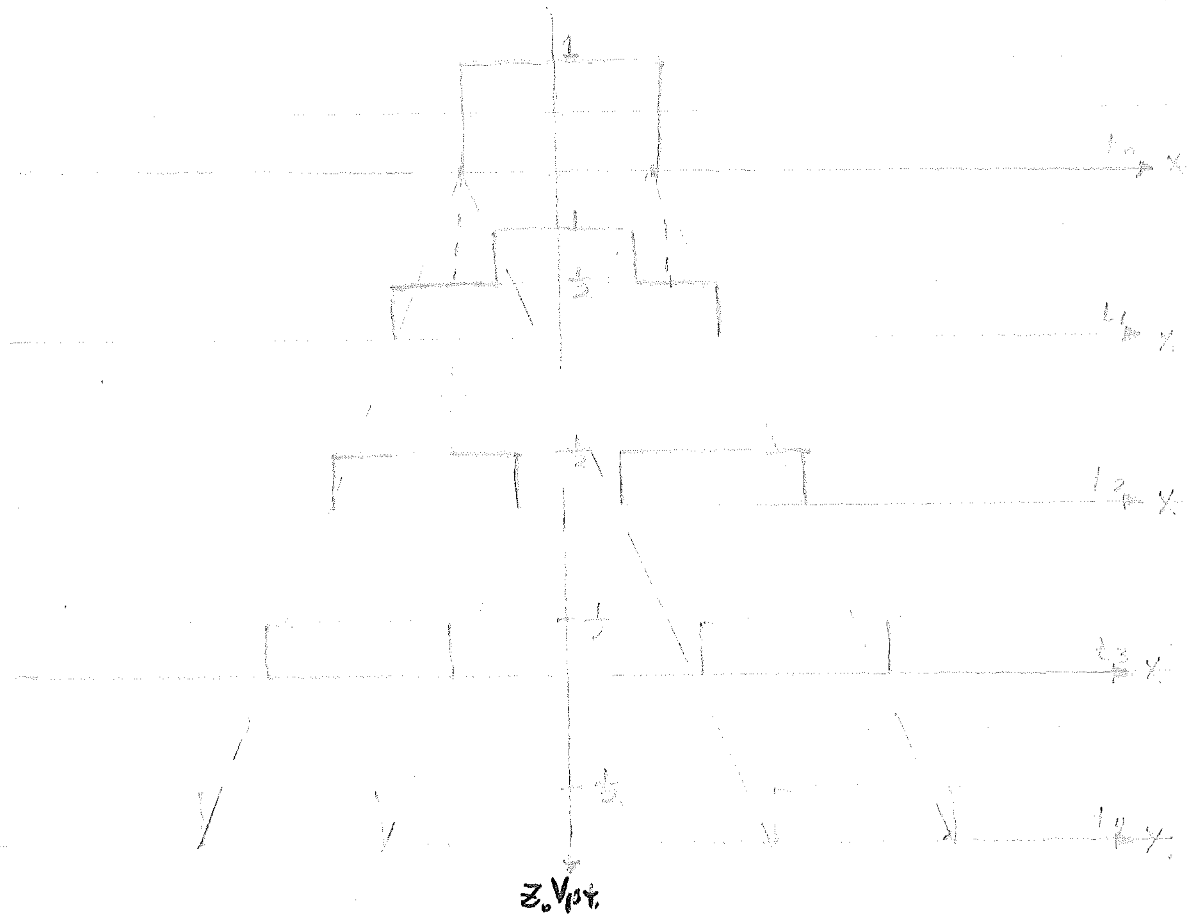
$$\therefore \quad i(t) = \text{velocity}(t) = \frac{2}{Z_0}\Big[\delta(t - l/u_p) - \frac{e^{-(t - l/u_p)/Z_0 C_L}}{Z_0 C_L}u_{-1}(t - l/u_p)$$
$$+ \delta(t - 3l/u_p) - \frac{e^{-(t - 3l/u_p)/Z_0 C_L}}{Z_0 C_L}\left(3 - \frac{2(t - 3l/u_p)}{Z_0 C_L}\right)\cdot u_{-1}\left(t - \frac{3l}{u_p}\right)$$
$$+ \cdots\Big].$$

BOB MARKS

20/30

$V_0(x) = 0$

$\Rightarrow v(x,t) = \frac{1}{2}\left[i_0(x - v_p t) - i_0(x + v_p t) \right]$ ✓



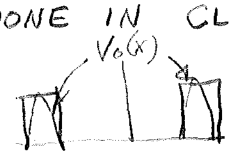
BOB MARKS

////

$$\therefore i(x,t) = \frac{1}{2Z_0}\left[V_0(x - V_p t) - V_0(x + v_p t)\right]$$

(DONE IN CLASS)

1.2)



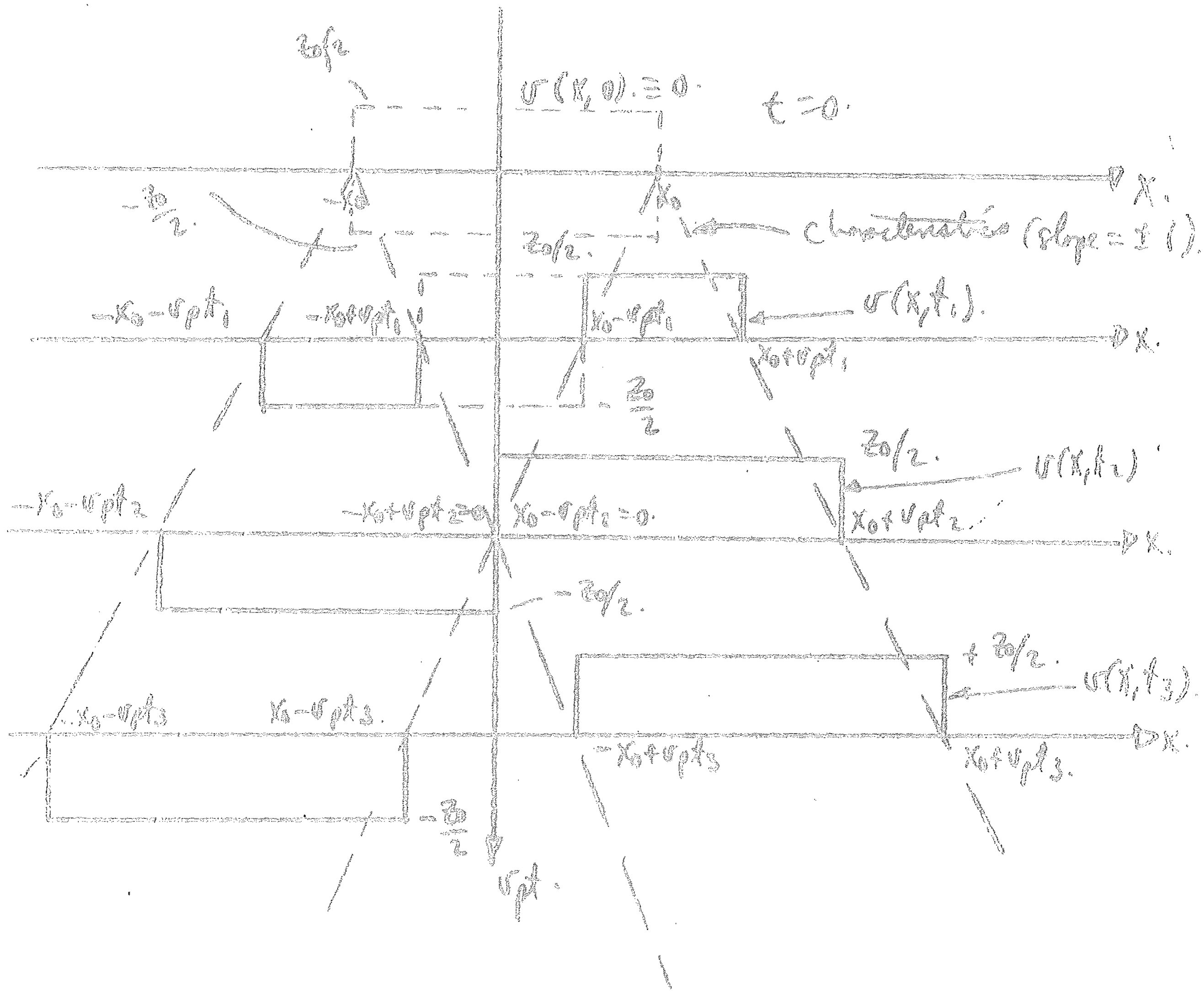
$$i(x,t) = \frac{1}{2Z_0}\left[V_0(x - V_p t) - V_0(x + V_p t)\right]$$

$$V_0(x) = V_0(-x) \quad (V_0 \text{ EVEN})$$

$$i(0,t) = \frac{1}{2Z_0}\left[V_p(-V_p t) - V_p(V_p t)\right] = 0$$

Traveling Waves: Solutions of (1-1) – (1-2).

(1-1). $v(x,t) = \frac{Z_0}{2}\left[i_0(x - v_p t) - i_0(x + v_p t) \right]$. Let current pulse extend from $-x_0 < x < x_0$.



(1-2). $\frac{\partial i}{\partial x}(x,t) = -C\frac{\partial v}{\partial t} = -\frac{C}{2}\left[-v_p v_0'(x - v_p t) - v_p Z_0 i_0'(x - v_p t) + v_p v_0'(x + v_p t) - v_p Z_0 i_0'(x + v_p t) \right].$

$= \frac{C v_p}{2}\left[v_0'(x - v_p t) + Z_0 i_0'(x - v_p t) - v_0'(x + v_p t) + Z_0 i_0'(x + v_p t) \right].$

$\therefore\ i(x,t) = \frac{1}{2}\left[\frac{v_0(x - v_p t)}{Z_0} - \frac{v_0(x + v_p t)}{Z_0} + i_0(x - v_p t) + i_0(x + v_p t) \right].$

Note that $i(x,0) = \frac{1}{2}\left[i_0(x) + i_0(x) \right] = i_0(x)$, as it should be.

(1-2) Now that we have done (1-4) we can follow the derivation on p. 5 and parallel the arguments using $i(x,t)$ as the basis.

Thus, for the open-circuited, semi-infinite line set

$$i(x,t) = \frac{1}{2}\left[I_0(x-v_p t) + \frac{\bar{V}_0(x-v_p t)}{Z_0} + I_0(x+v_p t) - \frac{\bar{V}_0(x+v_p t)}{Z_0} \right].$$

Then,
$$i(0,t) = \frac{1}{2}\left[I_0(-v_p t) + I_0(v_p t) + \frac{\bar{V}_0(-v_p t)}{Z_0} - \frac{\bar{V}_0(v_p t)}{Z_0} \right].$$

$$\frac{\partial i(0,t)}{\partial t} = \frac{-v_p}{2}\left[I_0'(-v_p t) - I_0'(v_p t) + \frac{\bar{V}_0'(-v_p t)}{Z_0} + \frac{\bar{V}_0'(v_p t)}{Z_0} \right].$$

Hence, if $I_0(\xi)$ is taken to be an odd function of its argument and $\bar{V}_0$ is even, then these expressions for $i(0,t)$ and $\frac{\partial i(0,t)}{\partial t}$ vanish identically.

Hence, take

$$I_0(x) = \begin{cases} i_0(x), & x > 0 \\ -i_0(-x), & x < 0. \end{cases}$$

~~$\bar{V}_0(x) = \begin{cases} v_0(x), & x > 0 \\ v_0(-x), & x < 0 \end{cases}$~~
$$\bar{V}_0(x) = \begin{cases} v_0(x), & x > 0 \\ v_0(-x), & x < 0 \end{cases}$$

~~where $\left.\frac{\partial i}{\partial t}\right|_{t=0} = -\frac{1}{L}\frac{\partial v(x,0)}{\partial x} = -\frac{1}{L}\frac{\partial v_0(x)}{\partial x} = \bar{I}_0$~~

~~$\therefore \bar{V}_0(x) = \int^x \bar{V}_0(\xi)\,d\xi$ = even function = $\begin{cases} v_0(x), & x > 0 \\ v_0(-x), & x < 0 \end{cases}$~~

Now if $i_0(x) \equiv 0$, then $I_0(x) \equiv 0$, and so the expression for $i(x,t)$ above becomes
$$i(x,t) = \frac{1}{2Z_0}\left[\bar{V}_0(x-v_p t) - \bar{V}_0(x+v_p t) \right] =$$

$$= \frac{1}{2Z_0}\left[v_0(x - v_p t) - v_0(x + v_p t) \right], \qquad x \geq v_p t.$$

$$= +\frac{1}{2Z_0}\left[v_0(v_p t - x) - v_0(v_p t + x) \right], \qquad x < v_p t.$$

Note that at $x = 0$, $i(0,t) = \frac{1}{2Z_0}\left[v_0(v_p t) - v_0(v_p t)\right] \equiv 0$ (using the second expression above, for $x < v_p t$).

Now we can easily compute $v(x,t)$ from this expression for $i(x,t)$:

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} = -\frac{L}{2Z_0}\left[-v_p v_0'(x - v_p t) - v_p v_0'(x + v_p t) \right], \qquad x > v_p t$$

$$= -\frac{L}{2Z_0}\left[v_p v_0'(-x + v_p t) - v_p v_0'(v_p t + x) \right], \qquad x < v_p t.$$

$$\therefore \quad v(x,t) = \frac{v_p L}{2Z_0}\left[v_0(x - v_p t) + v_0(x + v_p t) \right], \qquad x > v_p t.$$

$$= \frac{+v_p L}{2Z_0}\left[v_0(x + v_p t) + v_0(-x + v_p t) \right], \qquad x < v_p t.$$

$$= \frac{1}{2}\left[v_0(x - v_p t) + v_0(x + v_p t) \right], \qquad x > v_p t.$$

$$= \frac{1}{2}\left[v_0(x + v_p t) + v_0(v_p t - x) \right], \qquad x < v_p t.$$

Same picture as Figure 1-2 except that <u>image</u> <u>voltage</u> is <u>upright</u>, hence voltage at $x=0$ is $v_0(v_p t)$. If we had an initial value of <u>current</u> and asked for the current picture, it would have been identical to Figure 1-2. In our example the reflected <u>voltage</u> is <u>upright</u>.

(1-3) Eqn. (1-18) holds except that the upper limit is L:

$$V(x,s) = -\frac{s}{v_p^2}\int_0^L G(x|x')\, v_0(x')\,dx'.$$

The solution for the transformed current, $I(x,s)$, follows from (1-15)(a) with $i_0(x)$, the initial current distribution, set equal to zero:

$$\therefore\ I(x,s) = -\frac{1}{sL}\frac{dV}{dx} = -\frac{1}{sL}\cdot\frac{-s}{v_p^2}\int_0^L \frac{\partial G(x|x')}{\partial x}\, v_0(x')\,dx'$$

$$= C\int_0^L \frac{\partial G(x|x')}{\partial x}\, v_0(x')\,dx', \text{ where we have used } v_p^2 = \frac{1}{LC}.$$

Because of the open circuit at $x=L$, $I(L,s)=0$

$$\therefore\ I(L,s) = 0 = C\int_0^L \left.\frac{\partial G(x|x')}{\partial x}\right|_{x=L} v_0(x')\,dx'$$

$$\therefore\ \boxed{\left.\frac{\partial G(x|x')}{\partial x}\right|_{x=L} = 0.}$$

This is the condition that replaces the vanishing of $G(x|x')$ at $x=L$ in Example 4. All other conditions remain in effect in (1-22). Thus two functions which satisfy (1-22)(a), (c), as well as the condition that $\left.\frac{\partial G(x|x')}{\partial x}\right|_{x=L}=0$ are:

$$G(x|x') = A\sinh\left(\frac{sx}{v_p}\right), \quad x<x'$$

$$= B\cosh\left[\frac{s(L-x)}{v_p}\right], \quad x>x'.$$

...ity of $G(x|x')$ as $x \to x'_{\pm}$ requires that

$$A \sinh\left(\frac{sx'}{v_p}\right) = B \cosh\left[\frac{s(L-x')}{v_p}\right],$$

and the discontinuity in slope condition (1-22)(b) implies that

$$-\frac{Bs}{v_p}\sinh\left[\frac{s(L-x')}{v_p}\right] - A\frac{s}{v_p}\cosh\left(\frac{sx'}{v_p}\right) = 1.$$

Solution of these two equations for A and B yields

$$A = \frac{-\left(\frac{v_p}{s}\right)\cosh\frac{s(L-x')}{v_p}}{\cosh\left(\frac{sL}{v_p}\right)}, \quad B = \frac{-\left(\frac{v_p}{s}\right)\sinh\left(\frac{sx'}{v_p}\right)}{\cosh\left(\frac{sL}{v_p}\right)},$$

with the consequent expression for $G(x|x')$:

$$G(x|x') = \frac{-\left(\frac{v_p}{s}\right)\cosh\frac{s(L-x')}{v_p}\cdot\sinh\left(sx/v_p\right)}{\cosh\left(\frac{sL}{v_p}\right)}, \quad x < x'$$

$$= \frac{-\left(\frac{v_p}{s}\right)\sinh\left(\frac{sx'}{v_p}\right)\cosh\frac{s(L-x)}{v_p}}{\cosh\left(\frac{sL}{v_p}\right)}, \quad x > x'.$$

Note that reciprocity continues to hold.

The solution for $V(x,s)$, corresponding to (1-27) is:

$$V(x,s) = \frac{1}{v_p}\left\{\int_0^x \frac{\cosh\frac{s(L-x')}{v_p}\cdot\sinh\left(\frac{sx'}{v_p}\right)}{\cosh\left(\frac{sL}{v_p}\right)}\sigma_0(x')\,dx' + \int_x^L \frac{\sinh\left(\frac{sx}{v_p}\right)\cosh\frac{s(L-x')}{v_p}}{\cosh\left(sL/v_p\right)}\sigma_0(x')\,dx'\right\}.$$

Bob Marks
Traveling Waves I

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VALUES ASSUMED ON PAGE 1

2-10) $L = \frac{\mu}{2\pi} \ln \frac{b}{a}$

$C = \frac{2\pi\epsilon}{\ln b/a}$

$R = \frac{1}{2}\sqrt{f\mu/\pi\sigma}\left(\frac{1}{a}+\frac{1}{b}\right)$

5/10 $G = 0$

$\mu = 4\pi \times 10^{-7}\ \frac{H}{M}$

$\epsilon = \frac{1}{36\pi} \times 10^{-9}\ \frac{F}{M}$

$\sigma = 5.8 \times 10^{7}\ \frac{\mho}{M}$

$b = 2.7$ mm

$a = 1$ mm

$f = 10^{10}$ Hz $\Rightarrow \omega = 2\pi \times 10^{10}\ \frac{RAD}{SEC}$

$L = 6.241 \times 10^{-7}$ H/M

$C = 5.593 \times 10^{-11}$ F/M

$R = 5.690\ \Omega/M$

$\gamma = [(R + j\omega L)(j\omega C)]^{1/2}$

$= [-\omega^2 LC + j\omega CR]^{1/2}$

$= \left[(\omega^4 L^2 C^2 + \omega^2 C^2 R^2)^{\frac{1}{2}} \angle \tan^{-1} \frac{-\omega CR}{\omega^2 CL}\right]^{1/2}$

$= \left[\{\omega^2 C^2 (L^2\omega^2 + R^2)\}^{1/2} \angle \tan^{-1} \frac{-R}{LC}\right]^{\frac{1}{2}}$

$= (\omega C)^{\frac{1}{2}} (L^2\omega^2 + R^2)^{\frac{1}{4}} \angle \frac{1}{2} \tan^{-1} \frac{-R}{LC}$

$\alpha = \sqrt{\omega C}\,(\omega^2 L^2 + R^2)^{\frac{1}{4}} \cos \frac{1}{2} \tan^{-1} \frac{-R}{LC}$

$\tan^{-1} \frac{-R}{LC} = -90^\circ$

$\Rightarrow \alpha = 1.7~~2~~3 \times 10^3$ ~~NEPERS/SEC~~ **NEPERS/M**

$\beta = \sqrt{\omega C}\,(\omega^2 L^2 + R^2)^{\frac{1}{4}} \sin \frac{1}{2} \tan^{-1} \frac{-R}{LC}$

FROM DEFN. OF α & β

$\Rightarrow \beta = 1.7~~2~~3 \times 10^3$ ~~RADIANS/SEC~~ **RADIANS/M**

$\Rightarrow V_p = \frac{\omega}{\beta}$ ✓ $= 9.4~~4~~1 \times 10^6$ m/SEC

$\Rightarrow \lambda = \frac{2\pi}{\beta}$ ✓ $= 9.4~~4~~1 \times 10^{-4}$ m 2.98 cm

$Z_0^2 = \frac{R + j\omega L}{j\omega C}$

$Z_0 = \left(\frac{R^2 + \omega^2 L^2}{\omega^2 C^2}\right)^{\frac{1}{4}} \angle \frac{1}{2} \tan \frac{\omega L}{R} - \frac{\pi}{4}$

$= 1.885 \angle -1.305 \times 10^{-2\circ}\ \Omega$

5/10

2.12) $R = 2.56\ \Omega/km$

$L = 1.94 \times 10^{-3}\ H/km$

$C = 6.22 \times 10^{-9}\ F/km$

$G = 6.8 \times 10^{-8}\ S/km$

$f = 10^3\ Hz \Rightarrow \omega = 6.2832 \times 10^3\ \frac{RAD}{SEC}$

$\gamma = [(R + j\omega L)(G + j\omega C)]^{1/2}$

$= [(2.56 + j(6.28\times10^3)(1.94\times10^{-3}))(6.8\times10^{-8} + j(6.28\times10^3)(6.22\times10^{-9}))]^{1/2}$

$= [(2.56 + j12.2)(.68\times10^{-7} + j39.1\times10^{-7})]^{1/2}$

$= [(12.45 \angle 78.15^\circ)(39.1\times10^{-7} \angle 89.00^\circ)]^{1/2}$

$= [4.86\times10^{-6} \angle 167.15^\circ]^{1/2}$

$= 6.90\times10^{-3} \angle 83.57^\circ$

$= .754\times10^{-3} + j6.45\times10^{-3}$

$\Rightarrow \alpha = .75 \times 10^{-3}\ \frac{NEPERS}{km}$ .0023 NP/KM

$\Rightarrow \beta = 6.45 \times 10^{-3}\ \frac{RAD}{km}$ .022 RAD/KM

$V_p = \frac{\omega}{\beta} = \frac{6.28\times10^3}{6.45\times10^{-3}}$

$\Rightarrow V_p = .97 \times 10^6\ \frac{km}{SEC}$

$\lambda = \frac{2\pi}{\beta} = \frac{6.28}{6.45\times10^{-3}}$

$\Rightarrow \lambda = 973\ km$ 287 KM

$Z_0^2 = \frac{R + j\omega L}{G + j\omega C}$

$= \frac{12.45 \angle 78.15^\circ}{3.91\times10^{-6} \angle 89.00^\circ}$

$= 3.18\times10^6 \angle -10.85^\circ\ \Omega$

$\Rightarrow Z_0 = 1.785 \angle -5.42\ M\Omega$ $562 \angle -6^\circ\ \Omega$

10/20

2-13,M) $\alpha = \left[(R^2+\omega^2L^2)(G^2+\omega^2C^2)\right]^{1/4}\cos\frac{1}{2}\left[\tan^{-1}\frac{\omega L}{R}+\tan^{-1}\frac{\omega C}{G}\right]$

$$= \left[(R^2+4\pi^2L^2f^2)(G^2+4\pi^2C^2f^2)\right]^{1/4}\cos\frac{1}{2}\left[\tan^{-1}\frac{2\pi L}{R}f+\tan^{-1}\frac{2\pi C}{G}f\right]$$

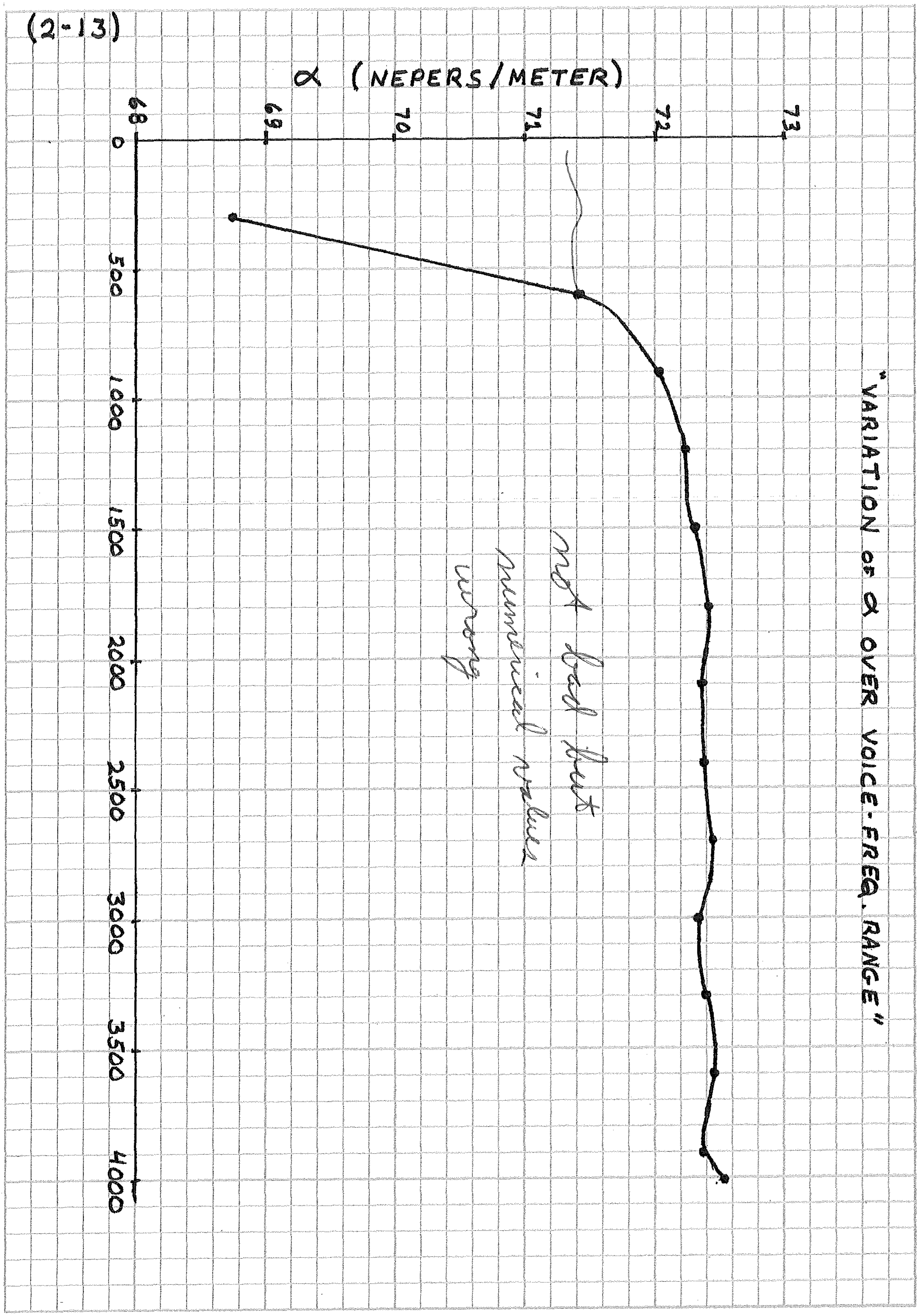
$$V_P = \frac{\omega}{\beta} = 2\pi f\left[(R^2+4\pi^2L^2f^2)(G^2+4\pi^2C^2f^2)\right]^{-1/4}$$
$$\left(\sin\frac{1}{2}\left[\tan^{-1}\frac{2\pi L}{R}f+\tan^{-1}\frac{2\pi C}{G}f\right]\right)^{-1}$$

FROM MONROE:

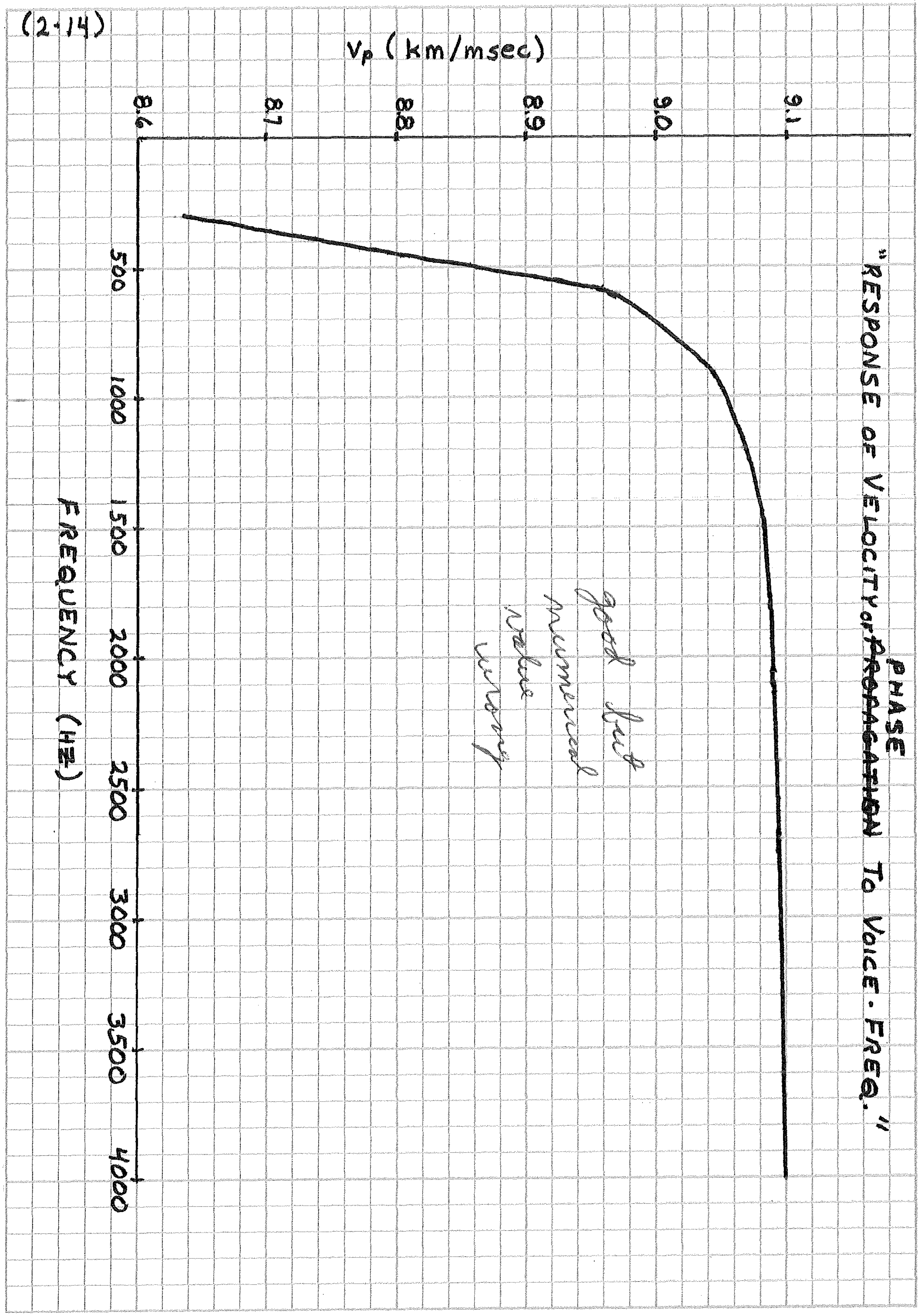
.0022 2.72×10^5

f	$\lvert\gamma\rvert \angle 2\psi$	$\alpha\left(\frac{N}{KM}\right)$	$\beta\left(\frac{RAD}{KM}\right)$	$V_P\left(\frac{KM}{SEC}\right)$
300	.22378∠145.01°	.068776	.21819	8638.76
600	.42626∠160.71°	.071417	.42023	8970.97
900	.62947∠166.86°	.072022	.62534	9042.92
1200	.83451∠170.07°	.072224	.83134	9069.06
1500	1.04034∠170.03°	.072299	1.03783	9081.28
1800	1.24656∠173.34°	.072409	1.24446	9088.10
2100	1.45304∠174.29°	.072374	1.45123	9092.03
2400	1.65964∠175.00°	.072395	1.66806	9094.75
2700	1.86636∠175.55°	.072459	1.86495	9096.63
3000	2.07314∠176.00°	.072352	2.07187	9097.82
3300	2.27996∠176.36°	.072411	2.27881	9098.63
3600	2.48682∠176.66°	.072473	2.48577	9099.57
3900	2.69373∠176.92°	.072393	2.69276	9100.12
4000	2.76271∠176.99°	.072360	2.76175	9100.21

.0022 2.87×10^5


(2-13)
α (NEPERS/METER)
68
69
70
71
72
73
0
500
1000
1500
2000
2500
3000
3500
4000
"VARIATION OF α OVER VOICE-FREQ. RANGE"
not bad but numerical values wrong

(2-14)



"RESPONSE OF VELOCITY of PROPAGATION PHASE TO VOICE-FREQ."
V_p (km/msec)
8.6
8.7
8.8
8.9
9.0
9.1
500
1000
1500
2000
2500
3000
3500
4000
FREQUENCY (Hz)
Good but numerical value wrong

2-14) (CONT.)

$f_{HI} = 4\,kH \Rightarrow T_{LO} = \frac{1}{f_{HI}} = .25\ m\,SEC$

$\Rightarrow 10\%$ OF $T_{LO} = 2.5 \times 10^{-5}$ SEC

FROM COMPUTATIONS:

$V_P(f_{LO}) = 8.63 \times 10^6 \frac{m}{SEC}$

$V_P(f_{HI}) = 9.10 \times 10^6 \frac{m}{SEC}$

$\Delta V_P = V_P(f_{HI}) - V_P(f_{LO}) = .47 \times 10^6 \frac{m}{SEC}$

LET Δt_T = DIFFERENCE IN TRANSIENT TIMES BETWEEN $f_{HI} = 4\,kHz$, AND $f_{LO} = .3\,kHz$

ℓ = TRANSMISSION LINE LENGTH

THEN $\Delta t_T = 2.5 \times 10^{-5}$

AND $\ell = \Delta t_T \Delta V_P$

$\Rightarrow \ell = (2.5 \times 10^{-5})(4.7 \times 10^5)$

$= 11.7\ m$ 205 km

5/10

370 ∠ " Ω

8/10

2.21) $5.0\times10^7 = \frac{V^2}{1.4\times10^2 + j8.45\times10^2} \Rightarrow V = 1.36\times10^5 \angle 38.8$

$5.0\times10^7 = I^2 (3.69\times10^2 \angle 7.6) \Rightarrow I = 3.69\times10^2 \angle 33.8$

$\alpha = 1.08\times10^{-3}\times50 = 5.36\times10^{-2}$

$e^{.0536} = 1.055$

$V = (1.36\times10^5)(1.055) = 1.436\times10^5\ V$ ✓

$I = (3.69)\times10^2(1.055) = 3.89\times10^2\ A$ ✓

$P = VI = 5.57\ MW$

$P_{loss} = 5.58\ MW - 5.0\ MW = 5.8\times10^5\ W$

$(50)(.119) = 5.95\ \Omega$

$P = I^2R = (3.89\times10^2)^2(5.95) = .9\ MW$

should be = .81 MW

Solutions of 2-5 to 2-10.

-5).
$$\frac{\partial^2 \delta}{\partial t^2} = c^2 \frac{\partial^2 \delta}{\partial x^2} + w_c^2 \delta.$$

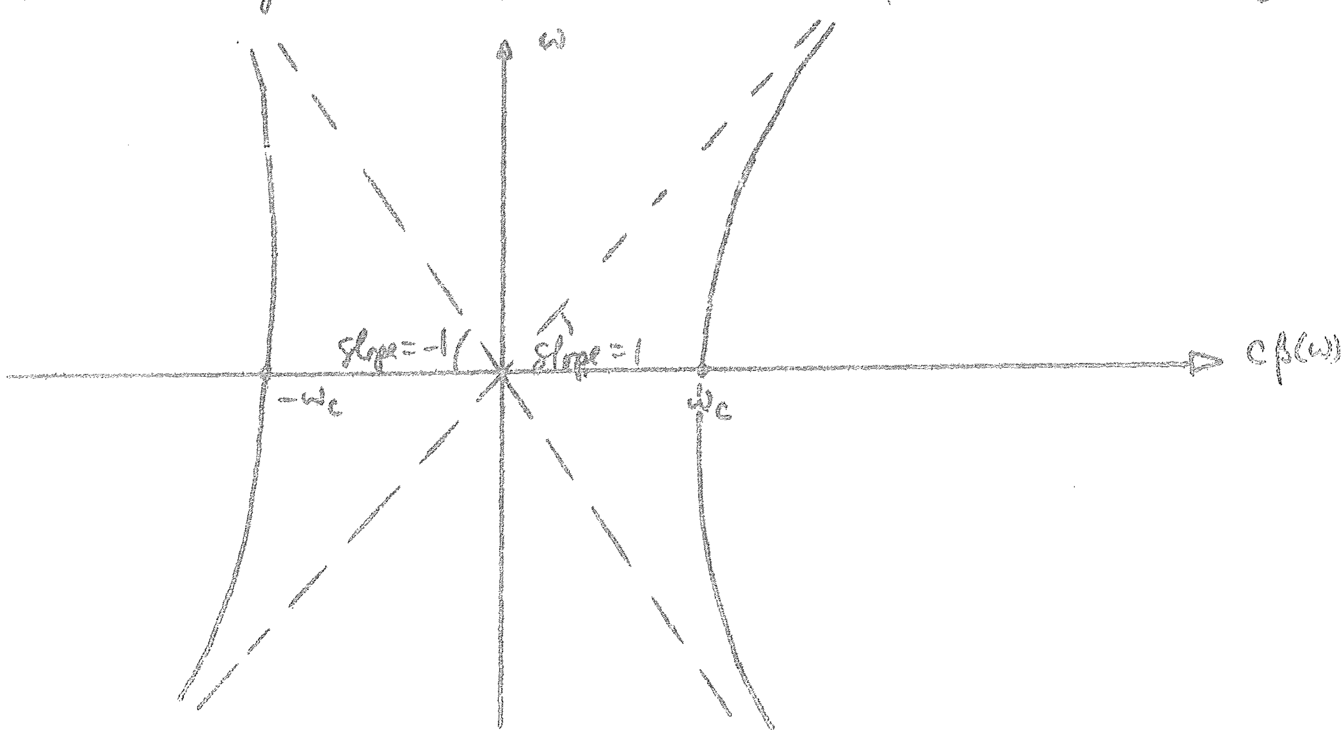
Take the Fourier transform:

$$-\omega^2 \tilde{\delta}(x,\omega) = c^2 \frac{d^2 \tilde{\delta}(x,\omega)}{dx^2} + w_c^2 \tilde{\delta}(x,\omega).$$

or
$$\frac{d^2 \tilde{\delta}(x,\omega)}{dx^2} = -\frac{\omega^2}{c^2}\left(1 + \frac{w_c^2}{\omega^2}\right)\tilde{\delta}(x,\omega).$$
$$= -\beta^2(\omega)\,\tilde{\delta}(x,\omega)$$

$$\therefore \quad \beta^2(\omega) = \frac{\omega^2}{c^2}\left(1 + \frac{w_c^2}{\omega^2}\right) = \frac{\omega^2}{c^2} + \frac{w_c^2}{c^2}.$$

or $\omega^2 = c^2\beta^2(\omega) - w_c^2$ which is the dispersion relation of the waveguide with the ω- and $c\beta(\omega)$- axes interchanged



(2-6). Take the Laplace transform of $\frac{\partial^2 \delta}{\partial t^2} = c^2 \frac{\partial^2 \delta}{\partial x^2} + \omega_c^2 \delta$:

$$s^2 \tilde{\delta}(x,s) - s\,\delta_0(x) - \dot{\delta}_0(x) = c^2 \frac{d^2 \tilde{\delta}(x,s)}{dx^2} + \omega_c^2 \tilde{\delta}(x,s).$$

But $\dot{\delta}_0(x) = 0$, and $\delta_0(x) = \delta_0 \sin \frac{\pi x}{L}$. Hence,

$$\frac{d^2 \tilde{\delta}(x,s)}{dx^2} + \frac{(\omega_c^2 - s^2)}{c^2} \tilde{\delta}(x,s) = -\frac{s}{c^2}\delta_0(x) = -\frac{s}{c^2}\delta_0 \sin \frac{\pi x}{L}.$$

Boundary conditions are $\tilde{\delta}(0,s) = \tilde{\delta}(L,s) = 0$, because $\delta(0,t) \equiv \delta(L,t) \equiv 0$, and the Laplace transform of $\delta(0,t)$, $\delta(L,t)$ must vanish identically as a result.

Let $\tilde{\delta}(x,s) = A \sin \frac{\pi x}{L}$, where A is an undetermined multiplier. Clearly this solution satisfies the boundary conditions. Substituting this proposed solution into the diff. eqn. yields

$$\left[-\left(\frac{\pi}{L}\right)^2 + \frac{\omega_c^2}{c^2} - \frac{s^2}{c^2}\right] A \sin \frac{\pi x}{L} = -\frac{s}{c^2}\delta_0 \sin \frac{\pi x}{L}.$$

$$\therefore\quad A = \frac{s}{s^2 - \left(\omega_c^2 - \left(\frac{\pi c}{L}\right)^2\right)}\delta_0$$

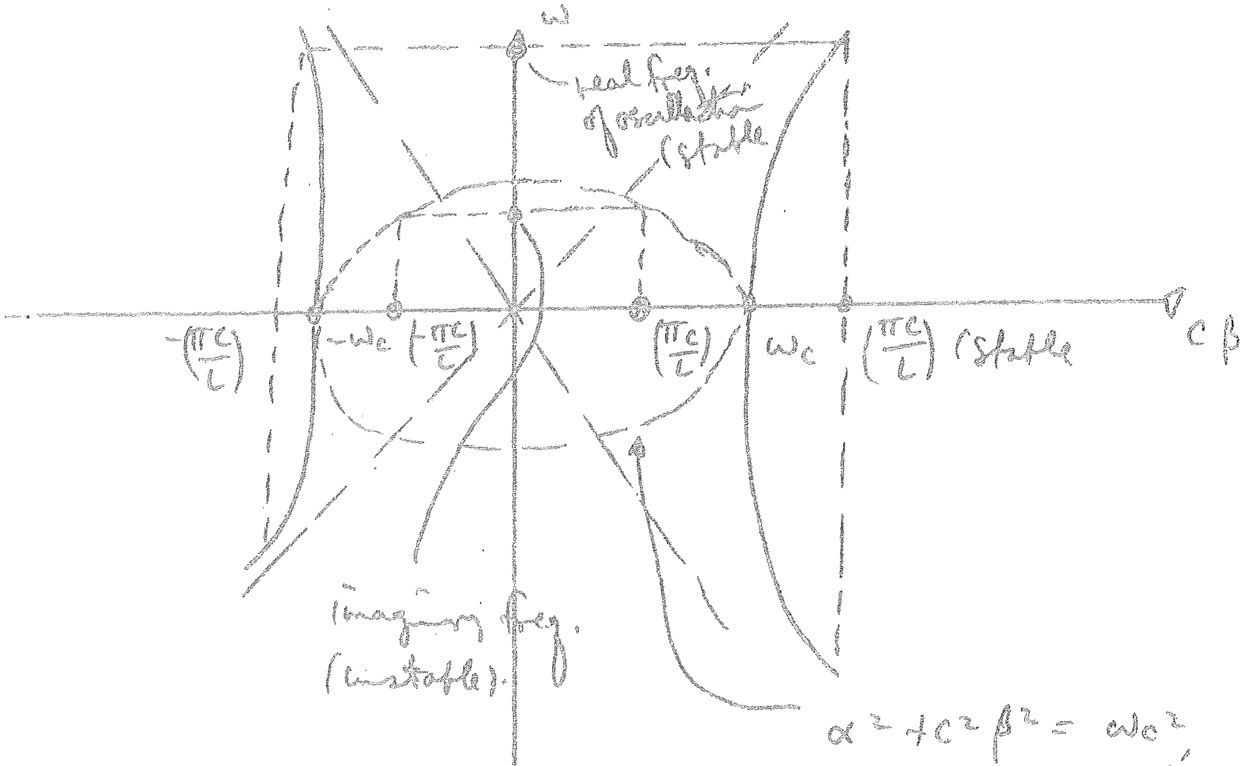
and

$$\tilde{\delta}(x,s) = \frac{s}{s^2 - \left(\omega_c^2 - \left(\frac{\pi c}{L}\right)^2\right)} \delta_0 \sin \frac{\pi x}{L}.$$

Poles are located where $s^2 - \left(\omega_c^2 - \left(\frac{\pi c}{L}\right)^2\right) = 0$

or $\boxed{s = \pm\left(\omega_c^2 - \left(\frac{\pi c}{L}\right)^2\right)^{1/2}}$. If $\omega_c^2 > \left(\frac{\pi c}{L}\right)^2$, then the poles are real, and because of the (+)-root there is instability.

If $\omega_c^2 < \left(\frac{\pi c}{L}\right)^2$, both poles are imaginary and there is undamped oscillation at frequency $\left[\left(\frac{\pi c}{L}\right)^2 - \omega_c^2\right]^{1/2}$. Hence, motion is stable.



$\alpha^2 + c^2\beta^2 = \omega_c^2$
where $\omega = j\alpha$ for imaginary frequencies. ~~(ellipse)~~
(circle of radius ω_c in $\alpha - c\beta$ plane)

(2-9) From (2-88') $\Omega(T) = \dfrac{1-S}{(S^2 - 2S + P^2)^{1/2}}$

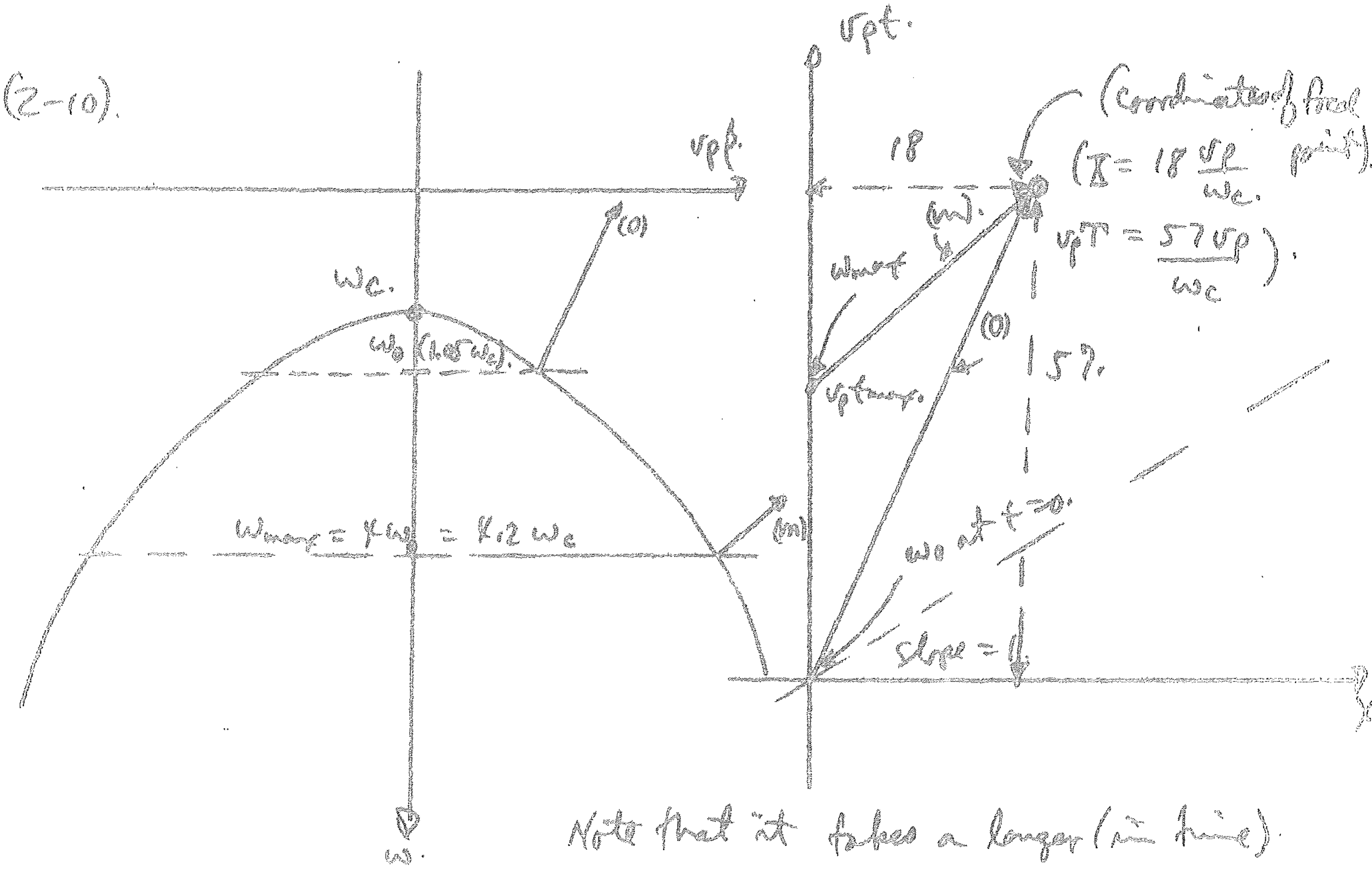
$$S = \frac{T}{Y}(1-P^2)^{1/2} = \frac{T}{18}(1-0.95^2)^{\frac{1}{2}} = T/57. \quad , P^2 = (0.95)^2 = 0.90$$

$$\therefore \quad \Omega(T) = \frac{1 - T/57}{\left[\frac{T^2}{(57)^2} - \frac{2T}{57} + 0.90\right]^{1/2}}$$

The normalized time for compression at Y = 18 is

$$\frac{18}{[1-(0.95)^2]^{1/2}} = \;\; = 57.$$

(2-10).



Note that it takes a longer (in time) signal to compress at Y = 18 than at Y = 15, keeping all other factors equal.

(1-5) A rod of length L, fixed at x=0 and x=L, is initially in torsion with $\varphi_0(x) = \Phi_0 \sin \pi x/L$. The rod is released from rest, i.e., the initial angular velocity is zero for all x.

(a) What is the initial distribution of torque along the rod? SHORT

(b) What is the differential equation satisfied by the Laplace transform $\Phi(x,s) = \mathcal{L}\{\varphi(x,t)\}$? SHORT

(c) Show that $\Phi(x,s) = \dfrac{s}{s^2 + \left(\frac{\pi v_p}{L}\right)^2} \Phi_0 \sin \pi x/L$, SHORT where $v_p = \sqrt{\frac{G}{\rho}}$

Do this two ways: (1) by solving directly the simple differential equation of part (b) and ~~(2) by substituting directly into (1-27) and evaluating the integral~~ → SKIP (Hint: make use of the identity $\sin a \sinh b = \dfrac{\cos(a-jb) - \cos(a+jb)}{2j}$).

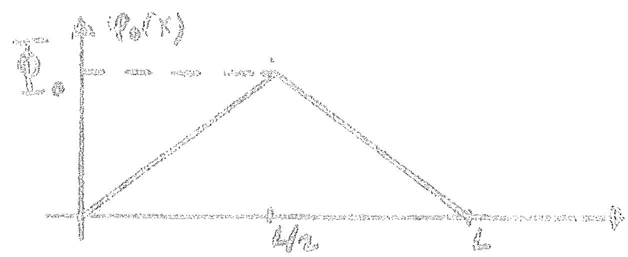
(d) Evaluate $\varphi(x,t) = \mathcal{L}^{-1}\{\Phi(x,s)\}$.
What is the frequency of vibration?

(e) Sketch the result of (d) for t=0, $\frac{T}{4}$, $\frac{T}{2}$, where T is the period of vibration. Why is this wave motion called a "Standing Wave"?

(f) Use the simple trigonometric identity $\sin\alpha \cos\beta = \dfrac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$ to show that $\varphi(x,t)$ can be decomposed into two equal and oppositely traveling waves. (TALK ABOUT)

(1-6) The same rod of Prob. (1-5) is in the initial torsion state $\varphi_0(x) = \Phi_0 \sin n\pi x/L$, n = 2, 3, ... Evaluate $\varphi(x,t)$ and determine the frequency of vibration. COMPARE WITH 1-5

(1-7) The same rod of Prob. (1-5) is in the initial torsional state shown in the figure



(a) What is the initial torque distribution along the rod?

(b) Determine $\Phi(x,s) = \sum_{n=0}^{\infty}$ by solving directly its differential equation. (Hint: Use Fourier series).

(c) Determine and plot $\varphi(x,t)$ for t=0, $\frac{T}{4}$, $\frac{T}{2}$, where T is the period of vibration. (Hint: you will have to sum a Fourier series on the computer).

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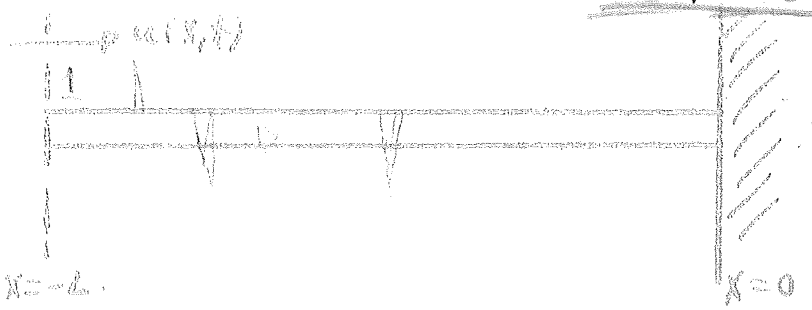


The left face of the (initially relaxed) rod is struck impulsively and thereafter remains free. The right end is fixed. Calculate u(x,t), the displacement, at x = - L, and sketch.

(1-9) The rod of Prob. (1-8) (left end free, right end fixed) is given an initial twist so that $\varphi_0(x) = kx$, and then released from rest at t=0. Calculate and sketch the angular displacement, $\varphi(-L,t)$, of the left end as a function of time.

(1-10) A rod of length L is free at the left end and connected to a lumped mass, M, at the right end. The left end is struck impulsively at t=0 and thereafter remains free. Determine the motion (displacement or velocity) of the mass. Draw an electrical transmission-line equivalent circuit.

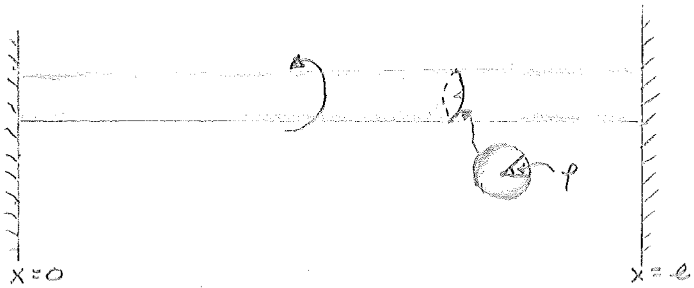
(1-11) A rod of length L is free at the left end and connected to a torsion spring at the right end. The torque-angular displacement characteristic of the spring is $\tau = G\varphi$, where G is a constant. The left end is given an impulsive torque at t=0 and thereafter remains free. Determine the motion of the right end. Draw an electrical transmission-line equivalent circuit.

(1-12) Verify that initial conditions (1-11) of the text are satisfied by (1-12) when use is made of the definitions of $V_0(x)$, $U_0(x)$ on p. 5.

BOB MARKS
TRAVELING WAVES I
9-27-72

90

1-5)



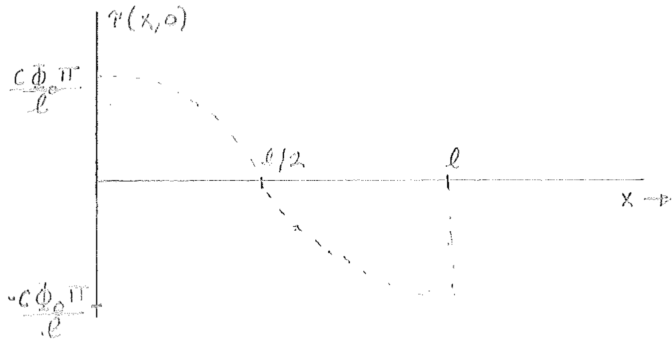
a) $\varphi_0(x) = \varphi(x,0) = \Phi_0 \sin\frac{\pi x}{\ell}$ = INITIAL ANGULAR DISPLACEMENT

$$\frac{\delta \varphi(x,0)}{\delta x} = \Phi_0 \frac{\pi}{\ell} \cos\frac{\pi x}{\ell} = \text{"TWIST"}$$

$$\Upsilon(x,0) = \frac{C\,\delta\varphi(x,0)}{\delta x} \ni \Upsilon(x,0) = \text{INITIAL TORQUE DISTRIBUTION}$$

C = COEFFICIENT OF TORSIONAL RIGIDITY

$$\therefore\ \Upsilon(x,0) = C\Phi_0\frac{\pi}{\ell}\cos\frac{\pi x}{\ell}$$ ✓



b) $\frac{\delta \Upsilon}{\delta x} + T = J\frac{\delta^2\varphi}{\delta t^2} \ni T$ = APPLIED BODY TORQUE/PER UNIT LENGTH DUE TO EXTERNAL OR NON-ELASTIC AGENTS

J = MOMENT OF INERTIA ABOUT X AXIS

$$\Rightarrow \frac{C\,\delta^2\varphi(x,t)}{\delta x^2} + T = J\frac{\delta^2\varphi(x,t)}{\delta t^2}$$

$$\mathcal{L}\left\{\frac{C\,\delta^2\varphi(x,t)}{\delta x^2} + T\right\} = \mathcal{L}\left\{J\frac{\delta^2\varphi(x,t)}{\delta t^2}\right\}$$

$$\frac{C\,d^2\Phi(x,s)}{dx^2} + \frac{T}{s} = J\left[s^2\Phi(x,s) - s\varphi(x,0) - \frac{\delta\varphi(x,0)}{\delta t}\right] \quad \omega_0(x)=0$$

$$\therefore\ C\frac{d^2\Phi(x,s)}{dx^2} + \frac{T}{s} = J\left[s^2\Phi(x,s) - s\Phi_0\sin\frac{\pi x}{\ell}\right]$$

T = "D.C" TORQUE VALUE = 0 ; $V_p = \sqrt{C/J}$

$$\Rightarrow \frac{d^2\Phi(x,s)}{dx^2} = \frac{1}{V_p^2}\left[s^2\Phi(x,s) - s\Phi_0\sin\frac{\pi x}{\ell}\right]$$ ✓

c) $\varphi(\ell,t) = \varphi(0,t) = 0 \Rightarrow \bar{\Phi}(\ell,s) = \bar{\Phi}(0,s) = 0$

$$\frac{d^2\bar{\Phi}}{dx^2} - \left(\frac{s}{v_p}\right)^2 \bar{\Phi} = -\frac{s}{v_p^2}\Phi_0 \sin\frac{\pi x}{\ell}$$

Let $\bar{\Phi}(x,s) = c_1 \sin\frac{\pi x}{\ell}$

$$-c_1\left[\left(\frac{\pi}{\ell}\right)^2 + \left(\frac{s^2}{v_p^2}\right)\right] \sin\frac{\pi x}{\ell} = -\frac{s}{v_p}\Phi_0 \sin\frac{\pi x}{\ell}$$

$$\Rightarrow c_1 = \frac{(s/v_p^2)\,\Phi_0}{(\pi/\ell)^2 + s^2/v_p^2}$$

$$\therefore \bar{\Phi}(x,s) = \frac{s/v_p^2}{\left(\frac{\pi}{\ell}\right)^2 + \left(s/v_p\right)^2}\Phi_0 \sin\frac{\pi x}{\ell}$$

$$= \frac{s}{s^2 + \left(\frac{\pi v_p}{\ell}\right)^2}\Phi_0 \sin\frac{\pi x}{\ell}$$

✓

d) $\bar{\Phi}(x,s) = \frac{s}{s^2 + \left(\frac{\pi v_p}{\ell}\right)^2} \Phi_0 \sin \frac{\pi x}{\ell}$

$$\Rightarrow \varphi(x,t) = \Phi_0 \cos \frac{\pi v_p}{\ell} t \sin \frac{\pi x}{\ell}; \quad \varphi(0,t) = \varphi(\ell,t) = 0$$

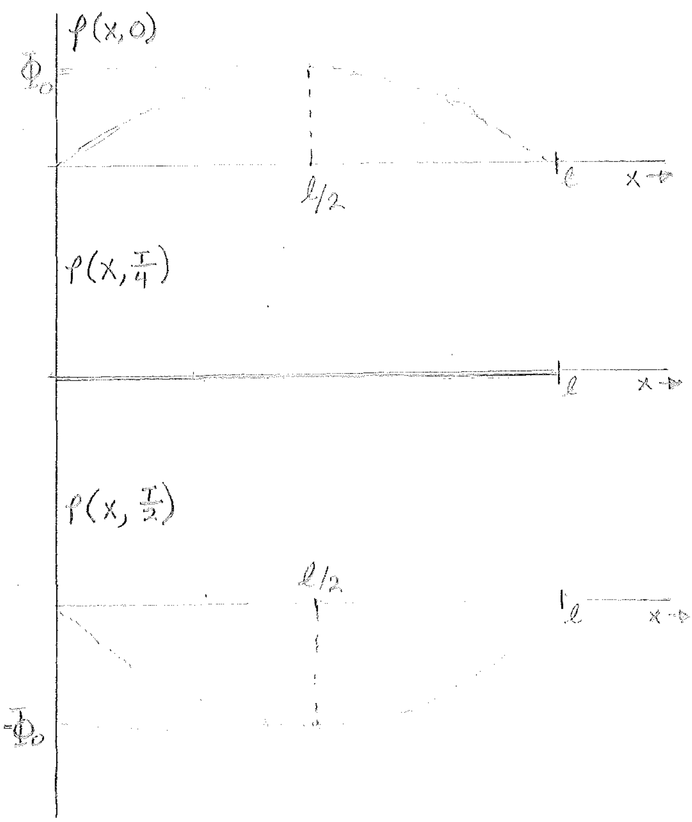
$$\omega(x,t) = \frac{\delta \varphi(x,t)}{\delta t} = -\frac{\Phi \pi v_p}{\ell} \sin \frac{\pi v_p}{\ell} t \sin \frac{\pi x}{\ell}$$

$$\omega_0 = \frac{\pi v_p}{\ell} \Rightarrow f_0 = \frac{\omega_0}{2\pi} = \frac{v_p}{2\ell} \left(\Rightarrow T = \frac{2\ell}{v_p}\right)$$

e) $\varphi(x,0) = \Phi_0 \sin \frac{\pi x}{\ell}$ (INITIAL CONDITION)

$\varphi(x, \frac{T}{4}) = \varphi(x, \frac{\ell}{2 v_p}) = 0$

$\varphi(x, \frac{T}{2}) = \varphi(x, \ell / v_p) = -\Phi_0 \sin \frac{\pi x}{\ell}$

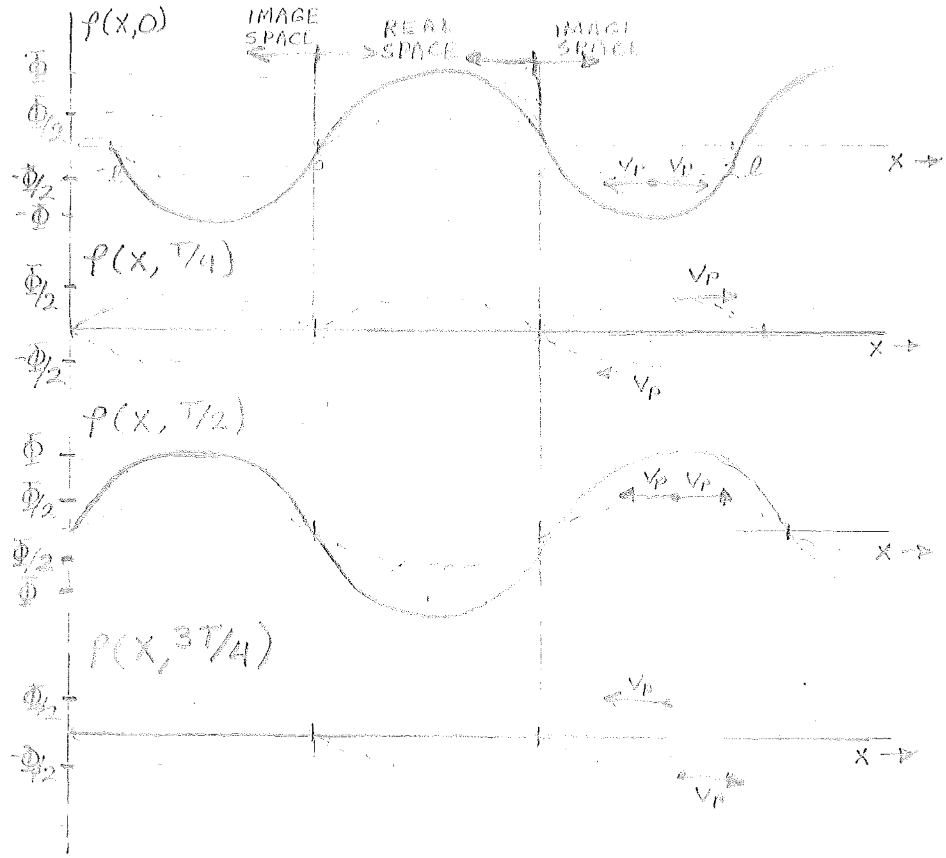


THE ENERGY REQUIRED FOR TWISTING THE ROD CANNOT PASS THE ANCHORED NODES AT $x = 0$ AND $x = \ell$. THE ENERGY THUS REMAINS "STANDING", ALTHOUGH IT ALTERNATES BETWEEN KINETIC ENERGY (MAXIMUM AT $t = (n+1)\frac{T}{4}$; $n = 0, 1, 2, \ldots$) AND ELASTIC POTENTIAL ENERGY (MAXIMUM AT $t = \frac{nT}{2}$; $n = 0, 1, 2, \ldots$). CHARACTERISTIC OF THE STANDING WAVE IS THAT THE AMPLITUDE OF ANGULAR DISPLACEMENT IS NOT THE SAME, BUT VARIES WITH x. (AS OPPOSED TO A WAVE PROPAGATED DOWN AN INFINITE LOSSLESS ROD, WHERE EVERY POINT WOULD ACHIEVE THE SAME MAXIMUM DISPLACEMENT AT DIFFERENT t's)

f) $\varphi(x,t) = \Phi_0 \cos\frac{\pi V_p}{\ell} t \sin\frac{\pi x}{\ell}$

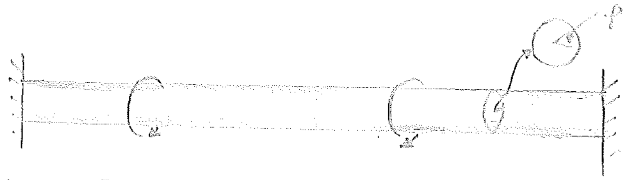
$$= \frac{\Phi_0}{2}\left[\sin\left(\frac{\pi x}{\ell} + \frac{\pi V_p}{\ell}t\right) + \sin\left(\frac{\pi x}{\ell} - \frac{\pi V_p}{\ell}t\right)\right]$$

$$= \frac{\Phi_0}{2}\left[\sin\frac{\pi}{\ell}(x + V_p t) + \sin\frac{\pi}{\ell}(x - V_p t)\right]; \quad T = \frac{2\ell}{V_p}$$



$\varphi(x,t)$ MAY BE THOUGHT OF AS TWO SINUSOIDS
PROPAGATED IN BOTH REAL AND IMAGE SPACE.
THE $\frac{\Phi}{2}\sin\frac{\pi}{\ell}(x - V_p t)$ COMPONENT TRAVELING
TO THE RIGHT ON THE X AXIS, AND THE
$\frac{\Phi}{2}\sin\frac{\pi}{\ell}(x + V_p t)$ COMPONENT TRAVELING TO
THE LEFT, WITH THE SAME SPEED, FREQUENCY,
AND MAXIMUM AMPLITUDE. AT $t=0$, BOTH
COMPONENTS ARE EQUAL, THUS COMPRISING A
CONSTRUCTIVE SINUSOID OF TWICE THE
MAXIMUM AMPLITUDE OF EITHER COMPONENT.
AT $t = T/4 = \ell/2V_p$, THE TWO COMPONENTS
ARE 180° OUT OF PHASE, CANCELING
OUT RESULTANT WAVE. $t = T/2 = \ell/V_p$ YIELDS
A WAVE OPPOSITE IN SIGN TO THE WAVE
AT $t=0$ BY A SIMILAR ARGUMENT.
SIMILARLY, AT $t = 3T/4$, $\varphi(x,t) = 0$.
$\varphi(x,t) = \varphi(x, t+T)$

1-6) $\phi_0(x) = \Phi_0 \sin \frac{n\pi x}{\ell}$; $n = 2, 3, 4, \ldots$

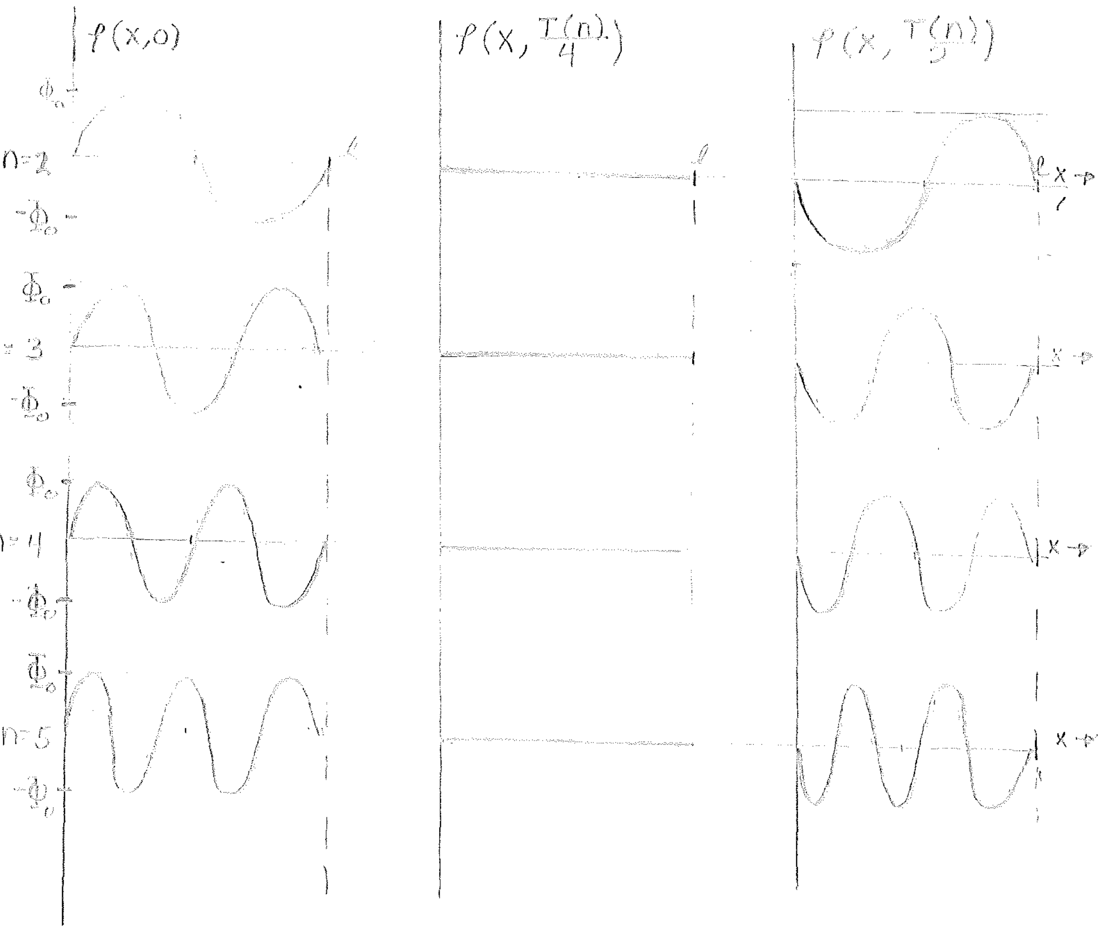


$$\Rightarrow \frac{d^2\Phi(x,s)}{dx^2} = \frac{1}{v_p^2}\left[s^2\Phi(x,s) - s\Phi_0 \sin\frac{n\pi x}{\ell} - \omega_\phi(t)\right] \; ; \; v_p = \sqrt{\frac{G}{\rho}}$$

(with $\omega_\phi(t) \to 0$)

$$\Phi(x,s) = \frac{s}{s^2 + \left(\frac{n\pi v_p}{\ell}\right)^2}\Phi_0 \sin\frac{n\pi x}{\ell}$$

$$\Rightarrow \phi(x,t) = \Phi_0 \cos\frac{n\pi v_p}{\ell}t \, \sin\frac{n\pi x}{\ell}$$



THE EXPRESSION $\phi(x,t,n)$ IS A MORE GENERAL CASE OF (1-5), WHERE $n=1$. THE FREQUENCY OF OSCILLATION FROM $\phi(x,t,n)$ IS $\frac{n v_p}{2\ell} = f(n)$
THE VALUE OF $T(n)$ USED IN ABOVE GRAPHS:

$$T(n) = \frac{1}{f(n)} = \frac{2\ell}{n v_p}$$

CLEARLY, AS n INCREASES, $T(n)$ DECREASES.
$f(n)$ GIVES THE HARMONIC FREQUENCIES OF $f(1)$, COMPUTED IN (1·5) AS $V_P/2\ell$.
THE WAVELENGTH AS A FUNCTION OF n IS:

$$\lambda(n) = \frac{\ell}{n}$$

$\Rightarrow$ THE NUMBER OF WAVELENGTHS IN $\ell = \frac{n}{2}$

$$f(x,t) = \Phi_0 \sin\frac{n\pi x}{\ell} \cos\frac{n\pi V_P}{\ell} t$$

$$= \frac{\Phi_0}{2}\left[\sin\frac{n\pi}{\ell}(x - V_p t) + \sin\frac{n\pi}{\ell}(x - V_p t)\right]$$

EACH CASE MAY THEN BE ARGUED, BY AN ANALYSIS SIMILAR TO (1-5f) TO BE A STANDING WAVE, THE SPEED AT WHICH THE TWO WAVES TRAVEL BEING V_P, WHERE $V_P = \sqrt{C/J}$, AND THE WAVE FREQ. BEING $f(n) = \frac{n V_P}{2\ell}$

MORE INTERESTING IS THE FACT THAT THE NODES IMPOSED AT $m\ell/n$ $(m = 0, 1, 2, \ldots n)$ DO NOT MOVE IN SPACE OR TIME.

$$f(x,t,n) = \Phi_0 \sin\frac{n\pi x}{\ell} \cos\frac{n\pi V_P}{\ell} t$$

$$f(x,t,n)\Big|_{x = m\ell/n} = \Phi_0 \sin \pi m \cos\frac{n\pi V_P}{\ell} t$$

$$= 0 \quad \forall t$$

EACH $\frac{1}{2}\lambda$ SEGMENT BETWEEN NODES ACTS AS AN INDEPENDENT ROD, AS IN (1-5), WITH $\ell_s = \frac{\ell_0}{n}$.

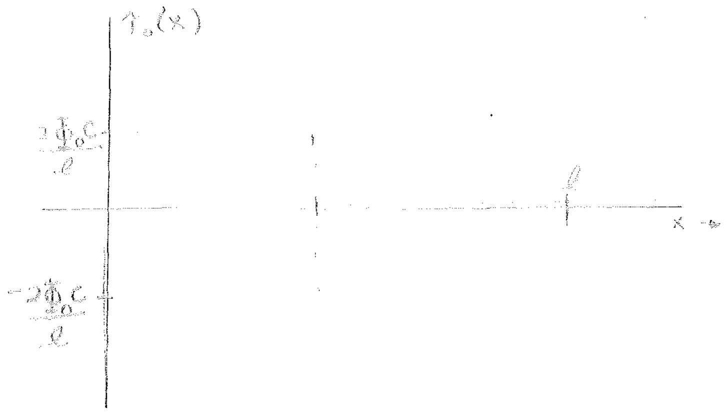
1-7) a) $\rho_0(x) = 2\Phi_0 \frac{x}{\ell} \quad ; \quad 0 < x < \ell/2$

$\rho_0(x) = -2\Phi_0\left[\frac{x}{\ell} - 1\right] \quad ; \quad \ell/2 < x < \ell$

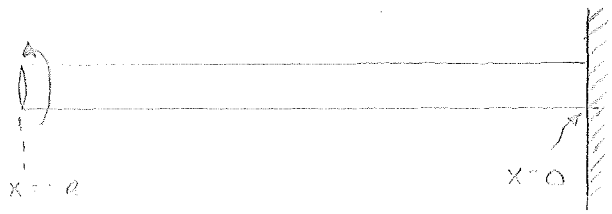
$$\frac{1}{c}\gamma_0(x) = \frac{d\rho_0(x)}{dx}$$

$$\Rightarrow \gamma_0(x) = \frac{2\Phi_0 c}{\ell} ; \quad 0 < x < \ell/2$$

$$= \frac{-2\Phi_0 c}{\ell} ; \quad \frac{\ell}{2} < x < \ell$$



1-9)



$$\varphi(x,0) = kx \Rightarrow \varphi(0,0) = 0$$

$$C\frac{\delta \varphi(x,0)}{\delta x} = \tau(x,0) = Ck$$

FROM (1-5)

$$\frac{C\,\delta^2 \varphi(x,t)}{\delta x^2} = J\frac{\delta^2 \varphi(x,t)}{\delta t^2}$$

$$\frac{d^2\Phi(x,s)}{dx^2} = \frac{1}{V_p^2}\left[s^2\Phi(x,s) - s\varphi(x,0) - \frac{\delta \varphi(x,0)}{\delta t}\right]$$

$$\ni V_p^2 = \frac{C}{J}$$

$$\Rightarrow \frac{d^2\Phi(x,s)}{dx^2} = \frac{1}{V_p^2}\left[s^2\Phi(x,s) - skx - k\right]$$

$$= \frac{s^2}{V_p^2}\Phi(x,s) - \frac{skx}{V_p^2} - \frac{k}{V_p^2}$$

$$\Phi(x,s) = \frac{kx}{s} + \frac{k}{s^2} + C_1 e^{-\frac{s}{V_p}x} + C_2 e^{\frac{s}{V_p}x}$$

$$\frac{d\Phi(x,s)}{\delta x} = \frac{sk}{V_p^2} - C_1\frac{s}{V_p}e^{-\frac{s}{V_p}x} + C_2\frac{s}{V_p}e^{\frac{s}{V_p}x}$$

$$\frac{\delta^2\Phi(x,s)}{\delta x} = C_1\left(\frac{s}{V_p}\right)^2 e^{-\frac{s}{V_p}x} + C_2\left(\frac{s}{V_p}\right)^2 e^{\frac{s}{V_p}x}$$

CHECK:

$$C_1\left(\frac{s}{V_p}\right)^2 e^{-\frac{s}{V_p}x} + C_2\left(\frac{s}{V_p}\right)^2 e^{\frac{s}{V_p}x} = \frac{s^2}{V_p^2}\left[\frac{kx}{s} + \frac{k}{s^2} + C_1 e^{-\frac{s}{V_p}x} + C_2 e^{\frac{s}{V_p}x}\right] - \frac{skx}{V_p^2} - \frac{k}{V_p^2}$$

O.K.

$$@\, x = 0,\ \varphi_0 = 0 \Rightarrow \Phi_0 = 0,$$

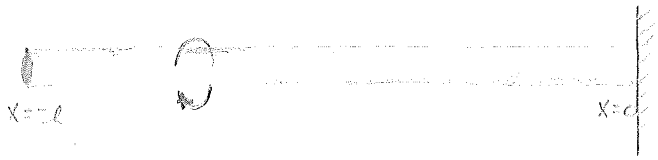
$$0 = \frac{k}{s^2} + C_1 + C_2$$

$$\Phi(x,s) = \frac{kx}{V_p^2} + \frac{k}{s^2} - \frac{k}{s^2}e^{-\frac{s}{V_p}x}$$

$$= \frac{k}{s^2}\left[1 - e^{-\frac{s}{V_p}x}\right] + \frac{kx}{V_p^2}$$

$$\Rightarrow \varphi(x,t) = \frac{kx}{V_p^2}\delta(t) + t$$ (ARG!)

1-9)



$$f(x,0) = kx$$

$$\frac{\delta f(x,0)}{\delta x} = k$$

$$\frac{\delta^2 f(x,t)}{\delta x^2} = \frac{1}{V_p^2}\frac{\delta^2 f(x,t)}{\delta t^2}$$

$$\frac{d^2\Phi(x,s)}{dx^2} = \frac{1}{V_p^2}\left[s^2\Phi(x,s) - skx - k\right]$$

$$= \frac{s^2}{V_p^2}\Phi(x,s) - \frac{skx}{V_p^2} - \frac{k}{V_p^2}$$

$$\Rightarrow \Phi(x,s) = \frac{-k(sx+1)}{V_p^2} + c_1 e^{-\frac{s}{V_p}x} + c_2 e^{\frac{s}{V_p}x}$$

$$\Phi(0,s) = 0 = \frac{-k}{V_p^2} + c_1 + c_2 \rightarrow c_2 = \frac{k}{V_p^2} - c_1)$$

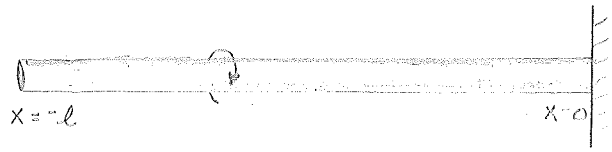
$$\Rightarrow \Phi(x,s) = \frac{-k(sx+1)}{V_p^2} + c_1 e^{-\frac{s}{V_p}x} + \left(\frac{k}{V_p^2} - c_1\right)e^{\frac{sx}{V_p}}$$

$$= \frac{-k(sx+1)}{V_p^2} + c_1 e^{-\frac{s}{V_p}x} - c_1 e^{\frac{sx}{V_p}} + \frac{k}{V_p^2}e^{\frac{sx}{V_p}}$$

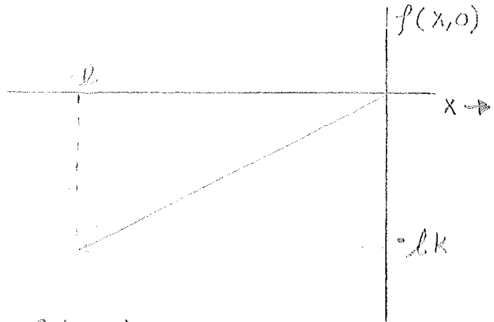
$$= \frac{-k(sx+1)}{V_p^2} + \frac{k}{V_p^2}e^{\frac{sx}{V_p}} + C_3 \sinh\frac{sx}{V_p}$$

(ARG!)

1-9)



$f(x,0) = kx$



$$\frac{\delta f(x,0)}{\delta x} = k$$

$$\frac{\delta^2 f(x,t)}{dx^2} = \frac{1}{V_p^2}\frac{\delta^2 f(x,t)}{\delta t^2}$$

$$\Rightarrow \frac{d^2 \Phi(x,s)}{dx^2} = \frac{1}{V_p^2}\left[s^2\Phi(x,s) - s f(x,0) - \frac{d f(x,0)}{\delta t^2}\right]$$

$$= \frac{1}{V_p^2}\left[s^2\Phi(x,s) - skx - k\right]$$

$$= \left(\frac{s}{V_p}\right)^2 \Phi(x,s) - \frac{k(sx+1)}{V_p^2}$$

$$\Rightarrow \frac{d^2\Phi(x,s)}{dx^2} - \left(\frac{s}{V_p}\right)^2\Phi(x,s) = \frac{-k(sx+1)}{V_p^2}$$

$$\Phi_x'' - \left(\frac{s}{V_p}\right)^2\Phi = \frac{-k(sx+1)}{V_p^2}$$

$$\Phi(x,s) = \frac{k(sx+1)}{s^2} + C_1 e^{-\frac{sx}{V_p}} + C_2 e^{\frac{sx}{V_p}}$$

$$f(0,t) = \Phi(0,s) = 0$$

$$\Phi(0,s) = \frac{k}{s^2} + C_1 + C_2 = 0 \Rightarrow C_2 = -\left(C_1 + \frac{k}{s^2}\right)$$

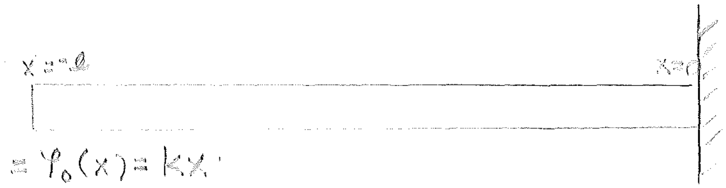
$$\Phi(x,s) = \frac{k(sx+1)}{s^2} + C_1 e^{-\frac{sx}{V_p}} - C_1 e^{\frac{sx}{V_p}} - \frac{k}{s^2} e^{\frac{sx}{V_p}}$$

$$= \frac{k}{x} \frac{(s + \frac{1}{x})}{s^2} - C_1 \sinh \frac{sx}{V_p} - \frac{k}{s^2} e^{\frac{sx}{V_p}}$$

$$= \frac{kx}{s} + \frac{k}{s^2} - C_1 \sinh \frac{sx}{V_p} - \frac{k}{s^2} e^{\frac{sx}{V_p}}$$

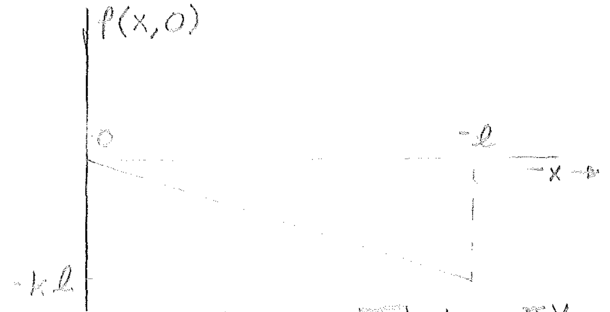
(ARG!)

1-9)



$\varphi(x,0) = \varphi_0(x) = kx$

AT $X = -l$, $\varphi(-l,t)$ WILL BE A SINUSOID IN TIME WITH INITIAL VALUE $\varphi(-l,0) = -lk \Rightarrow \varphi(-l,t) = -lk \cos \omega t$



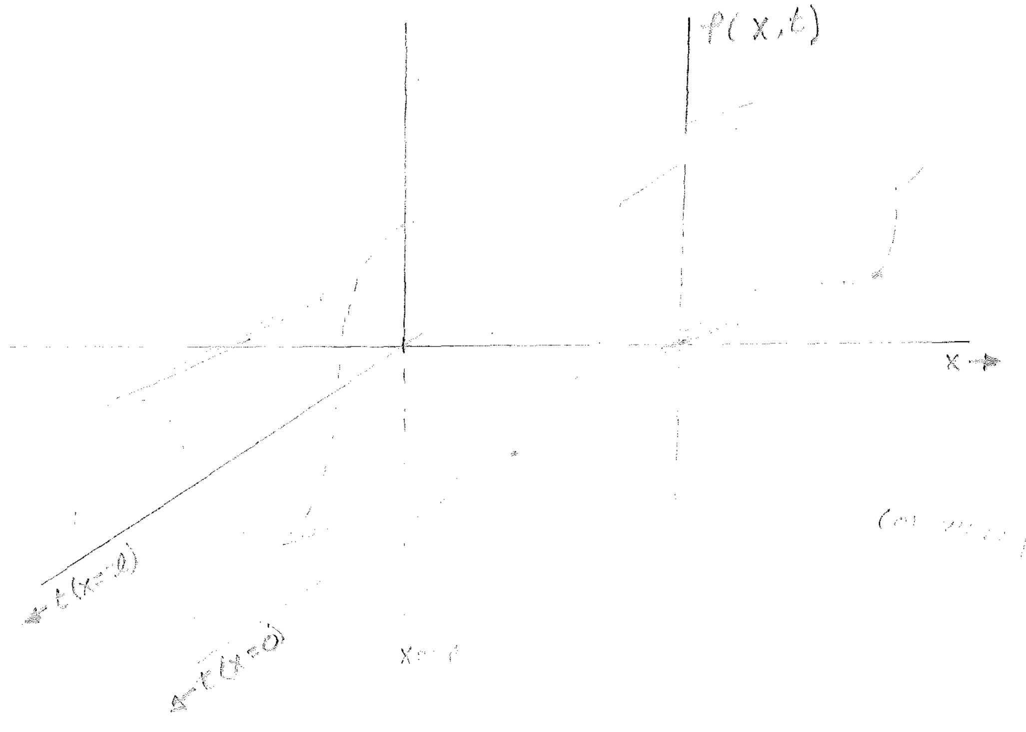
$\omega = 2\pi \frac{V_P}{2l} = 2\pi \sqrt{\frac{C}{J}} \, \frac{1}{2l} = \frac{\pi V_P}{l}$

$\ni$ C = COEFFICIENT OF TORSIONAL

J = MOMENT OF INERTIA ABOUT X AXIS

$2l$ IS USED INSTEAD OF l, IN THAT ROD ACTS AS $\frac{1}{4}\lambda$

$\Rightarrow \varphi(-l,t) = -lk \cos \frac{\pi V_P}{l} t$



(1-5) (a) $C \frac{\partial \varphi_0}{\partial x} = \boxed{C\, \Phi_0 \cdot \frac{\pi}{L} \cos \pi x/L.}$

(b) $$\boxed{\frac{d^2 \Phi}{dx^2} - \left(\frac{s}{v_p}\right)^2 \Phi = -\frac{s}{v_p^2} \varphi_0(x) = -\frac{s}{v_p^2} \Phi_0 \sin \frac{\pi x}{L}.}$$

(c) Upon substitution, we have

$$\frac{d^2 \Phi}{dx^2} = -\left(\frac{\pi}{L}\right)^2 \Phi_0 \sin \frac{\pi x}{L} \cdot \frac{s}{s^2 + \left(\frac{\pi v_p}{L}\right)^2}$$

$$\left(\frac{s}{v_p}\right)^2 \Phi = \frac{s^3}{v_p^2\left(s^2 + \left(\frac{\pi v_p}{L}\right)^2\right)} \Phi_0 \sin \frac{\pi x}{L}.$$

$$\therefore \quad \frac{d^2 \Phi}{dx^2} - \left(\frac{s}{v_p}\right)^2 \Phi = \frac{-s}{s^2 + \left(\frac{\pi v_p}{L}\right)^2} \Phi_0 \sin \frac{\pi x}{L} \left[\left(\frac{\pi}{L}\right)^2 + \frac{s^2}{v_p^2}\right].$$

$$= \frac{-s}{s^2 + \left(\frac{\pi v_p}{L}\right)^2} \Phi_0 \sin \frac{\pi x}{L} \cdot \frac{1}{v_p^2}\left[s^2 + \left(\frac{\pi v_p}{L}\right)^2\right]$$

$$= -\frac{s}{v_p^2} \Phi_0 \sin \frac{\pi x}{L} \qquad \text{Q.E.D.}$$

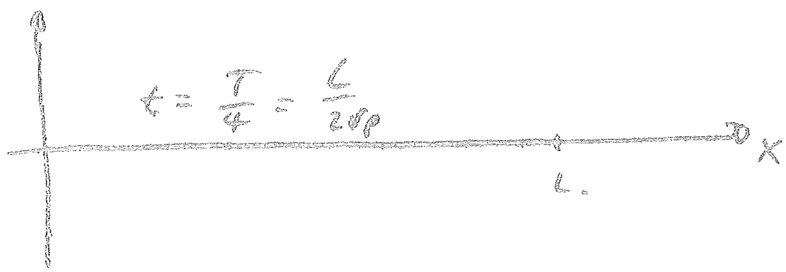
Note that the solution $\frac{s}{s^2 + \left(\frac{\pi v_p}{L}\right)^2} \Phi_0 \sin \frac{\pi x}{L}$ also satisfies the requisite boundary conditions $\Phi(0, s) = \Phi(L, s) = 0$.

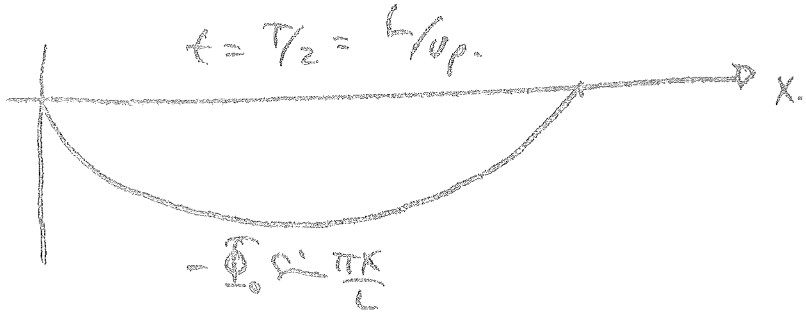
(d) $$\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \left(\frac{\pi v_p}{L}\right)^2} \Phi_0 \sin \frac{\pi x}{L}\right\} = \Phi_0 \sin \frac{\pi x}{L}\, \mathcal{L}^{-1}\left\{\frac{s}{s^2 + \left(\frac{\pi v_p}{L}\right)^2}\right\}$$

$$= \boxed{\Phi_0 \sin \frac{\pi x}{L} \cos\left(\frac{\pi v_p}{L}\right) t}$$

$$\boxed{f = \frac{1}{2\pi} \cdot \frac{\pi v_p}{L} = \frac{v_p}{2L}}$$

, $T = \frac{1}{f} = \frac{2L}{v_p}$





The wave does not propagate in the sense that the peak value does not move along the x-axis with time. The peak simply "stands" in place.

(f). $\Phi_0 \sin \frac{\pi x}{L} \cdot \cos\left(\frac{\pi v_p}{L}\right)t = \frac{\Phi_0}{2}\left\{ \sin\left(\frac{\pi x}{L} + \frac{\pi v_p}{L}t\right) + \sin\left(\frac{\pi x}{L} - \frac{\pi v_p t}{L}\right)\right\}$

$= \frac{\Phi_0}{2}\left\{ \sin \frac{\pi}{L}(x + v_p t) + \sin \frac{\pi}{L}(x - v_p t)\right\}$

(first term) traveling to the left; (second term) traveling to the right.

(1-6). $\frac{d^2 \Phi(x,s)}{dx^2} - \left(\frac{s}{v_p}\right)^2 \Phi(x,s) = -\frac{s}{v_p^2}\Phi_0 \sin \frac{n\pi x}{L}$.

$\therefore \Phi(x,s) = \Phi_0 \cdot \frac{s}{s^2 + \left(\frac{n\pi v_p}{L}\right)^2} \cdot \sin \frac{n\pi x}{L} \Rightarrow \varphi(x,t) = \boxed{\Phi_0 \cdot \sin\frac{n\pi x}{L} \cdot \cos\left(\frac{n\pi v_p}{L}\right)t}$

$\boxed{f_n = \frac{n v_p}{2L}}$. Note the higher frequencies as the "mode number" n increases.

(1-9). Laplace transform satisfies:

$$\frac{d^2\bar{\Phi}(x,s)}{dx^2} - \left(\frac{s}{v_p}\right)^2 \bar{\Phi}(x,s) = -\frac{s}{v_p^2} kx. \quad , v_p^2 = \frac{C}{J}.$$

$\varphi(x,t)$ satisfies: $\varphi(0,t) = 0$, $\left.\frac{\partial\varphi}{\partial x}\right|_{x=-l} = 0$. The latter condition follows from the equation $\tau = C\frac{\partial\varphi}{\partial x}$ and the fact that $\tau = 0$ at the free end $x = -l$.

Thus, $\bar{\Phi}(x,s)$ satisfies: $\bar{\Phi}(0,s) = 0$, $\frac{d\bar{\Phi}}{dx} = 0$ at $x = -l$.

The solution of the above equation consists of a particular integral plus a complementary function.

$$\bar{\Phi}_{part} = \frac{kx}{s}, \quad \bar{\Phi}_{comp} = Ae^{+sx/v_p} + Be^{-sx/v_p}$$

$$\therefore \bar{\Phi}(x,s) = \frac{kx}{s} + Ae^{sx/v_p} + Be^{-sx/v_p}$$

Boundary conditions require that

$$\bar{\Phi}(0,s) = A + B = 0; \quad \tau(l) = C\frac{\delta\varphi(l,t)}{\delta x} = 0$$

$$\frac{d\bar{\Phi}}{dx}(l,s) = 0 = \frac{s}{v_p}\left[Ae^{-sl/v_p} - Be^{sl/v_p}\right] + \frac{k}{s}$$

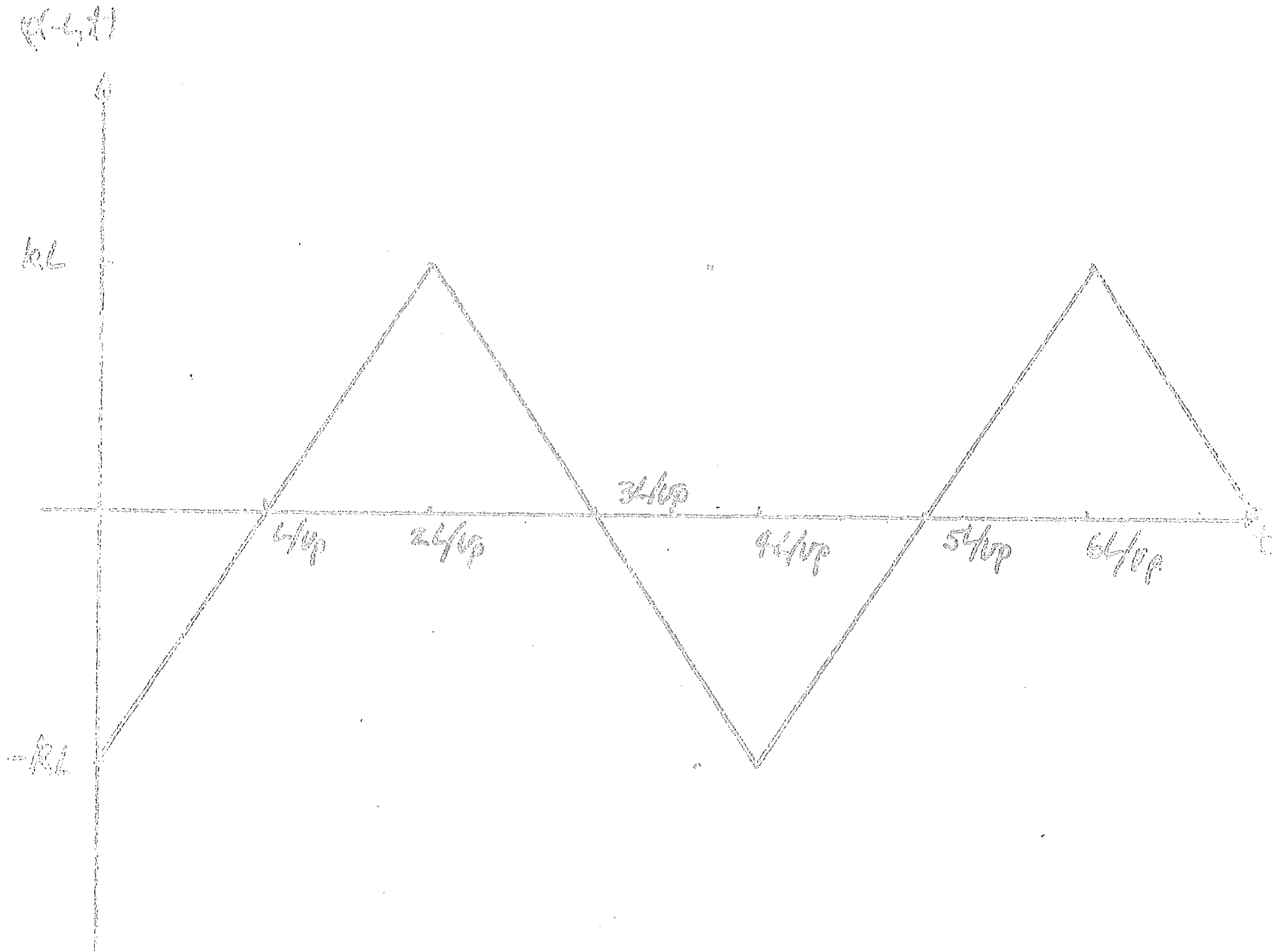
Thus, $A(s) = \dfrac{-kv_p}{s^2(e^{-sl/v_p} + e^{sl/v_p})}$, $B(s) = \dfrac{kv_p}{s^2(e^{-sl/v_p} + e^{sl/v_p})}$.

IN SOLVING FOR COMPLEMENTRY ~~FUNCTION~~ SOLUTION, SET FORCIN FUNCTION TO O,

$$\therefore \quad \Phi(-L,s) = -\frac{kL}{s} + \frac{kv_p}{s^2}\left\{\frac{e^{sL/v_p} - e^{-sL/v_p}}{e^{sL/v_p} + e^{-sL/v_p}}\right\}$$

$$= -\frac{kL}{s} + \frac{kv_p}{s^2}\left\{1 - 2e^{-2sL/v_p} + 2e^{-4sL/v_p} - 2e^{-6sL/v_p} + \cdots\right\}$$

and $\varphi(-L,t) = \mathcal{L}^{-1}\{\Phi(-L,s)\}$ is sketched below.



Note that the initial condition $\varphi(-L,0) = -kL$ is satisfied.

BOB MARKS

57/60 Excellent paper.

PROBLEMS

2-5. The equation of motion of a wire carrying current in an external magnetic field is given as

$$\frac{\partial^2 \delta}{\partial t^2} = c^2 \frac{\partial^2 \delta}{\partial x^2} + \omega_c^2 \delta$$

where δ is the displacement of the wire.

Sketch the dispersion relation (ω vs. β) for this system and compare it with the dispersion relation for the waveguide. Comment on the asymptotic (large ω or large β) behavior of the dispersion relation and on such things as cutoff frequencies, wave numbers, etc.

Hint: Recall a movie you saw recently.

2-6. The wire of problem 2-5 is given an initial displacement $\delta_0(x) = \delta_0 \sin \frac{\pi x}{L}$. In addition it is fixed at x=0 and x=L so that the displacement, $\delta(x,t)$, vanishes at those two ends. Using the ideas of problem 1-5, show that the Laplace transform of $\delta(x,t)$ satisfies

$$\tilde{\delta}(x,s) = \frac{s}{s^2 - \omega_c^2 + (c\pi/L)^2}\, \delta_0 \sin \pi x/L ,$$

if the wire is released from rest (i.e., zero initial velocity). Under what conditions is the resulting motion stable? Unstable? Relate these results to the dispersion relation of problem 2-5. Hint: Recall a movie you saw recently.

2-7. According to the theory of magneto-elastic vibrations (see, e.g. H.A. Sabbagh, "Thermal Noise in Spin-Phonon Systems", IEEE Transactions on Sonics and Ultrasonics, Vol. SU-16, No. 3, July 1969, pp. 147-156), the displacement, u, of an elastic system is related to the components, S_x, S_y, of the magnetic dipole moment of "atomic spin" by the equations

(i) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial z^2} - A_1 H_0 \frac{\partial S_x}{\partial z}$$

(ii) $$\frac{\partial S_x}{\partial t} = -\omega_0 S_y , \quad \frac{\partial S_y}{\partial t} = \omega_0 S_x + A_2 H_0 \frac{\partial u}{\partial z} ,$$

where A_1 and A_2 are "magneto-strictive" coupling constants and H_0 is an externally applied d.c. magnetic field.

(a) Set $A_2 = 0$ in (ii) and interpret the significance of ω_0.

(b) Take the Fourier transform of (i) and (ii), then eliminate S_x and S_y to show that the Fourier transform, $U(z,\omega)$ of $u(z,t)$ satisfies $$\frac{d^2 U}{dz^2} = -\left\{ \frac{\omega^2}{c^2} \cdot \frac{\omega^2 - \omega_0^2}{\omega^2 - (\omega_0^2 + A_1 A_2 H_0^2 \omega_0 / c^2)} \right\} U = -\beta^2(\omega)\, U$$

(c) Sketch $c\,\beta(\omega)$ vs. ω obtained from (b). In addition sketch $c\,\partial\beta/\partial\omega$ and v_g/c , where v_g is the group velocity.

Make these sketches similar to Figure 2-9.

(d) Interpret these sketches by locating cutoff and propagating frequency ranges. Also indicate (qualitatively) how the frequency of the received impulse response of such a system varies with time.

2-8. Determine the incremental equivalent circuit of an analogous electrical transmission-line for the spin-phonon system of Problem 2-7.

2-9. Utilizing the ideas developed in Section 2-8(B), design a signal which, when propagated through a waveguide whose normalized length, $\gamma \equiv \omega_c x / v_p$ is 18 units, will be compressed at the output. Let $\rho \equiv \omega_c / \omega_0 = 0.95$, $T_0 \equiv$ (normalized pulse length) be such that $\Omega_{max} / \Omega(0) = 4$. For what value of normalized time, $T (\equiv \omega_c t)$ will the output compressed peak occur?

2-10. Draw a space-time ray diagram for the design of Problem 2-9. Use this diagram to interpret the dispersive effects which occur.

2-5) $\frac{\partial^2 \delta}{\partial t^2} = c^2 \frac{\partial^2 \delta}{\partial x^2} + \omega_c^2 \delta$

$\frac{\partial^2 \delta}{\partial x^2} = \frac{1}{c^2}\left[\frac{\partial^2 \delta(x,t)}{\partial t^2} - \omega_c^2 \delta(x,t)\right]$

$\frac{d^2 \Delta(x,s)}{dx^2} = \frac{1}{c^2}\left[s^2 \Delta(x,s) - \omega_c^2 \Delta(x,s)\right]$

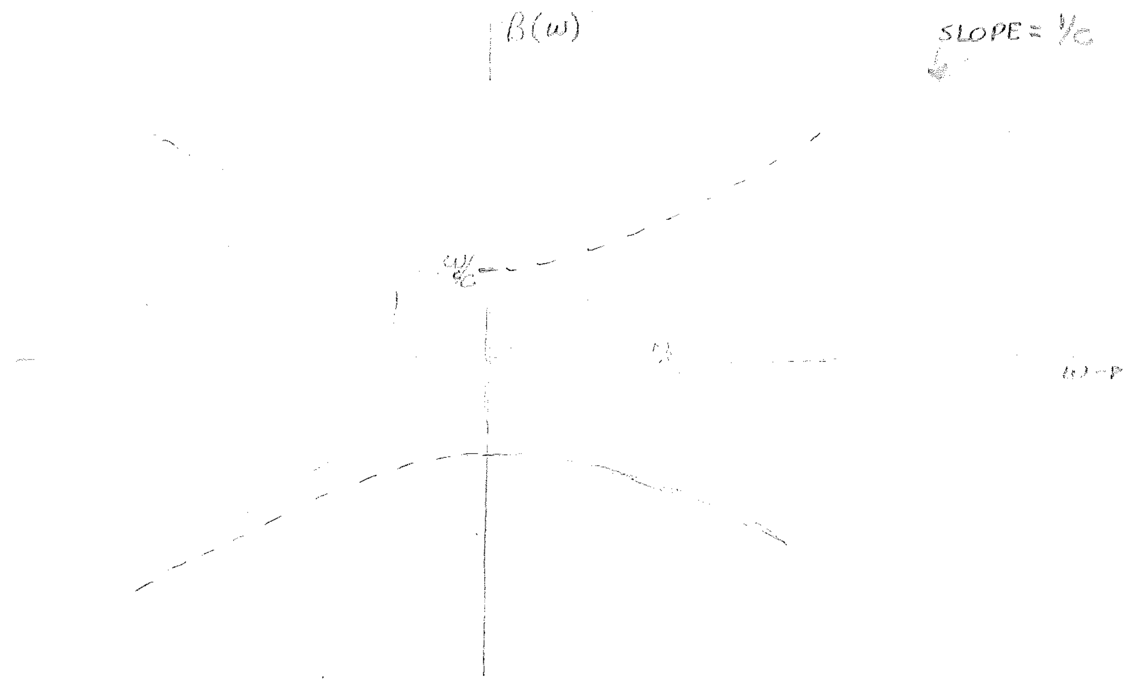
$= \frac{1}{c^2}(s^2 - \omega_c^2)\Delta(x,s)$

$\therefore \frac{d^2 \Delta(x,\omega)}{dx^2} = \frac{-1}{c^2}(\omega^2 + \omega_c^2)^{\frac{1}{2}} \Delta(x,\omega) = \frac{-\omega}{c^2}\left(1 + \left(\frac{\omega}{\omega_c}\right)^2\right)^{\frac{1}{2}} \Delta(x,\omega)$

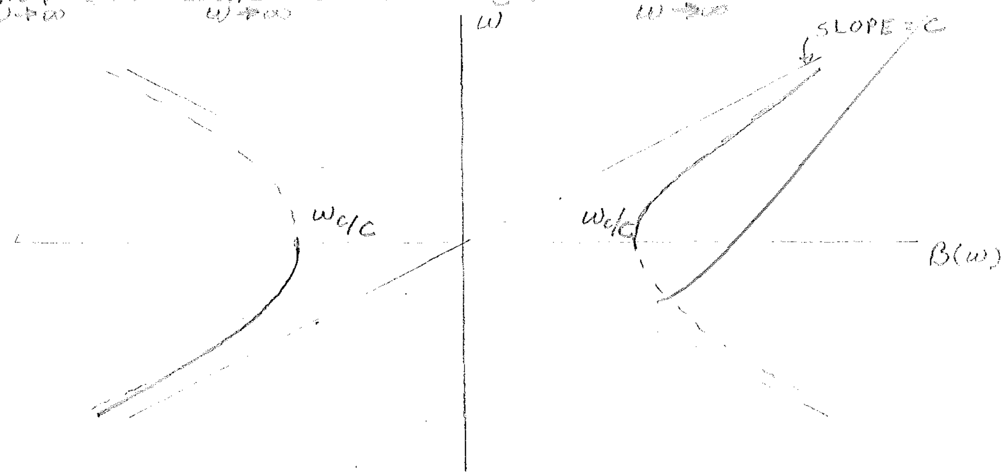
$\Rightarrow \Delta(x,\omega) = A e^{\beta(\omega)x} + B e^{-\beta(\omega)x}$

$\ni$ A AND B ARE ARBITRARY CONSTANTS

AND $\beta(\omega) = \frac{1}{c}(\omega^2 + \omega_c^2)^{\frac{1}{2}} = \frac{\omega}{c}\left(1 + \left(\frac{\omega_c}{\omega}\right)^2\right)^{\frac{1}{2}}$



$\lim_{\omega \to \infty} \beta(\omega) = \lim_{\omega \to \infty} \frac{1}{c}(\omega^2 + \omega_c^2)^{\frac{1}{2}} = \lim_{\omega \to \infty} \frac{\omega}{c}$



DISCUSSION

(−3)

THE RUSULTANT RELATIONSHIP BETWEEN $B(\omega)$ AND ω, SUGGESTS THERE IS A PURE IMAGINARY PROPAGATION CONSTANT FOR VALUES OF ω BELOW THE CUT-OFF FREQUENCY, ω_c. THERFORE, FOR VALUES OF ω BELOW ω_c, THE SYSTEM ENTERS AN INSTABILITY MODE. THE LARGER ω_c (AND THUS $k_c = \omega_c / v_p$), THE LARGER THE INTERVAL OVER WHICH $B(\omega)$ IS PURE IMAGINARY. FOR VERY LARGE VALUES OF B OR ω, THE SYSEM CHARACTERISTICS APROACH THOSE OF A ~~LOSSLESS~~ non-dispersive SYSTEM (LINEAR RELATION BETWEEM FREQUENCY AND PROPAGATION CONSTANT). THIS CAN BE SEEN FROM ASYMPTOTIC BEHAVIOR ON $B(\omega)$ vs. ω GRAPH. THIS, AGAIN IMPLIES HIGHER FREQUENCIES TRAVEL FASTER IN ~~LOSSY~~ non-dispersive SYSTEMS. THE ~~LOSSLESS~~ SYSTEM IS THUS APROACHED, FOR THE VELOCITY OF THE HIGHER FREQUENCY COMPONENTS APPROACHES V_p, THE VELOCITY OF ALL FREQUENCY COMPONENTS IN A LOSSLESS, NON-DISPERSIVE SYSTEM.

cut-off wavenumber, $\frac{\omega_c}{c}$

ω_c cut-off frequency

?

2-6) $\frac{\partial^2 \delta}{\partial x^2} = \frac{1}{c^2}\left[\frac{\partial^2 \delta}{\partial t^2} - \omega_c^2 \delta\right]$

$$\frac{d^2 \Delta(x,s)}{dx^2} = \frac{1}{c^2}\left[s^2 \Delta(x,s) - s\delta(x,0) - \frac{\partial \delta(x,0)}{\partial t} - \omega_c^2 \Delta(x,s)\right]$$
$$= \frac{1}{c^2}\left[(s^2 - \omega_c)^2 \Delta(x,s) - s\delta_0 \sin\frac{\pi x}{l}\right]$$
$$= \beta^2(\omega)\Delta(x,s) - \frac{s\delta_0}{c^2}\sin\frac{\pi x}{l} \ni \beta^2(\omega) = \left(\frac{s^2 - \omega_c^2}{c}\right)$$

SOLUTION OF DIFFERENTIAL

$$\Delta(x,s) = Ae^{\beta(\omega)x} + Be^{-\beta(\omega)x} + \left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1}\frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$

VERIFICATION: (A AND B CONSTANTS WITH RESPECT TO X)

$$\frac{d\Delta(x,s)}{dx} = A\beta(\omega)e^{\beta(\omega)x} - B\beta(\omega)e^{-\beta(\omega)x} + \frac{\pi}{l}\left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1}\frac{s\delta_0}{c^2}\cos\frac{\pi x}{l}$$
$$\frac{d^2\Delta(x,s)}{dx^2} = A\beta^2(\omega)e^{\beta(\omega)x} + B\beta^2(\omega)e^{-\beta(\omega)x} - \left(\frac{\pi}{l}\right)^2\left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1}\frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$
$$\stackrel{?}{=} \beta^2(\omega)\Delta(x,s) - \frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$
$$\stackrel{?}{=} A\beta^2(\omega)e^{\beta(\omega)x} + B\beta^2(\omega)e^{-\beta(\omega)x} + \beta^2(\omega)\left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1}\frac{s\delta_0}{c^2}\sin\frac{\pi x}{l} - \frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$
$$\stackrel{?}{=} A\beta^2(\omega)e^{\beta(\omega)x} + B\beta^2(\omega)e^{-\beta(\omega)x} + \left[\left(\beta^2(\omega)\left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1} - 1\right]\frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$
$$= A\beta^2(\omega)e^{\beta(\omega)x} + B\beta(\omega)e^{-\beta(\omega)x} - \left(\frac{\pi}{l}\right)^2\left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1}\frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$
$$\therefore \Delta(x,s) = Ae^{\beta(\omega)x} + Be^{-\beta(\omega)x} + \left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1}\frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$

EMPLOYING BOUNDARY CONDITIONS:

$$\Delta(0,s) = 0 = A + B \Rightarrow B = -A$$
$$\Rightarrow \Delta(x,s) = 2B\sinh\beta(\omega)x + \left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1}\frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$
$$\Delta(l,s) = 0 = 2B\sinh\beta(\omega)l \Rightarrow B = 0$$
$$\Rightarrow \Delta(x,s) = \left[\left(\frac{\pi}{l}\right)^2 + \beta^2(\omega)\right]^{-1}\frac{s\delta_0}{c^2}\sin\frac{\pi x}{l}$$
$$= \frac{s}{c^2\left[\left(\frac{\pi}{l}\right)^2 + \frac{s^2 - \omega_c^2}{c^2}\right]}\delta_0\sin\frac{\pi x}{l}$$
$$\therefore \Delta(x,s) = \frac{s}{s^2 - \omega_c^2 + (c\pi/l)^2}\delta_0\sin\frac{\pi x}{l}$$

$$\Delta(x,s) = \delta_0 \frac{s}{s^2 + \left((c\pi/\ell)^2 - \omega_c^2\right)} \sin\frac{\pi x}{\ell}$$

$$= \delta_0 \frac{s}{s^2 + \left[\{(c\pi/\ell)^2 - \omega_c^2\}^{1/2}\right]^2}$$

$$\Rightarrow \delta(x,t) = \delta_0 \cos\left[\left(\frac{c\pi}{\ell}\right)^2 - \omega_c^2\right]^{\frac{1}{2}} t \sin\frac{\pi x}{\ell}$$

LET $\frac{c\pi}{\ell} = \omega_0$

$$\therefore \delta(x,t) = \delta_0 \cos\left[\omega_0^2 - \omega_c^2\right]^{\frac{1}{2}} t \sin\frac{\pi x}{\ell}$$

(STANDING WAVE)

DISCUSSION:

THE SYSTEM SOLUTION IS A STRING OF LENGTH ℓ, VIBRATING IN ITS FUNDAMENTAL HARMONIC, WITH ANGULAR FREQUENCY $\omega_v = \sqrt{\omega_o^2 - \omega_c^2}$, $\omega_o = \frac{c\pi}{\ell}$. IF THE STANDING WAVE BETWEEN $X = 0$ AND $X = \ell$ IS THOUGHT OF AS TWO SINUSOIDS OF AMPLITUDE $\delta_o/2$ TRAVELING WITH EQUAL AND OPPOSITE VELOCITY, C MAY BE THOUGHT OF AS THE VELOCITY OF THESE SINUSOIDS PROPAGATING IN REAL AND IMAGE SPACE. THE FREQUENCY ω_o IS THE FREQUENCY AT WHICH THE STRING (WIRE) WOULD VIBRATE IN ABSENCE OF CURRENT IN THE WIRE. ($\omega_c^2\delta$ TERM DISAPPEARS IN ABSENCE OF CURRENT)

phase vel $= \pm \frac{(\omega_o^2 - \omega_c^2)^{1/2}}{\frac{\pi}{\ell}}$

$= \pm \left[c^2 - \left(\frac{\omega_c}{\frac{\pi}{\ell}}\right)^2\right]^{1/2}$

THIS CONCLUSION AGREES WITH OBSERVATIONS MADE IN PREVIOUS PROBLEM (2-5), WHERE FOR FREQUENCIES BELOW ω_c, THE PROPAGATION CONSTANT BECOMES PURE IMAGINARY, AT WHICH TIME:

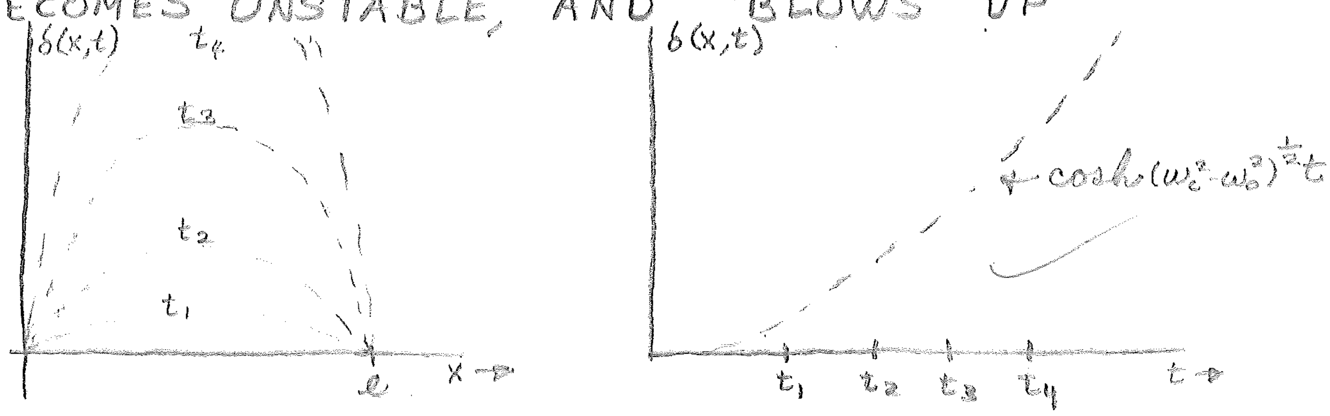
$$\delta(x,t) = \delta_o \cos\,(\omega_o^2 - \omega_c^2)^{\frac{1}{2}} t \,\sin \tfrac{\pi x}{\ell}$$

$$= \delta_o \cos j\,(\omega_c^2 - \omega_o^2)^{\frac{1}{2}} t \,\sin \tfrac{\pi x}{\ell} \quad (\omega_c^2 - \omega_o^2)^{\frac{1}{2}} \text{ IS REAL}$$

$$= \delta_o \cosh\,(\omega_c^2 - \omega_o^2)^{\frac{1}{2}} t \,\sin \tfrac{\pi x}{\ell} \;;\; |\omega_c| > \omega_o$$

$\omega_c > \omega_o \Rightarrow$ unstable.

FOR $\omega < \omega_c$ THE SYSTEM BECOMES UNSTABLE, AND "BLOWS UP"



2-7) $\frac{\delta^2 U}{\delta t^2} = c^2 \frac{\delta^2 U}{\delta z^2} - A_1 H_0 \frac{\delta S_x}{\delta z}$

$\frac{\delta S_x}{\delta t} = -\omega_0 S_y$

$\frac{\delta S_y}{\delta t} = \omega_0 S_x + A_2 H_0 \frac{\delta U}{\delta z}$

Not graded.

a) $A_2 = 0 \Rightarrow \frac{\delta S_y}{\delta t} = -\frac{\delta S_x}{\delta t} = \omega_0 S_y$

ω_0 IS THE ANGULAR VELOCITY OF THE MAGNETIC DIPOLE MOMENT OF THE ATOMIC SPIN AT RESONANCE

b) $-\omega^2\, U(S_p,\omega) = c^2 \dfrac{d^2 U(S_p,\omega)}{dz^2} - A_1 H_0 \dfrac{d S_x(S_p,\omega)}{dz}$

$j\omega\, S_x(S_p,\omega) = -\omega_0\, S_y(S_p,\omega)$

$j\omega\, S_y(S_p,\omega) = \omega_0\, S_x(S_p,\omega) + A_2 H_0 \dfrac{dU(S_p,\omega)}{dz}$

HENCE:

$U(S_p,\omega) = \dfrac{1}{\omega^2} A_1 H_0 \dfrac{d S_x(S_p,\omega)}{dz} - \left(\dfrac{c}{\omega}\right)^2 \dfrac{d^2 U(S_p,\omega)}{dz^2}$

$S_x(S_p,\omega) = \dfrac{-\omega_0}{j\omega}\, S_y(S_p,\omega)$

$S_y(S_p,\omega) = \dfrac{\omega_0}{j\omega}\, S_x(S_p,\omega) + \dfrac{1}{j\omega} A_2 H_0 \dfrac{dU(S_p,\omega)}{dz}$

ERGO:

$S_x(S_p,\omega) = \left(\dfrac{\omega_0}{\omega}\right)^2 S_x(S_p,\omega) + \dfrac{\omega_0}{\omega^2} A_2 H_0 \dfrac{dU(S_p,\omega)}{dz}$

$S_x(S_p,\omega)\left[1 - \left(\dfrac{\omega_0}{\omega}\right)^2\right] = \dfrac{\omega_0}{\omega^2} A_2 H_0 \dfrac{dU(S_p,\omega)}{dz}$

$S_x(S_p,\omega) = \left[1 - \left(\dfrac{\omega_0}{\omega}\right)^2\right]^{-1} \dfrac{\omega_0}{\omega^2} A_2 H_0 \dfrac{dU(S_p,\omega)}{dz}$

$\dfrac{d S_x(S_p,\omega)}{dz} = \left[1 - \left(\dfrac{\omega_0}{\omega}\right)^2\right]^{-1} \dfrac{\omega_0}{\omega^2} A_2 H_0 \dfrac{d^2 U(S_p,\omega)}{dz^2}$

THUS:

$U(S_p,\omega) = \left[\left(1 - \left(\dfrac{\omega_0}{\omega}\right)^2\right)^{-1} \dfrac{\omega_0}{\omega^4} A_1 A_2 H_0^2 - \left(\dfrac{c}{\omega}\right)^2\right] \dfrac{d^2 U(S_p,\omega)}{dz^2}$

$$\frac{d^2 U(S_p,\omega)}{dz^2} = \frac{\left(1 - \left(\frac{\omega_0}{\omega}\right)^2\right) U(S_p,\omega)}{\frac{\omega_0}{\omega^4} A_1 A_2 H_0^2 - \left(1 - \left(\frac{\omega_0}{\omega}\right)^2\right)\left(\frac{c}{\omega}\right)^2}$$

$$= \frac{\omega^2(\omega^2 - \omega_0^2)\, U(S_p,\omega)}{\omega_0 A_1 A_2 H_0^2 - (\omega^2 - \omega_0^2) c^2}$$

$$= \frac{\omega^2\, (\omega^2 - \omega_0^2)\, U(S_p,\omega)}{c^2\left[\omega_0 A_1 A_2 H_0^2/c^2 - \omega^2 + \omega_0^2\right]}$$

$$= -\left(\frac{\omega^2}{c^2}\right) \frac{(\omega^2 - \omega_0^2)}{\omega^2 - \omega_0(\omega_0 + A_1 A_2 H_0^2\, \omega_0/c^2)}\, U(S_p,\omega)$$

THEREFORE:

$$\beta^2(\omega) = \left(\frac{\omega}{c}\right)^2 \frac{(\omega^2 - \omega_0^2)}{\omega^2 - \omega_0(\omega_0 + A_1 A_2 H_0^2/c^2)}$$

$$\Rightarrow \beta(\omega) = \frac{\omega}{c} \sqrt{\frac{\omega^2 - \omega_0^2}{\omega^2 - \omega_0^2 - \omega_0 A_1 A_2 H_0^2/c^2}}$$

$$c\,\beta(\omega) = \omega\, (\omega^2 - \omega_0^2)^{\frac{1}{2}} \left(\omega^2 - \omega_0^2 - \omega_0 A_1 A_2 H_0^2/c^2\right)^{-\frac{1}{2}}$$

c) $cB(\omega) = \omega(\omega^2-\omega_0^2)^{\frac{1}{2}}(\omega^2-(\omega_0^2+\omega_0 A_1 A_2 H_0^2/c^2))^{-\frac{1}{2}}$

$$\lim_{\omega\to\infty}\sqrt{\frac{\omega^2-\omega_0^2}{\omega^2-F^2(\omega_0)}} = \left(\lim_{\omega\to\infty}\frac{\omega^2-\omega_0^2}{\omega^2-F^2(\omega_0)}\right)^{\frac{1}{2}} = 1$$

$$(F^2(\omega_0) = \omega_0^2+\omega_0 A_1 A_2 H_0^2/c^2)$$

$\therefore \lim_{\omega\to\infty} cB(\omega) = \lim_{\omega\to\infty}\omega$

$cB(0) = 0$

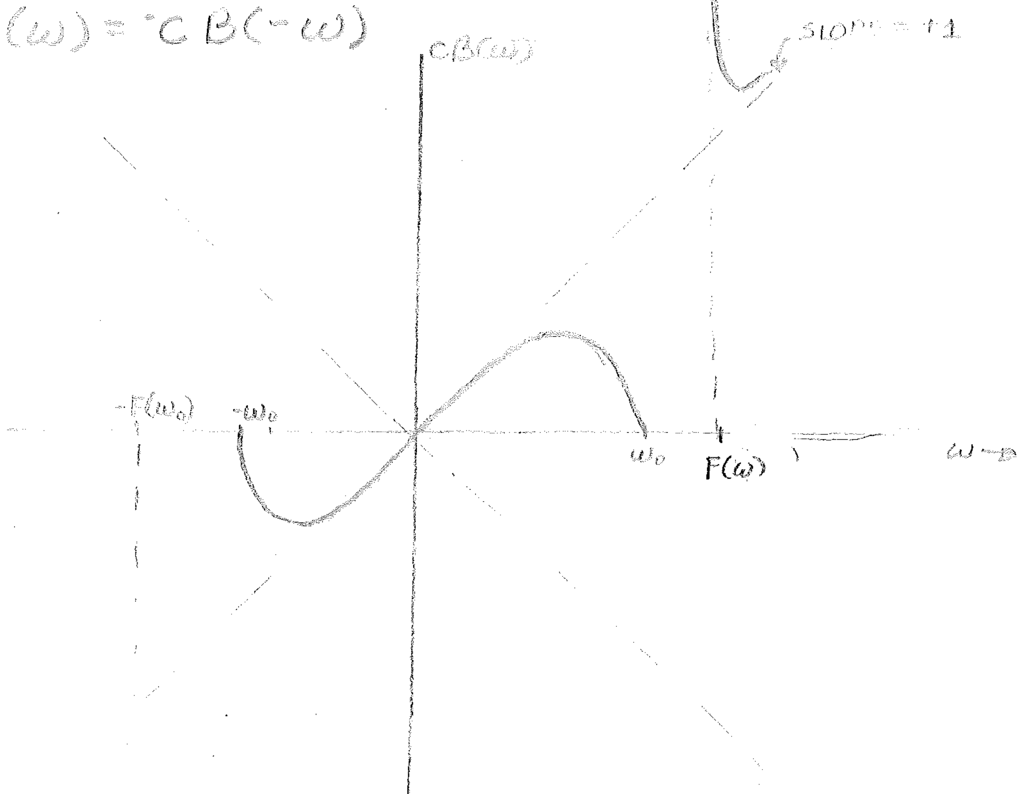
$cB(\omega)$ IS PURE IMAGINARY FOR $\omega_0^2 < \omega^2 < \omega_0^2+\omega_0 A_1 A_2 H_0^2/C^2$

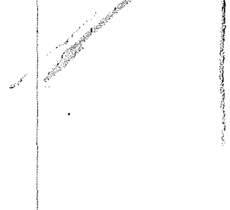
(ASSUMING $\omega_0 A_1 A_2 > 0$) $(\omega_0^2 < \omega^2 < F^2(\omega_0))$

$cB(\omega)$ IS POSITIVE REAL FOR $0 < \omega^2 < \omega_0^2$

$cB(\omega)$ IS POSITIVE REAL FOR $\omega^2 > \omega_0^2+\omega_0 A_1 A_2 H_0^2/C^2$

$cB(\omega) = -cB(-\omega)$





RESONATES @ $F(\omega_0)$

CUT·OFF @ ω_0

$$V_{gp} = \frac{\omega}{\beta(\omega)} = \frac{c\,\omega\,(\omega^2 - F^2(\omega_0))^{\frac{1}{2}}}{\omega\,(\omega^2 - \omega_0^2)^{1/2}}$$

$$\frac{V_{gp}}{c} = \left(\frac{\omega^2 - F^2(\omega_0)}{\omega^2 - \omega_0^2}\right)^{\frac{1}{2}}$$

(AGAIN, ASSUMING $|F(\omega_0)| > |\omega_0|$)

V_{gp}/c IS POSITIVE REAL: $0 < \omega < \omega_0$

V_{gp}/c IS PURE IMAGINARY: $\omega_0 < \omega < F(\omega_0)$

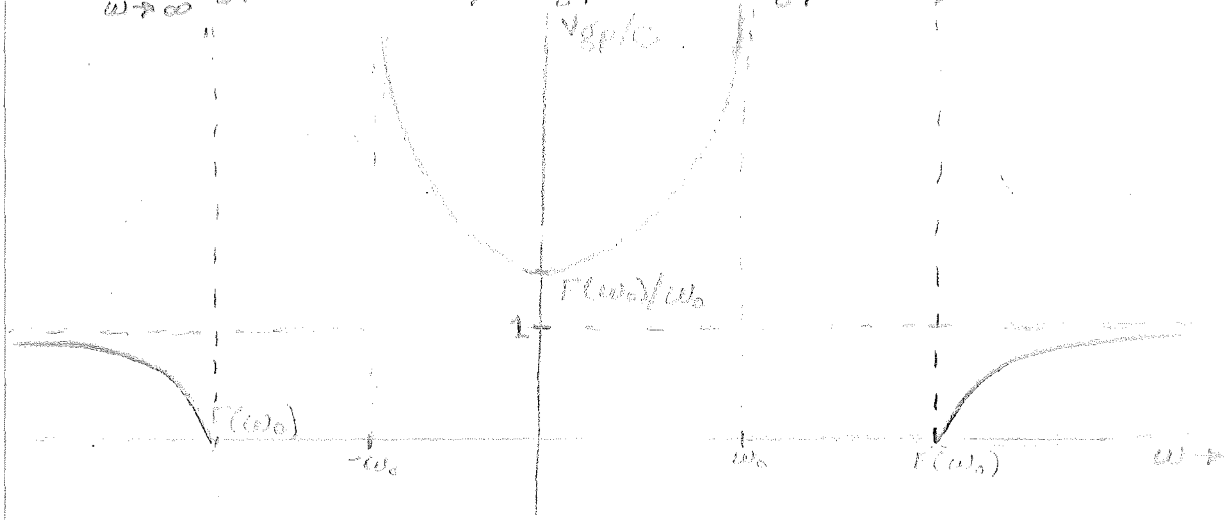
V_{gp}/c IS POSITIVE REAL: $\omega > F(\omega_0)$

$V_{gp}(0)/c = F(\omega_0)/\omega_0$

$V_{gp}(\omega_0)/c = \infty$

$V_{gp}(F(\omega_0))/c = 0$

$\lim_{\omega \to \infty} V_{gp}/c = 1$; $V_{gp}(\omega) = V_{gp}(-\omega)$



$$c\beta(\omega) = \frac{\omega\sqrt{\omega^2-\omega_0^2}}{\sqrt{\omega^2-F^2(\omega)}} \quad \ni F^2(\omega_0) = \omega_0^2 + A_1A_2H_0^2\omega_0/c^2\omega$$

$$U' = (\omega^2-\omega_0^2)^{\frac{1}{2}} \Rightarrow dU' = \omega(\omega^2-\omega_0^2)^{-\frac{1}{2}}d\omega = (\omega^2-\omega_0^2)^{\frac{1}{2}}$$

$$U = \omega(\omega^2-\omega_0^2)^{\frac{1}{2}} \Rightarrow dU = \left[\omega^2(\omega^2-\omega_0^2)^{-\frac{1}{2}} + (\omega^2-\omega_0^2)^{\frac{1}{2}}\right]d\omega$$

$$= \frac{2\omega^2-\omega_0^2}{(\omega^2-\omega_0^2)^{\frac{1}{2}}}d\omega$$

$$V = (\omega^2-F^2(\omega))^{\frac{1}{2}} \Rightarrow dV = \frac{\omega}{(\omega^2-F^2(\omega))^{\frac{1}{2}}}d\omega$$

$F(\omega_0) > \omega_0 \Rightarrow U' > V \quad \forall \ \omega > 0$

$F(\omega_0) > \omega_0 \Rightarrow dU' < dV \quad \forall \ \omega > 0$

$$c\frac{\delta\beta(\omega)}{\delta\omega} = \frac{VdU - UdV}{V^2}$$

$0 < \omega < \omega_0$

U IS $+I_m$; $U(0) = 0$; $U(\omega_0) = +\infty$

dU IS $\pm I_m$; $dU(0) = j\omega_0$; $dU(\omega_0) = +\infty$

V IS $+I_m$; $V(0) = jF(\omega)$; $V(\omega_0) = j\,(F^2(\omega)-\omega_0^2)^{\frac{1}{2}}$

dV IS $-I_m$; $dV(0) = 0$; $dV(\omega_0) = \frac{-j\omega_0}{(F^2(\omega)-\omega_0^2)^{1/2}}$

V^2 IS $-R_e$; $V^2(0) = -F^2(\omega)$; $V^2(\omega_0) = (\omega_0^2 - F^2(\omega_0))$

$F^2(\omega_0) - \omega_0^2 = A_1A_2H_0^2\omega_0/c^2 = G^2(\omega_0)$

$$\Rightarrow c\frac{\delta\beta(\omega)}{\delta\omega}\bigg|_{\omega=0} = \frac{-\omega_0F(\omega_0)}{\omega_0^2-F^2(\omega_0)}$$

$$c\frac{\delta\beta(\omega)}{\delta\omega}\bigg|_{\omega=\omega_0} = 0$$

$\omega_0 < \omega < F(\omega_0)$ $\qquad\qquad$ $F^2(\omega_0) < \omega < \infty$

U IS I_m $\quad U(F(\omega_0)) = \frac{\omega_0}{G(\omega_0)}$ $\qquad U(>F^2(\omega_0))$ IS $+R_e$

dU IS I_m $\quad U(F(\omega_0)) = \frac{F^2(\omega_0)-G^2(\omega_0)}{G(\omega_0)}$ $\qquad dU(>F^2(\omega_0))$ IS $+R_e$

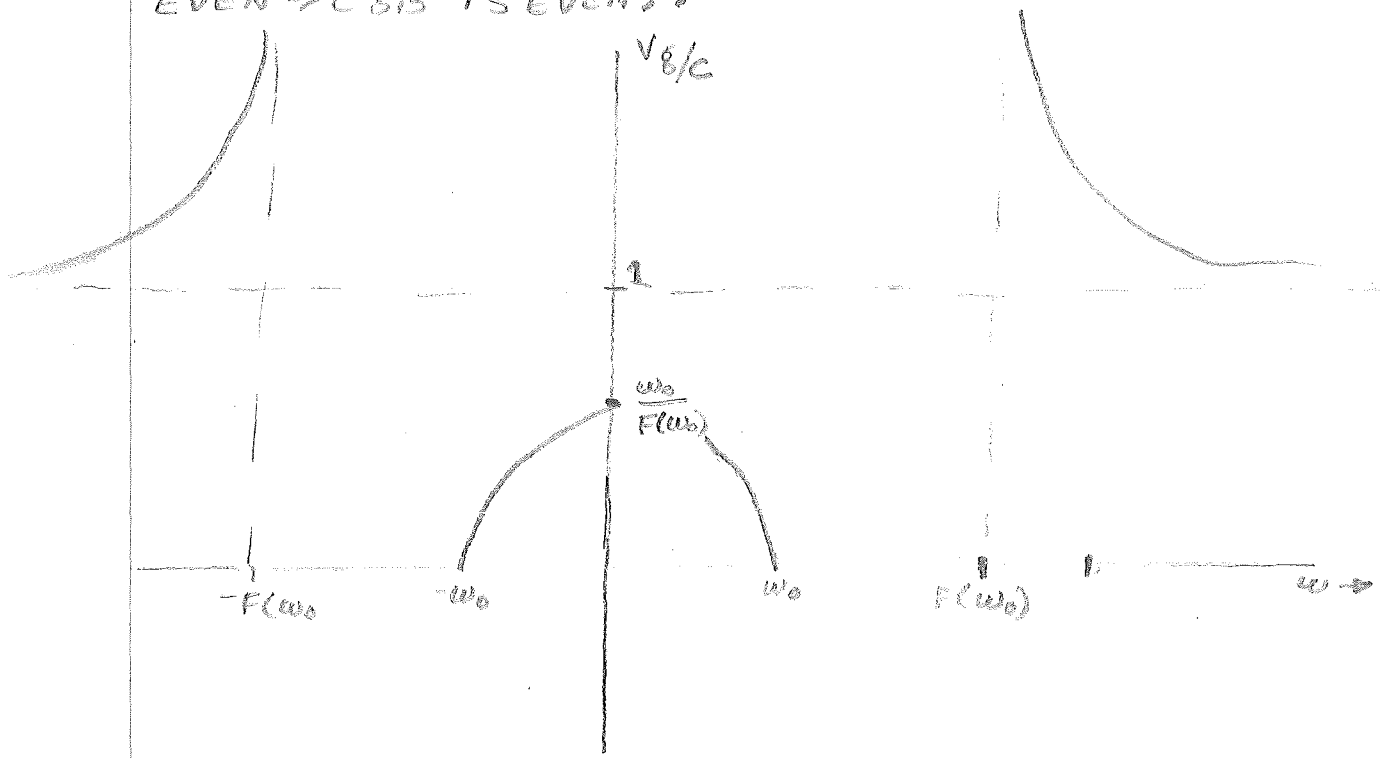
V IS I_m $\quad V(F(\omega_0)) = 0$ $\qquad V(>F(\omega_0))$ IS $+R_e$

dV IS I_m $\quad dV(F(\omega_0)) = \infty$ $\qquad dV(>F(\omega_0))$ IS $+R_e$

$\Rightarrow c\frac{\delta\beta}{\delta\omega}\Big|_{\omega=F(\omega_0)} = 0$ $\qquad \Rightarrow c\frac{\delta\beta}{\delta\omega}\Big|_{\omega=\infty} = \infty$

$\beta(\omega)$ IS ODD $\Rightarrow \frac{\delta\beta}{\delta\omega}$ IS EVEN $\Rightarrow \frac{\delta\omega}{\delta\beta}$ IS EVEN

AS CAN BE SEEN, DERIVATION OF V_g THRU THE EQUATION $\frac{V_g}{C} = \left(\frac{\delta B}{\delta \omega}\right)^{-1}$ IS A HAIRY PROCESS. HOWEVER, BY NOTING $\frac{V_g}{C}\Big|_{\omega=0} = \frac{\omega_0}{F(\omega_0)}$, AND REALIZING THAT $\frac{V_g}{C}$ FOLLOWS THE SAME ASYMPTOTES AS DOES V_{gp}/C, BUT IN AN ALMOST IN VERSE FASHION, THE V_g/C vs ω RELATIONSHIP MAY BE CONCLUDED TO BE THE FOLLOWING (B IS ODD $\Rightarrow \frac{\delta B}{\delta \omega}$ IS EVEN $\Rightarrow \frac{1}{C}\frac{\delta \omega}{\delta B}$ IS EVEN):



d) IN EACH CASE, VALUES OF V_{gp}, V_g, AND $\beta(\omega)$ BECAME PURE IMAGINARY IN THE INTERVAL $\omega_0 < \omega < F(\omega_0) = (\omega_0^2 + \omega_0 A_1 A_2 H_0^2/C^2)^{\frac{1}{2}}$ THIS CONCLUSION IS VALID IFF $\omega_0 < F(\omega_0)$, $(A_1 A_2 > 0)$. ~~IF THIS BE NOT THE CASE ($F(\omega_0) < \omega_0$), THE CUT OFF INTERVAL WOULD BE $F(\omega_0) < \omega < \omega_0$, AS OPPOSED TO $\omega_0 < \omega < F(\omega_0)$.~~

IF AN IMPULSE IS TIME WERE THE INPUT TO THIS SYSTEM (IMPLYING A CONSTANT FREQUENCY DISTRIBUTION), THOSE FREQUENCIES IN THE CUT OFF INTERVAL WOULD NOT PROPAGATE, BUT WOULD ENTER THE EVANECSENT MODE. AGAIN, HIGHER FREQUENCY'S BEHAVIOR WOULD APPROACH THAT OF A NON-DISPERSIVE SYSTEM. THE SYSTEM HAS CUT OFF FREQUENCIES @ $\omega = \omega_0$ AND $\omega = F(\omega_0)$. IF ON THE RECEIVING END, FREQUENCIES DIRECTLY BELOW ω_0 WOULD BE HEARD FIRST, FOLLOWED BY THE LOWER FREQUENCIES DOWN TO D.C. NEXT WOULD BE HEARD THE VERY HIGH FREQUENCIES, FOLLOWED BY LOWER FREQUENCIES DOWN TO CUT-OFF @ $F(\omega_0)$.

(CONT.)

AN INTERESTING ASPECT OF THIS SYSTEM IS THE INFINITE GROUP VELOCITY @ $\omega = \omega_0$. IF A COMMUNICATION SYSTEM COULD BE MODELED AFTER THIS SYSTEM, NEAR INSTANTANEOUS COMMUNICATION COULD BE REALIZED. EITHER SOME MISTAKE HAS BEEN MADE BETWEEN PROBLEM AND SOLUTION, OR ONCE AGAIN, THE CONCEPTS OF ATOMIC PHYSICS AGAIN DEFY LOGIC.

It was my error, but a good observation on your part!

2-8) $jB(s) = (Y(s)Z(s))^{\frac{1}{2}}$

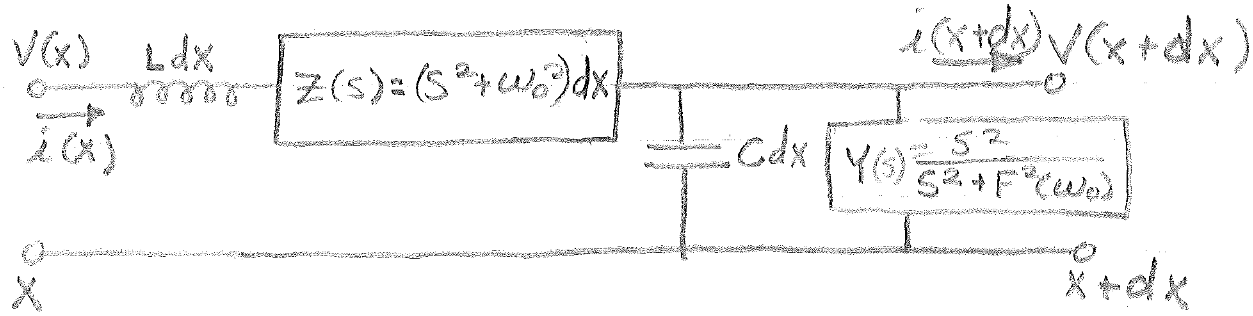
NOW:

$$cB(\omega) = \omega\sqrt{\frac{\omega^2-\omega_0^2}{\omega^2-F^2(\omega_0)}}$$

$$cB(s) = j\,s\sqrt{\frac{s^2+\omega_0^2}{s^2+F^2(\omega_0)}}$$

$$\Rightarrow jB(s) = \frac{s}{c}\sqrt{\frac{s^2+\omega_0^2}{s^2+F^2(\omega_0)}} \Rightarrow Y(s)Z(s) = \frac{s^2}{c^2}\frac{(s^2+\omega_0^2)}{(s^2+F^2(\omega_0))}$$

$$c = V_p = \sqrt{\frac{L}{C}}$$



Not graded.

2-9) $\theta(T) = \frac{\gamma}{(1-\rho^2)^{\frac{1}{2}}}\left[\rho - (S^2 - 2S + \rho^2)^{\frac{1}{2}}\right]$

$\rho = .95\ ;\ \gamma = 18$

$S = \frac{T}{\gamma}(1-\rho^2)^{\frac{1}{2}} = 1.73\times10^{-2}\,T$

$\theta(T) = 57.8\left[.95 - (3\times10^{-4}T^2 - 3.48\times10^{-2}T + .903)^{\frac{1}{2}}\right]$

$\Rightarrow \theta(0) = 55.0$

$\rho = \frac{1}{\Omega(0)} \Rightarrow \Omega(0) = 1.05$

$\frac{\Omega_M}{\Omega_0} = 4 \Rightarrow \Omega_M = 4.20$

$\Omega(T) = \frac{1-S}{(S^2 - 2S + \rho^2)}$

$\Omega(T)S^2 - \Omega(T)2S + \Omega(t)\rho^2 = 1 - S$

$\Omega(T)S^2 - [\Omega(T)2 + 1]S + (\Omega(T)\rho^2 - 1) = 0$

@ $T = T_0,\ \Omega(T) = \Omega_{MAX} = 4.2$

FROM QUADRATIC EQUATION;

$$S_0 = \frac{(2\Omega_M + 1) \pm (4\Omega_M^2 - 4\Omega_M(\Omega_M\rho^2 - 1))^{\frac{1}{2}}}{2\Omega_M + 1}$$

$$= \frac{(2 + \frac{1}{\Omega_M}) \pm (4 - 4\rho^2 + 4/\Omega_M)^{\frac{1}{2}}}{2 + 1/\Omega_m}$$

$\frac{1}{\Omega_m} = .238 \quad \rho^2 = .903$

$\Rightarrow S_0 = 1.73\times10^{-2}\,T = \frac{2.24 \pm \sqrt{9.81}}{2.24} = 2.31 \qquad (T_0 > 0)$

$\Rightarrow T_0 = 133$

$\therefore V_0(T) = V_0\, I_m(e^{j\theta(T)})[\mu(T) - \mu(T - T_0)]$

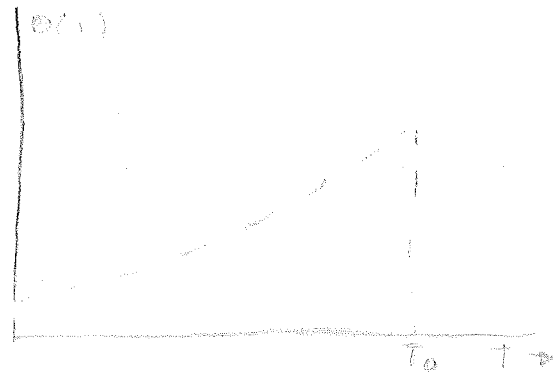
$= V_0 \sin\left[57.8(.95 - (3\times10^{-4}T^2 - 3.48\times10^{-2}T + .903)^{\frac{1}{2}})\right][\mu(T) - \mu(T - 133)]$

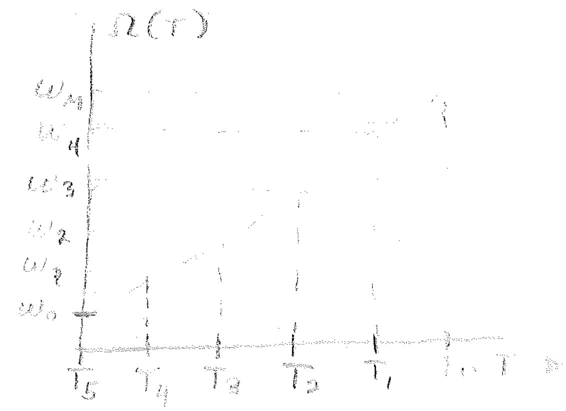
$\Rightarrow V_0$ IS THE ARBITRARILY CHOSEN "CHIRP" AMPLITUDE

T_p = NORMALIZED TIME FOR COMPRESSION TO OCCUR:

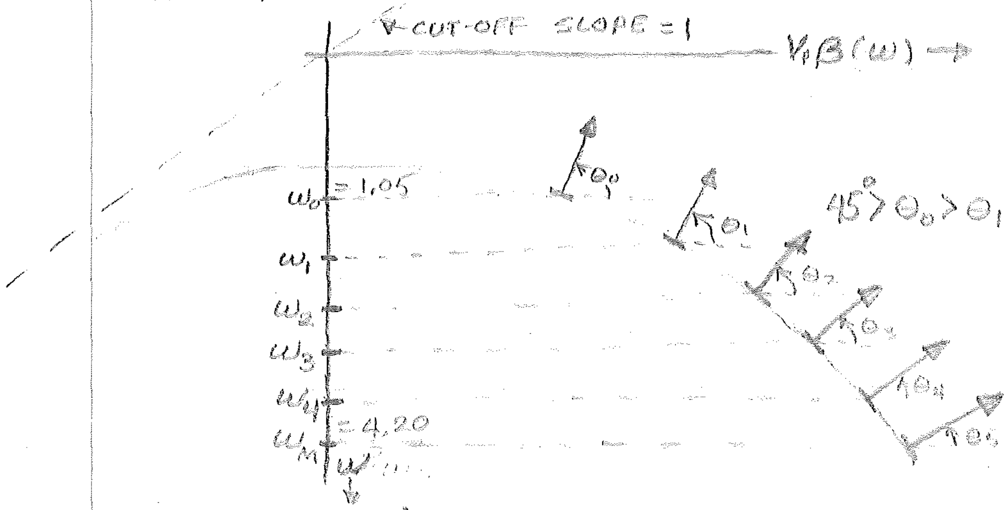
$T_p = \frac{\gamma}{(1-\rho^2)^{\frac{1}{2}}} = 57.8$

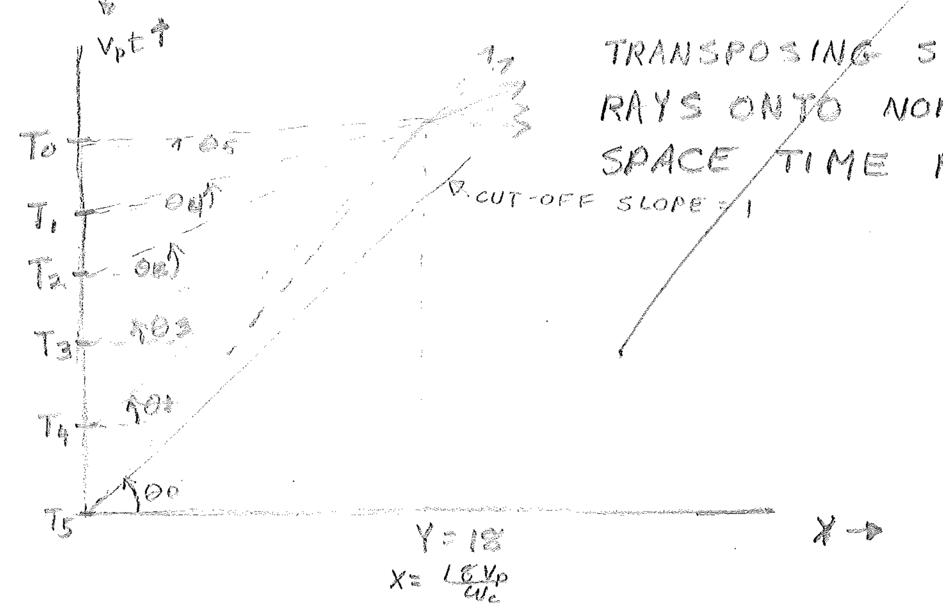
2-10)





WAVEGUIDE $\beta(\omega)-\omega$ RELATION:





TRANSPOSING SPACE TIME RAYS ONTO NORMALIZED SPACE TIME PLANE.

DISCUSSION:

$T_0, T_1, \ldots T_5$ ARE ARBITRARY INSTANTS OF TIME, IN THE INTERVAL $0 < T < T_0 = 18$, FROM WHICH CORRESPONDING VALUES OF ω MAY BE COMPUTED FROM THE EQUATION DERIVED IN THE PREVIOUS PROBLEM (2-9) RELATING ω AND T. THE VALUES OF ω MAY THEN BE TRANSPOSED TO A $\beta(\omega)$ VS ω GRAPH OF THE WAVEGUIDE, WHERE UPON CORRESPONDING SPACE-TIME RAYS MAY BE DRAWN. THESE RAYS, IN TURN MAY BE TRANSPOSED ONTO THE NORMALIZED SPACE-TIME PLANE, WHERE THEY RETAIN THEIR SLOPE (COMPUTED ON $\beta(\omega)$-ω PLANE) THE DESIGN OF THE SYSTEM INPUT, AS SHOWN, FORCES THE SPACE-TIME RAYS TO CONVERGE AT $Y = 18$, RESULTING IN PULSE COMPRESSION, AFTERWHICH, THEY DIVERGE, THE HIGHER FREQUENCY COMPONENTS PASSING THE LOWER FREQUENCY COMPONENTS.

BOB MARKS

57/60 Good work.

470 k
15 V

2-1. Plot $\frac{Z_0}{2T}$ vs. $\frac{T}{\sqrt{X_1\beta_0''}}$ using log-log paper, in Eqn (2-16) and show the regions of no pulse definition and good pulse definition. Interpret the results.

2-2. In Appendix A we showed that the characteristic impedance of the waveguide is $Z_c = \frac{Z_0}{[1-(\omega_c/\omega)^2]^{1/2}}$ where $Z_0 = \sqrt{\frac{L}{C}}$

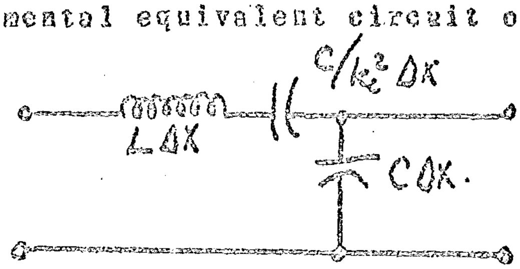
If an infinitely long waveguide is connected to a generator with frequency $\omega_0 < \omega_c$ an evanescent mode (wave) is produced. Explain physically what is occurring as far as energy propogation, storage, etc. is concerned.

2-3. In Appendix A the statement is made that

$$Z_c = \sqrt{Z/Y}\,, \quad j\beta = \sqrt{ZY}\,,$$

where Z is the series impedance-per-unit length and Y is the shunt admittance per-unit-length of the line. Draw an incremental equivalent circuit showing Z and Y of such a line, derive the transmission-line equations and verify the above statement.

2-4. The incremental equivalent circuit of a certain transmission-line is



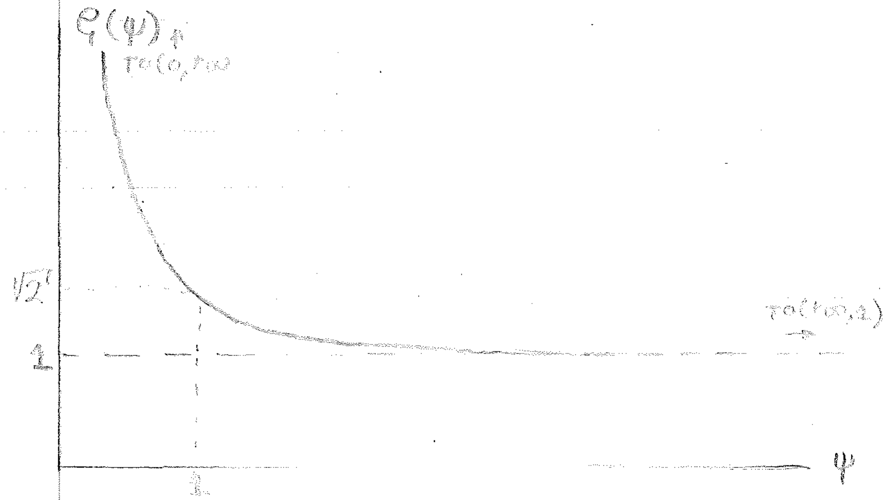
Determine $Z_c(\omega)$ and $\beta(\omega)$ vs. ω. Sketch them, along with the phase velocity, $v_{gp}(\omega)$ and group velocity $v_g(\omega)$.

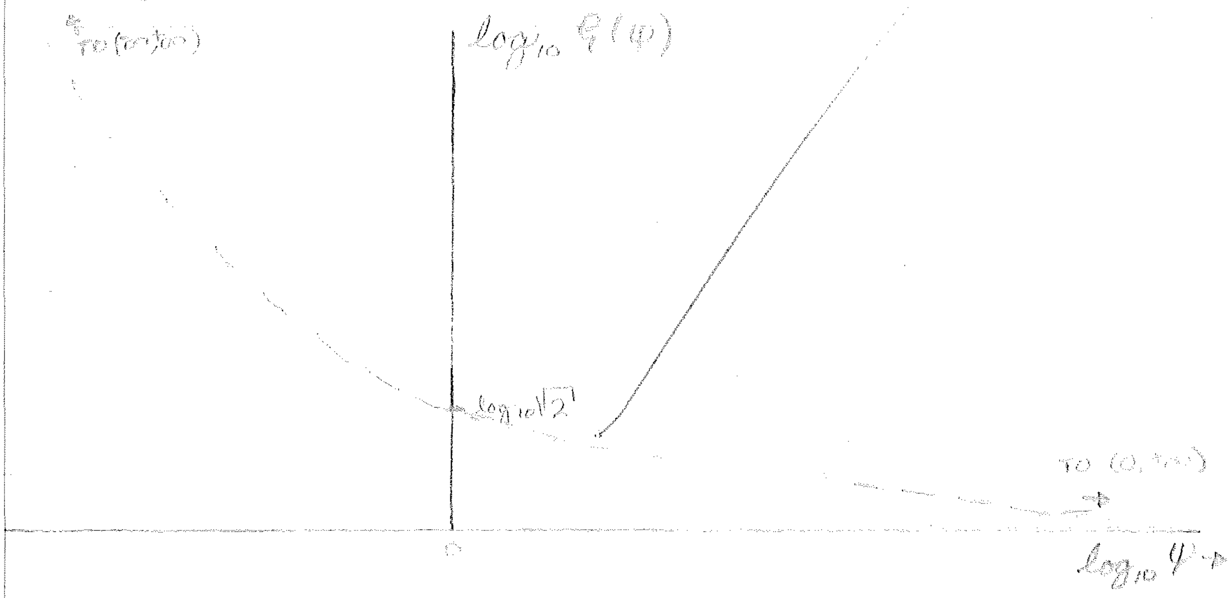
2-1) $\frac{\tau_0}{2T} = \left[1 + \left(\frac{X_1 B_0''}{T^2} \right)^2 \right]^{1/2}$

LET $\varphi = \frac{\tau_0}{2T}$ (φ DIMENSIONLESS)

LET $\psi = \frac{T}{\sqrt{X_1 B_0''}}$ (ψ DIMENSIONLESS)

$\Rightarrow \varphi = \left(1 + \frac{1}{\psi^4} \right)^{1/2}$





THE EXPRESSION $\frac{\tau_0}{2T} = \left[1+\left(\frac{X_1 B_0''}{T^2}\right)^2\right]^{1/2}$ DEFINES THE LOCUS OF POINTS OF AMBIGUITY, WHEREIN THE FINAL ADJACENT PULSE OVERLAP (τ_0) IS EQUIVALENT TO THE FINAL PULSE WIDTH ($2T\left[1+\left(\frac{X_1 B_0''}{T^2}\right)^2\right]^{1/2}$), AT WHICH POINT THE DISPERSED GAUSSIAN PULSES BEGIN TO TAKE THE FORM OF A NEARLY CONSTANT VALUE, THE RELATIVE MAXIMA AND MINIMA BEING ILL-DEFINED, RESULTING IN POOR PULSE DEFINITION, AND THUS EITHER POOR OR NON-EXISTANT COMMUNICATION. THE SITUATION WORSENS AS THE OVERLAP (τ_0) BECOMES LARGER THAN PULSE WIDTH ($2T\left[1+\left(\frac{X_1 B_0''}{T^2}\right)^2\right]^{1/2}$), OR

$$\frac{\tau_0}{2T} > \left[1+\left(\frac{X_1 B_0''}{T^2}\right)^2\right]^{1/2}$$

PULSE DEFINITION IS IMPROVED AS PULSE OVERLAP (τ_0) IS SMALL WITH RESPECT TO PULSE WIDTH, OR

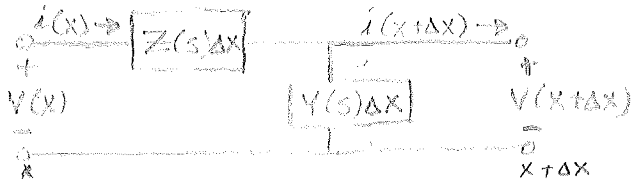
$$\frac{\tau_0}{2T} < \left[1+\left(\frac{X_1 B_0''}{T^2}\right)^2\right]^{1/2}$$

THE PLOT OF $\frac{\tau_0}{2T}$ VS. $\sqrt{\frac{T^2}{X_1 B_0''}}$ REPRESENTS A BOUNDRY CURVE, ABOVE WHICH REPRESENTS GOOD PULSE DEFINITION, BELOW WHICH REPRESENTS POOR PULSE DEFINITION. (AS ψ INCREASES, X_1 BECOMES SMALLER, RESULTING IN LESS DISPERSION, AND THUS BETTER PULSE DEFINITION) ROUGHLY, FOR $\psi < 1$, THERE IS POOR DEFINITION AND FOR $\psi > 1$, THERE IS GOOD DEFINITION.

2-2) $Z_c(j\omega) = \frac{Z_0}{[1-(\omega_c/\omega_0)^2]^{1/2}}$; $Z_0 = \sqrt{L/C}$

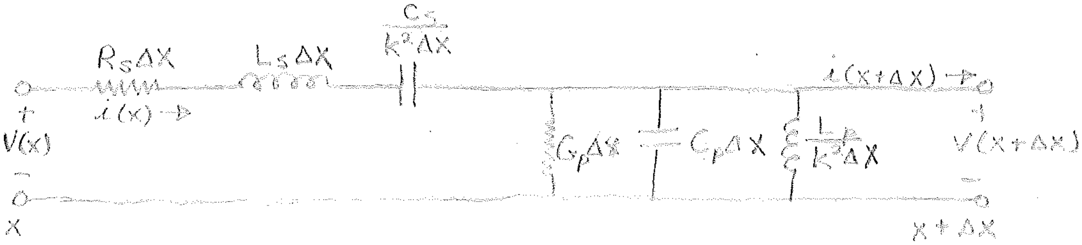
FOR $\omega_0 > \omega_c$, THE VALUE OF Z_c IS PURE REAL, OR PURELY RESISTIVE. AS ω_0 APPROACHES ω_c, Z_c BECOMES MORE POSITIVE, EQUALLING ∞, OR AN OPEN CIRCUIT AT $\omega_0 = \omega_c$. FOR VALUES OF ω_0 LESS THAN ω_c, $Z_c(j\omega)$ BECOMES PURE IMAGINARY, TAKING ON THE CHARACTERISTICS OF A ~~CAPACITIVE~~ inductive REACTANCE, OR DUALLY, AN INDUCTIVE SUSCEPTANCE. FOR $\omega_0 > \omega_c$, THERE THUS EXISTS WAVE PROPAGATION AND ENERGY TRANSFER DOWN THE WAVEGUIDE. FOR $\omega_c < \omega_0$, ENERGY IS NOT TRANSFERED DOWN THE WAVEGUIDE, BUT IS STORED IN AN ELECTRIC FIELD INSIDE THE EQUIVALENT OF A DISTRIBUTED SHUNT CONDUCTANCE, AND/OR IN A MAGNETIC FIELD AROUND THE EQUIVALENT SERIES INDUCTANCE. SUCH IS CHARACTERISTIC OF THE EVANESCENT MODE, ANALAGOUS TO POTENTIAL ENERGY AS OPPOSED TO WAVE PROPAGATION AND KINETIC ENERGY. (OVER 1/2 CYCLE, ENERGY IS STORED. OVER THE OTHER 1/2 CYCLE, ENERGY IS RESTORED TO THE SOURCE)

Not a good analogy. Even in a propagating wave there is "potential" and "kinetic" energy.

2-3) 

Z(S) MAY BE BROKEN DOWN INTO RESISTIVE, AND (CAPACATIVE AND INDUCTIVE) REACTIVE COMPONENTS. (Ω/m)

Y(S) MAY BE BROKEN DOWN INTO CONDUCTIVE AND (INDUCTIVE AND CAPACITIVE) SUSCEPTIVE COMPONENTS. (S/m)



$(k_c = \text{CUT-OFF WAVE NUMBER})$ $(C_S = C_P,\ L_S = L_P)$

$$\therefore \quad Z(S) = R_S + SL_S + \frac{k_c^2}{SC_S}\ ;\ Z(\omega) = R_S + j\omega L_S - \frac{j}{\omega C}$$
$$Y(S) = G_p + SC_p + \frac{k_c^2}{SL_p}\ ;\ Y(\omega) = G_p + j\omega C_p - \frac{j}{\omega L}$$

EMPLOYING KIRCHOFF'S VOLTAGE LAW:

$$V(x+\Delta x) - V(x) = -Z(\omega)\,\Delta x\, i(x)$$

DIVIDING BY ΔX, AND LETTING ΔX → 0

$$\frac{dV(x,\omega)}{dx} = \lim_{\Delta x \to 0} \frac{V(x+\Delta x) - V(x)}{\Delta x} = -Z(\omega)\, I(x,\omega)$$

$$\Rightarrow \frac{dV(x,\omega)}{dx} = -\overbrace{\left(R_S + j\omega L_S + \frac{k_c}{j\omega C_S}\right)}^{Z(S)} I(x,\omega)$$

NOTING DUALITY IN EXPRESSION AND DEFINITION OF Y(ω) AND Z(ω), EMPLOYING KIRCHOFF'S CURRENT LAW WOULD YIELD

$$\frac{dI(x,\omega)}{dx} = -Y(\omega)\,V(x,\omega)$$
$$= -\underbrace{\left(G_p + j\omega C_p + \frac{k_c^2}{j\omega L_p}\right)}_{Y(S)} V(x,\omega)$$

HENCE:

$$\frac{d^2 V(x,\omega)}{dx^2} = -Z(\omega)\frac{dI(x,\omega)}{dx}$$

$$= Z(\omega)Y(\omega)V(x,\omega)$$

$$\frac{d^2 I(x,\omega)}{dx^2} = -Y(\omega)\frac{dV(x,\omega)}{dx}$$

$$= Z(\omega)Y(\omega)I(x,\omega)$$

RECOGNISING THAT THE NEGATION OF THE SQUARE OF THE PROPAGATION CONSTANT $(-B^2(\omega))$ IS DEFINED AS THE PHASE SHIFT PER UNIT LENGTH COEFFICIENT $(Y(s)*Z(s))$ OF THE SECOND ORDER WAVE EQUATION:

$$-B^2(\omega) = Z(\omega)Y(\omega)$$

$\therefore jB(\omega) = \sqrt{Z(\omega)Y(\omega)}$, WITH UNITS $1/\text{length}$

A SOLUTION TO THE SECOND ORDER DIFFERENTIALS:

$$V(x,\omega) = Ae^{jB(\omega)x} + Be^{-jB(\omega)x}$$

$$I(x,\omega) = \frac{1}{Z_c(\omega)}\left[Ae^{jB(\omega)x} - Be^{-jB(\omega)x}\right]$$

$$\frac{dI(x,\omega)}{dx} = \frac{jB(\omega)}{Z_c(\omega)}\left[Ae^{jB(\omega)x} + Be^{-jB(\omega)x}\right]$$

$$= \frac{jB(\omega)}{Z_c(\omega)}V(x,\omega)$$

$$\therefore \frac{jB(\omega)}{Z_c(\omega)} = -Y(\omega)$$

$$\frac{-B^2(\omega)}{Z_c^2(\omega)} = Y^2(\omega)$$

$$\Rightarrow Z_c^2(\omega) = -B^2(\omega)/Y^2(\omega)$$

$\therefore Z_c(\omega) = \left(Z(\omega)/Y(\omega)\right)^{1/2}$ WITH UNITS OF Ω

2-4) FROM PROBLEM 2-3

$R_S = 0;\ L_S = L_S;\ C_S = C_S$

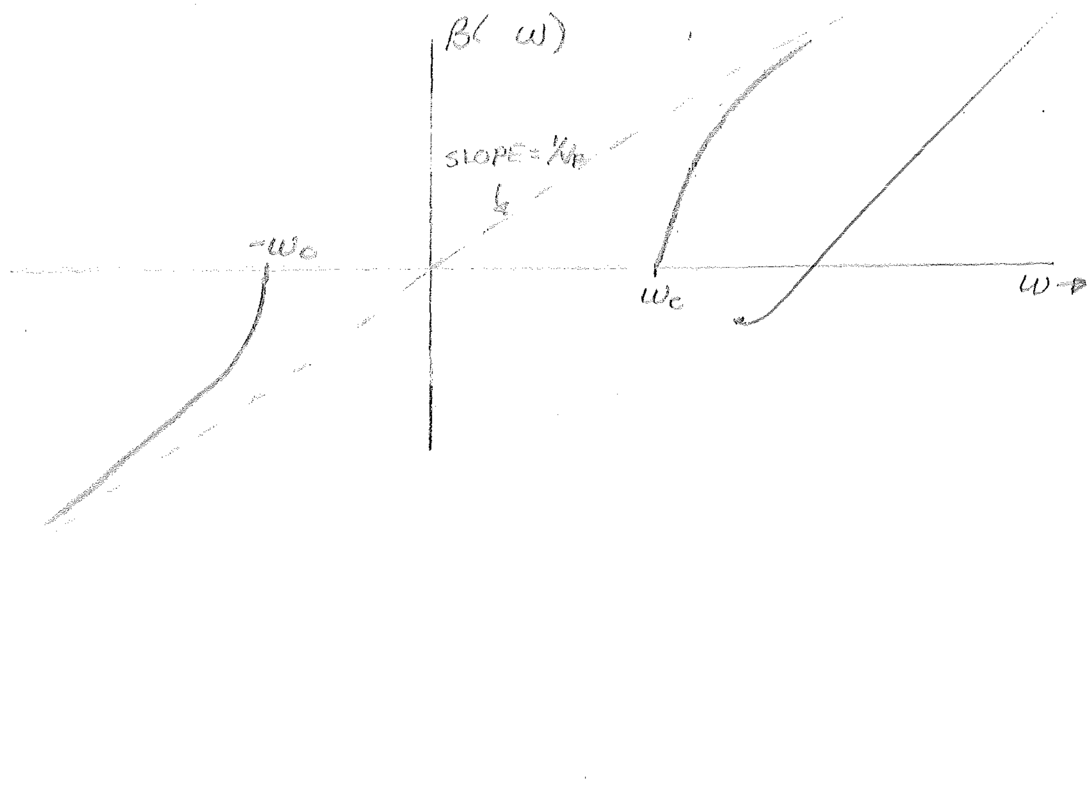
$G_P = 0;\ C_P = C_P;\ L_P = \infty$

THUS:

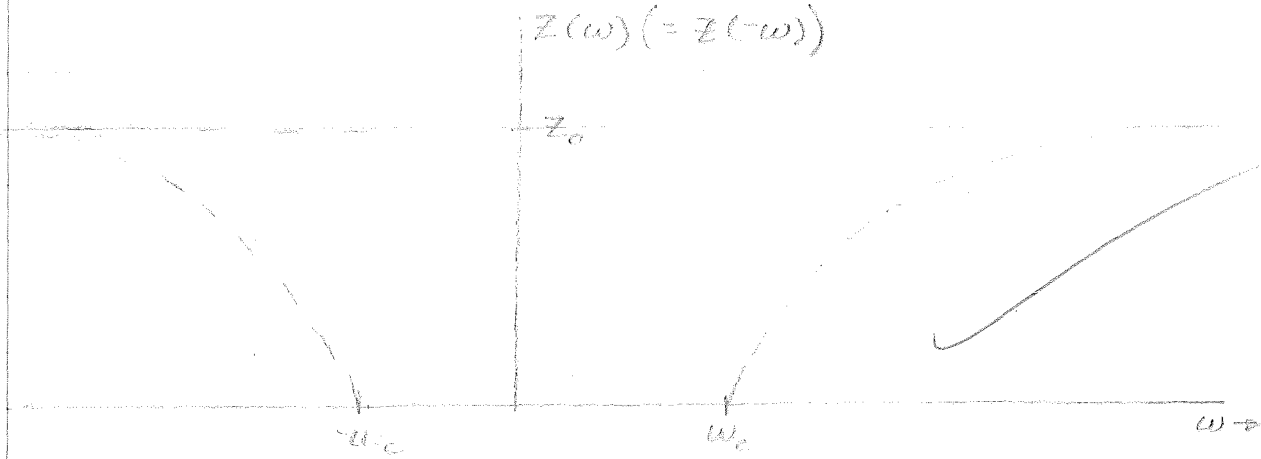
$$Z(j\omega) = j\omega L_S + \frac{k^2}{j\omega C_S}$$

$$Y(j\omega) = j\omega C_P$$

$$-\beta^2(j\omega) = Z(j\omega)\,Y(j\omega)$$
$$= \left(j\omega L_S + \frac{k_c}{j\omega C_{S_2}}\right) j\omega C_P$$
$$= -\omega^2 L_S C_S + k_c^2$$
$$\Rightarrow \beta(j\omega) = \sqrt{\omega^2 LC - k_c^2}$$
$$= \sqrt{(\omega/V_p)^2 - k_c^2}\ ; \quad \omega_c = k_c V_p$$
$$\therefore\ \beta(\omega) = \frac{1}{V_p}\sqrt{\omega^2 - \omega_c^2} = \frac{\omega}{V_p}\left(1 - (\omega_c/\omega)^2\right)^{\frac{1}{2}}$$



$$Z_c^2(\omega) = Z(\omega) / Y(\omega)$$
$$= \left(j\omega L + \frac{k^2}{j\omega C}\right)\left(\frac{1}{j\omega C}\right)$$
$$= \left(\frac{L}{C} - \left(\frac{k_c}{\omega C}\right)^2\right)$$
$$= \left(Z_0^2 - \left(\frac{k_c}{\omega C}\right)^2\right) \ni Z_0 = \sqrt{L/C}$$
$$Z_c(\omega) = Z_0\left(1 - \left(\frac{k_c}{\omega C Z_0}\right)^2\right)^{1/2}$$
$$= Z_0\left(1 - \left(\frac{k_c V_p}{\omega}\right)^2\right)^{1/2} \ni V_p = \sqrt{LC}$$
$$= Z_0\left(1 - (\omega_c/\omega)^2\right)^{1/2} \ni \omega_c = V_p k_c$$

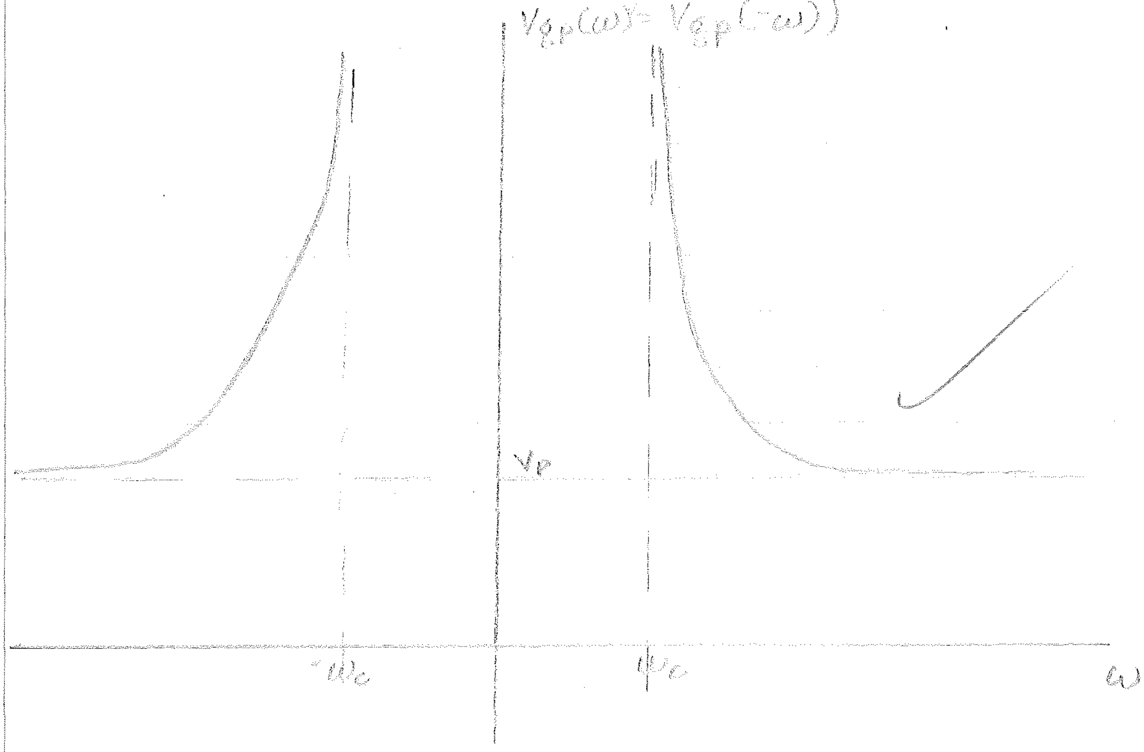


What is nature of $Z_c(\omega)$ for $\omega < \omega_c$?,

capacitive?
Inductive?,

(3)

$$V_{gp}(\omega) = \frac{\omega}{\beta(\omega)} = V_p\left(1 - \left(\omega_c/\omega\right)^2\right)^{-1/2}$$

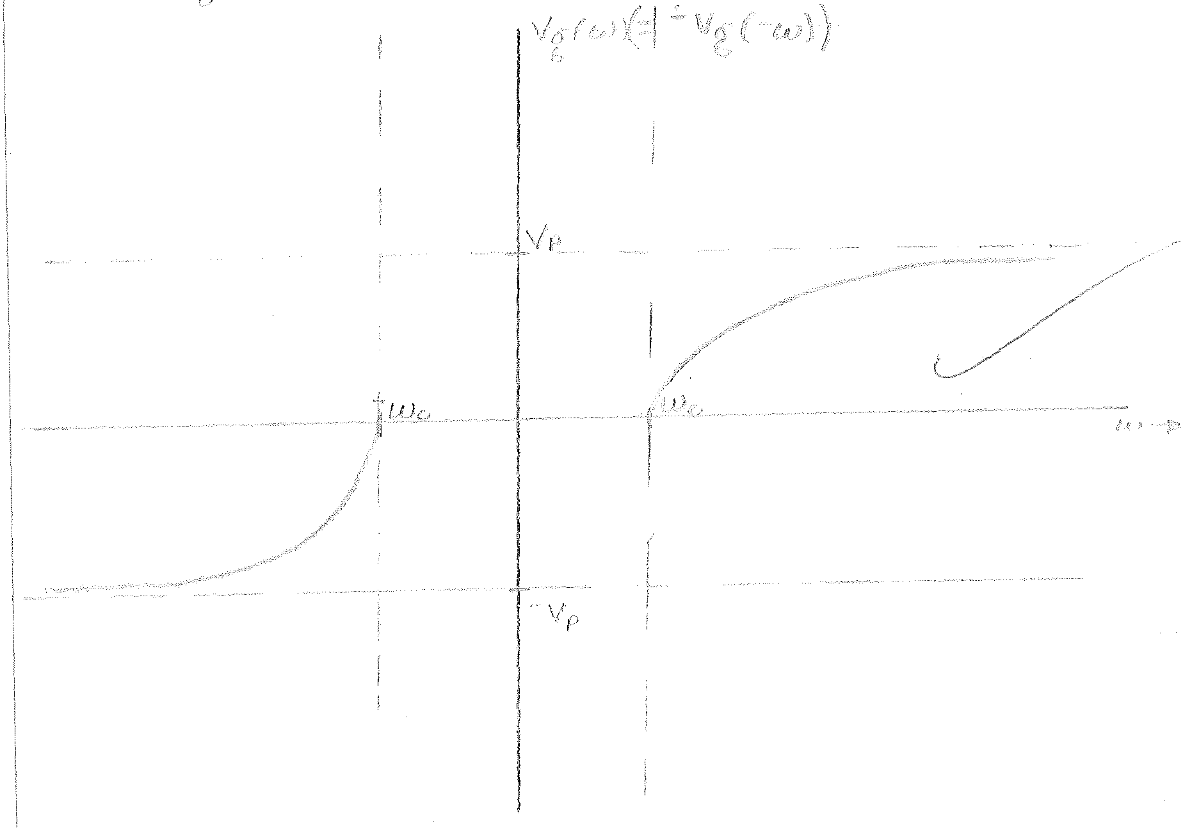


$$V_g(\omega) = \left(\frac{\delta \beta(\omega)}{\delta \omega} \Big|_{\omega = \omega_0} \right)^{-1}$$

$$\beta(\omega) = \frac{1}{V_p} (\omega^2 - \omega_c^2)^{\frac{1}{2}}$$

$$\frac{d\beta(\omega)}{d\omega} = \frac{\omega}{V_p} (\omega^2 - \omega_c^2)^{-\frac{1}{2}} = \frac{1}{V_g(\omega)}$$

$$\Rightarrow V_g(\omega) = \frac{V_p}{\omega} (\omega^2 - \omega_c^2)^{\frac{1}{2}}$$



DISCUSSION:

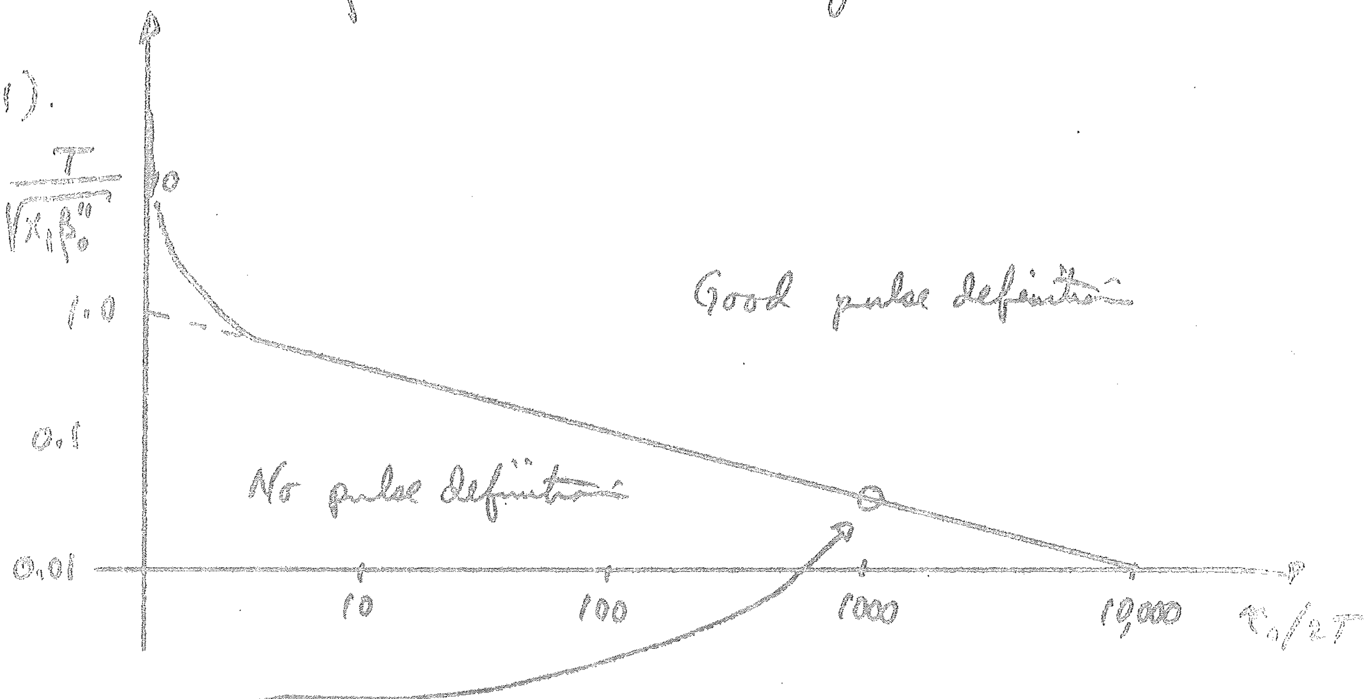
IN EACH OF THE FOUR CASES, REAL VALUES OF THE ARGUMENTS DO NOT RESULT UNTIL THE CUT-OFF FREQUENCY ω_c IS REACHED OR SURPASSED. FOR VALUES OF $|\omega|<|\omega_c|$, ARGUMENTS BECOME PURE IMAGINARY, AND THE SYSTEM ENTERS THE EVANESCENT MODE.

THE VALUES OF $\beta(\omega)$ APPROACH THOSE ON A NON-DISPERSIVE SYSTEM AT HIGHER FREQUENCIES. $Z(\omega)$ APPROACHES $Z_0=\sqrt{L/C}$ FOR HIGHER FREQUENCIES, AND BOTH $V_g(\omega)$ AND $V_{gp}(\omega)$ APROACH $V_p = \sqrt{LC}$ AT HIGHER FREQUENCIES. IT MAY THUS BE CONCLUDED THAT THE WAVEGUIDE APPROACHES A NON-DISPERSIVE SYSTEM AT HIGHER FREQUENCIES.

THE GUIDE PHASE $(V_{gp}(\omega))$ IS GREATER THAN V_p. IT IS INTERESTING TO NOTE THAT IN AN ELECTROMAGNETIC WAVEGUIDE OF THIS TYPE, THE GUIDE PHASE VELOCITY ALWAYS EXCEEDS THAT OF LIGHT. UNFORTUNATELY, INFORMATION CANNOT BE TRANSMITTED AT GUIDE PHASE VELOCITY, BUT IS RESTRICTED TO THE GROUP VELOCITY $(V_g(\omega))$ WHICH MAY NEVER EXCEED V_p

Traveling-Waves: Solutions of 2-1 – 2-8.

(2-1).



Example: for $\frac{\tau_0}{2T} = 1000$, then $T \gg 0.0316\sqrt{X_1\beta_0''}$

in order that there be good pulse definition after transmission over a path of length X_1.

(2-2) $$Z_c = \frac{Z_0}{\left[1-\left(\frac{\omega_c}{\omega}\right)^2\right]^{1/2}}.$$ According to (A-1),

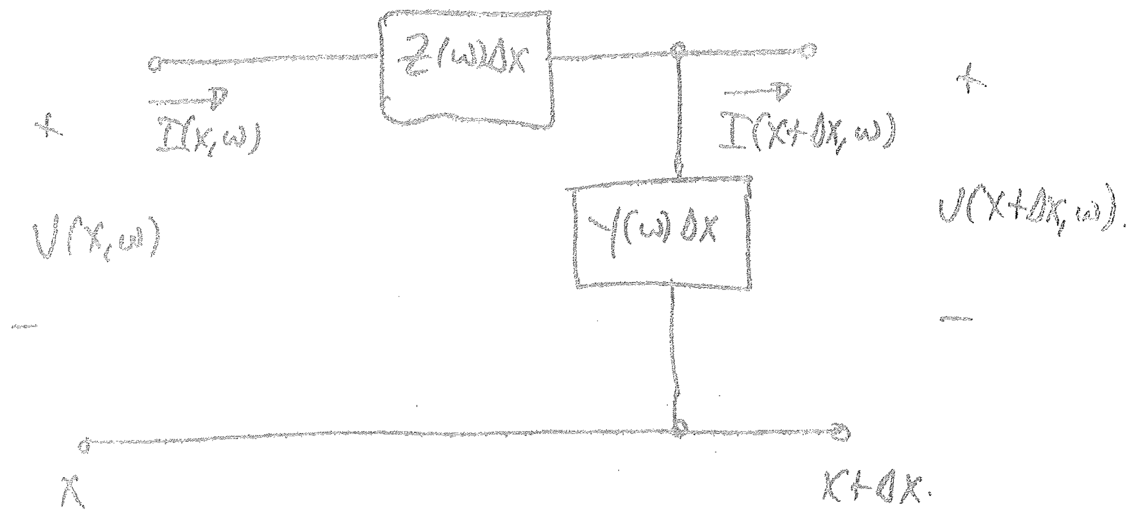
$\beta(\omega) = \left(\frac{\omega}{v_p}\right)\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2}$, so that $V(x) = V_+ e^{-j\beta(\omega)x}$

implies that when $\omega = \omega_0 < \omega_c$, we must take the negative square root in order to achieve an exponentially decaying evanescent wave. Thus

$$Z_c = \frac{Z_0}{-j\left[\left(\frac{\omega_c}{\omega_0}\right)^2-1\right]^{1/2}} = j\frac{Z_0}{\left[\left(\frac{\omega_c}{\omega_0}\right)^2-1\right]^{1/2}},$$ which is inductive.

Energy is stored in the waveguide and returned to the source every one-half cycle of the input. There is no propagation and, hence, no energy loss due to propagation. The evanescent exponential decay has *nothing* to do with copper losses, dielectric losses, etc.

(2-3).



Kirchhoff's laws yield: $V(x+\Delta x,\omega) - V(x,\omega) = -I(x,\omega)\,Z(\omega)\,\Delta x$

or $\dfrac{dV}{dx} = -I\,Z(\omega)$

$I(x+\Delta x,\omega) - I(x,\omega) = -V(x+\Delta x,\omega)\,Y(\omega)\,\Delta x.$

or $\dfrac{dI}{dx} = -Y(\omega)\,V(\omega).$

$\therefore \quad \dfrac{d^2V}{dx^2} = -Z(\omega)\dfrac{dI}{dx} = Y(\omega)\,Z(\omega)\,V(\omega). = -\beta^2(\omega)\,V(\omega)$

(by definition of $-\beta^2(\omega)$). $\therefore$ $\boxed{j\beta(\omega) = \left(Y(\omega)Z(\omega)\right)^{1/2}}$.

$V(x,\omega) = V(0,\omega)\,e^{-j\beta(\omega)x}$ (for a (+)-traveling wave)

$\therefore \quad I(x,\omega) = -\dfrac{1}{Z(\omega)}\dfrac{dV}{dx} = j\dfrac{\beta(\omega)}{Z(\omega)}\,V(0,\omega)\,e^{-j\beta(\omega)x} = \dfrac{V(0,\omega)\,e^{-j\beta(\omega)}}{Z_c(\omega)}$

$\therefore \quad \dfrac{1}{Z_c(\omega)} = \left(Y(\omega)Z(\omega)\right)^{1/2}/Z(\omega) \Rightarrow \boxed{Z_c(\omega) = \sqrt{\dfrac{Z(\omega)}{Y(\omega)}}}$

(2-4). $Z(\omega) = j\omega L + \frac{k_c^2}{j\omega C} = \frac{k_c^2 - \omega^2 LC}{j\omega C}$

$Y(\omega) = j\omega C$

$\therefore \quad Z_c(\omega) = \sqrt{\frac{Z(\omega)}{Y(\omega)}} = \sqrt{\frac{k_c^2 - \omega^2 LC}{-\omega^2 C^2}}$

$= \sqrt{\frac{L}{C} - \frac{k_c^2}{\omega^2 C^2}}$. Let $k_c^2 = \frac{\omega_c^2}{v_p^2} = \omega_c^2 LC$

Then $Z_c(\omega) = \sqrt{\frac{L}{C} - \frac{\omega_c^2 LC}{\omega^2 C^2}} = \boxed{Z_0\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{1/2}}$

$\beta(\omega) = -j\sqrt{Z(\omega)Y(\omega)} = -j\left[k_c^2 - \omega^2 LC\right]^{1/2}$

$= -j\left[-\omega^2 LC\left(1 - \frac{\omega_c^2}{\omega^2}\right)\right]^{1/2}$

$= \frac{\omega}{v_p}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{1/2}$, the same as for the waveguide described in the text. (TE-wave)

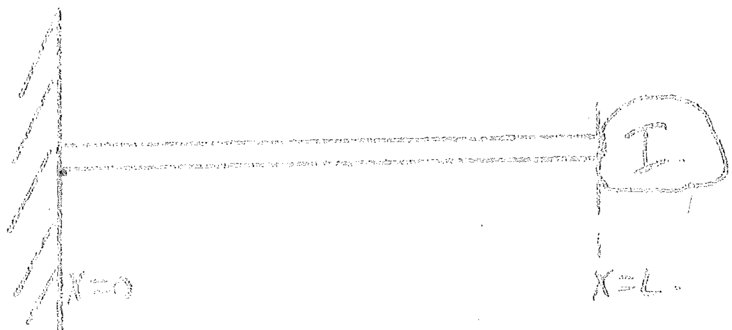
Thus, $v_{gp}(\omega)$ and $v_g(\omega)$ are identical to those sketched in the text. If $\omega < \omega_c$, we must again ~~the the~~ take the (−)-root to get

$Z_c(\omega) = -j Z_0\left(\frac{\omega_c^2}{\omega^2} - 1\right)^{1/2}, \quad \omega < \omega_c.$

This indicates a <u>capacitive</u> system for ω below cutoff. Thus, we conclude that while a TE-mode behaves <u>inductively below cutoff</u>, a TM-mode behaves <u>capacitively</u>.

MID-TERM EXAM

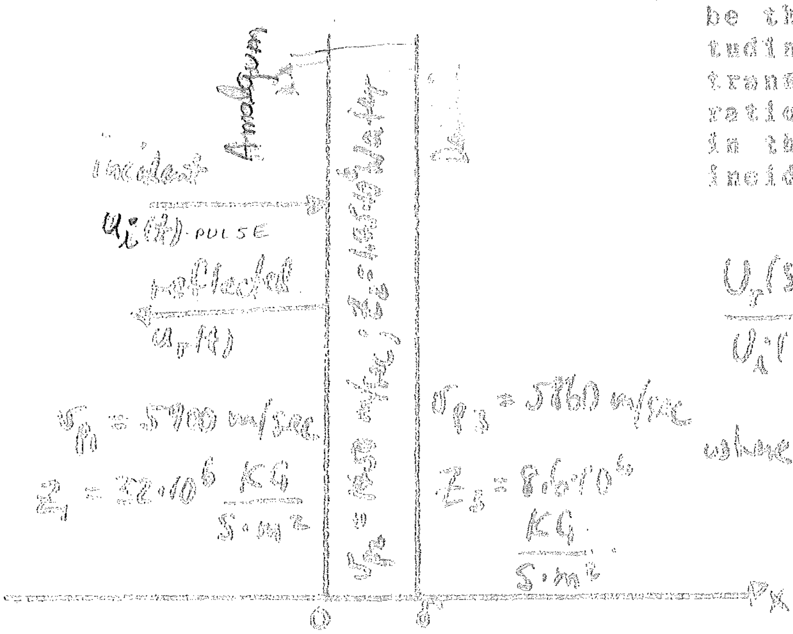
1.



V-STRESS
i-VELOCITY

The rod is terminated in a lumped element whose moment of inertia is I. It is given an initial twist such that $\varphi_0(X) = KX$ and then released. Specify an analogous electrical transmission line and compute the resulting motion of I.

2. In applying ultrasonics to the diagnosis of biological systems, one must often interpret reflected waveforms (given an incident waveform) from thin layers. Consider the layered system of dental amalgam (tooth filling), water, and tooth dentin shown. Assuming the laws of wave propagation to be those of a thin rod in longitudinal motion, show that the transfer function giving the ratio of reflected displacement in the amalgam, $U_r(s)$, to that incident in the amalgam, $U_i(s)$, is



$$\frac{U_r(s)}{U_i(s)} = \frac{\Gamma_1 + \Gamma_2 e^{-2s\delta/v_{p2}}}{1 + \Gamma_1\Gamma_2 e^{-2s\delta/v_{p2}}}$$

where $\Gamma_1 = \frac{Z_1 - Z_2}{Z_1 + Z_2}$, $\Gamma_2 = \frac{Z_2 - Z_3}{Z_2 + Z_3}$.

ULTRASONIC PROBE.
SCETCH FOR 2 CASES
$\delta = 40$ MICRON
$\delta = \frac{1}{2}$ MICRON

Let the input (incident) pulse be a square wave of unit amplitude, and time duration 80 n-sec. and let δ = 40 microns ($= 40.10^{-6}$ meters). Calculate the reflected pulse shape in time at x=0 (in the amalgam).

HINT: Use the usual infi[illegible] series [illegible] and re[illegible] the appropriate numbe[illegible] rms (y[illegible] that is appropriate)

For more details see Sidney [illegible], "Ult[illegible] of Thin Layers", IEEE Transacti[illegible] on Sonics and Ultrasonic Vol. SU-19, No. 2, April 197[illegible] p. 81-86.

DISCUSS RESULTS

DYNAMICS OF FLUID FLOW. A MODEL FOR CAPILLARY EXCHANGE.

A capillary is a part of the vascular system wherein mass is exchanged between blood and the tissue space surrounding the capillary. Imagine a cylindrical capillary of radius r within which the volume of some drug, whose mass density (or concentration) is C, moves with the blood velocity, u. In the external tissue space, the concentration of the drug is C_t. According to the law of mass transfer by Osmosis, the rate at which drug mass within the small volume δV inside the capillary increases is given by

$$(16) \qquad \delta x \cdot 2\pi r \cdot h \left(C_t - C \right),$$

where h is the capillary wall permeability coefficient (dimensions of L/T).

Equation (16) states that if $C_t > C$, i.e., the external concentration of the drug (or whatever else is being exchanged across the capillary wall) is greater than the intra-capillary concentration, the drug will osmose across the wall into the capillary.

Since the mass within δV increases by the amount $\delta x \cdot 2\pi r \cdot h \cdot (C_t - C)$, (3) must be modified to read

$$(17) \qquad \frac{d}{dt}\delta m = \left(\frac{dC}{dt} + C\frac{\partial u}{\partial x} \right)\delta V = \delta x \cdot 2\pi r \cdot h \cdot (C_t - C),$$

where ρ is replaced by C. Upon utilizing the fact that $\delta V = \pi r^2 \delta x$, (17) becomes

$$(18) \qquad \frac{dC}{dt} + C\frac{\partial u}{\partial x} = \frac{2h}{r}\left(C_t - C \right).$$

If we assume that the blood velocity, u, is independent of space and time, and recall the definition, (1), of the total time derivative, (18) becomes

$$(19)(a) \qquad \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \frac{2h}{r}\left(C_t - C \right).$$

This is one equation of a coupled system in which C_t is the other unknown. Let us now derive the second equation, assuming C_t is spatially uniform in the tissue space surrounding the capillary.

If V is the volume tissue space, then VC_t is the total mass of drug in the tissue space. Hence, the rate at which this mass increases is $V\frac{\partial C_t}{\partial t}$. Equation 16, on the other hand, is the negative of the rate at which drug enters the tissue space from the capillary at point x. Hence, upon integrating the negative of (16) over the capillary length, ℓ we arrive at our second equation

$$(19)(b) \qquad V\,\partial C_t/\partial t = \int_0^{\ell} 2\pi r h \left(C - C_t \right) dx.$$

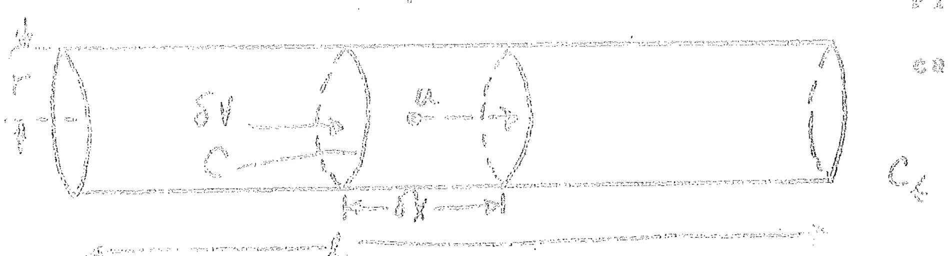


Figure A-3. Modeling capillary exchange.

3. (Determining transient response of a capillary).* Starting with (19)(a)(b), p. 5

C - CONCENTRATION
U - BLOOD VELOCITY
C_T - CONCENTRATION OF DRUG IN TISSUE SPACE
h = PERMEABILITY COEFFICIENT
V = VOLUME
r = CAPILARY RADIUS

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \frac{2h}{r}(C_t - C)$$

$$V\frac{\partial C_t}{\partial t} = \int_0^l 2\pi r h(C - C_t)\,dx = \int_0^l 2\pi r h\, C\,dx - 2\pi r h\, l\, C_t,$$

and with the boundary and initial conditions $C(o,t)=1$, $C(x,o)=0$, $C_t(0)=0$, introduce the dimensionless variables $\xi = x/l$, $\tau = ut/l$ to get

(1) (a) $$\frac{\partial C}{\partial \xi} + \frac{\partial C}{\partial \tau} = \alpha(C_t - C)$$

(b) $$K\frac{\partial C_t}{\partial \tau} = \alpha\int_0^1 C\,d\xi - \alpha C_t .$$

$C(\xi,\tau=0)=0$, $C_t(\tau=0)=0$, $C(0,\tau)=u_{-1}(\tau)$, Here $\alpha = 2hl/ru$, $K = V/\pi r^2 l$.

Note that C_t is independent of x, and, hence, ξ .

The system (1) (a)-(b) is a differential-integral equation system.

Define Laplace transform with respect to τ

(2) (a) $$\hat{C}(\xi,s) = \mathcal{L}\{C(\xi,\tau)\} = \int_0^\infty e^{-s\tau}C(\xi,\tau)\,d\tau$$

(b) $$\hat{C}_t(s) = \mathcal{L}\{C_t(\tau)\} = \int_0^\infty e^{-s\tau}C_t(\tau)\,d\tau.$$

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* J. A. Johnson and T. A. Wilson, "A Model for Capillary Exchange", Am. J. of Physiology, Vol. 210, No. 6., June 1966, pp. 1299-1303.
R. Bellman, J. Jacquez and R. Kalaba, "Some Mathematical Aspects of Chemotherapy. I: One-organ models. Bull. Math. Biophysics 22:181-198, 1960.

[illegible] conditions" $C(\xi,\tau=0)=0$, $C_f(\tau=0)=0$ to get

$$(3)\quad (a)\ \frac{d\hat{C}(\xi,s)}{d\xi}+s\hat{C}(\xi,s)=\alpha\left(\hat{C}_f(s)-\hat{C}(\xi,s)\right)$$

$$(b)\ sK\hat{C}_f(s)=\alpha\int_0^1\hat{C}(\xi,s)d\xi-\alpha\hat{C}_f(s).$$

Solving (3) (b) for $\hat{C}_f(s)$ yields

$$\hat{C}_f(s)=\left(\frac{\alpha}{sK+\alpha}\right)\int_0^1\hat{C}(\xi,s)d\xi,$$

which, when substituted into (3) (a) yields, after rearrangement

$$(4)\quad \frac{d\hat{C}}{d\xi}(\xi,s)+(s+\alpha)\hat{C}(\xi,s)=\left(\frac{\alpha^2}{sK+\alpha}\right)\int_0^1\hat{C}(\xi,s)d\xi,\quad \hat{C}(0,s)=\frac{1}{s}.$$

To solve the functional (differential-integral) equation

$$(5)\quad \frac{d\omega(y)}{dy}+a\,\omega(y)=b\int_0^1\omega(y)dy\ ,\quad \omega(0)=\frac{1}{s},$$

in which we note that the right-hand side is <u>independent</u> of y (because the y-variable is "integrated out"), set $\int_0^1\omega(y)dy=m$.

Then

$$\frac{d\omega}{dy}+a\omega=bm$$

is solved by $\omega(y)=fe^{-ay}+bm/a$, where f is a constant of integration. The first term is the complementary function and the second term is the particular integral.

Substitute this solution into the definition of m:

$$\int_0^1\omega(y)dy=m\Rightarrow\int_0^1\left[fe^{-ay}+\frac{bm}{a}\right]dy=-\frac{f}{a}e^{-a}+\frac{f}{a}+\frac{bm}{a}=m,$$

or, upon solving for m:

$$(6)\quad m=\frac{f(1-e^{-a})}{(a-b)}.$$

Thus,
$$\omega(y) = f e^{-ay} + \frac{b}{a} \cdot \frac{f(1-e^{-a})}{(a-b)} .$$

The boundary condition, $\omega(0) = 1/s$, yields

$$\frac{1}{s} = f + \frac{b}{a} \cdot \frac{f(1-e^{-a})}{(a-b)} ,$$

so that
$$f = \frac{1}{s} \cdot \frac{a(a-b)}{[a(a-b) + b(1-e^{-a})]} .$$

Hence, $\omega(y)$ is completely determined

(7) $$\omega(y) = \frac{1}{s} \cdot \frac{a(a-b)e^{-ay}}{[a(a-b)+b(1-e^{-a})]} + \frac{b}{s} \cdot \frac{(1-e^{-a})}{[a(a-b)+b(1-e^{-a})]} .$$

When we set $a = (s+\alpha)$, $b = \alpha^2/sK+\alpha$ and rearrange terms, the solution for the Laplace transform $\hat{C}(\xi,s)$ becomes

(8) $$\hat{C}(\xi,s) = \frac{1}{s} e^{-(s+\alpha)\xi} + \frac{(1-e^{-(s+\alpha)})(1-e^{-(s+\alpha)\xi})}{s[P(s) - e^{-(s+\alpha)}]}$$

where

(9) $$P(s) = \frac{Ks^3}{\alpha^2} + \frac{(1+2K)}{\alpha} s^2 + (1+K)s + 1 .$$

Upon taking the inverse transform of $\hat{C}(\xi,s)$, we obtain the time response of the drug concentration within the capillary:

(10) $$C(\xi,\tau) = \mathcal{L}^{-1}\{\hat{C}(\xi,s)\} .$$

Start by expanding (8)

(11) $$\hat{C}(\xi,s) = \frac{1}{s} e^{-(s+\alpha)\xi} + \frac{(1-e^{-(s+\alpha)})(1-e^{-(s+\alpha)\xi})}{sP(s)\left[1 - \frac{e^{-(s+\alpha)}}{P(s)}\right]}$$

$$= \frac{1}{s} e^{-(s+\alpha)\xi} + \frac{(1-e^{-(s+\alpha)})(1-e^{-(s+\alpha)\xi})}{sP(s)} \left[1 + \frac{e^{-(s+\alpha)}}{P(s)} + \frac{e^{-2(s+\alpha)}}{P^2(s)} + \cdots\right].$$

If we retain only the first term in the infinite series expansion, and simultaneously set $e^{-(s+\alpha)} \ll 1$, we get the short-time response immediately following the step-injection of the drug at t=0 (explain why this statement is true). Doing this we have

(12) $$\hat{C}(\xi,s) = \frac{1}{s} e^{-(s+\alpha)\xi} + \frac{1}{sP(s)} - \frac{e^{-(s+\alpha)\xi}}{sP(s)} .$$

Then,

(13) $$C(\xi,\tau) = f(\tau) + u_{-1}(\tau-\xi)\cdot e^{-\alpha\xi}\left[1 - f(\tau-\xi)\right],$$

where

(14) $$f(\tau) = \mathcal{L}^{-1}\left[\frac{1}{s\,P(s)}\right].$$

<u>Determine</u> $f(\tau)$ and sketch $C(\xi,\tau)$ as a function of ξ, for $0 \le \xi \le 1$ and $\tau = 0.2$, 0.4, 0.6, 0.8 and 1.0. Take $K=5$, $\alpha=1$. Discuss the physical meaning of the results.

4. A brick sliding across a table with a velocity of 10 m/sec comes to rest at a uniform rate (starting at $t=0$) in a distance of 1 meter.

(a) Find the rise in temperature of the brick-table interface at the time the brick stops.

(b) Find the <u>maximum</u> temperature of the brick-table interface, and calculate the time at which it occurs.

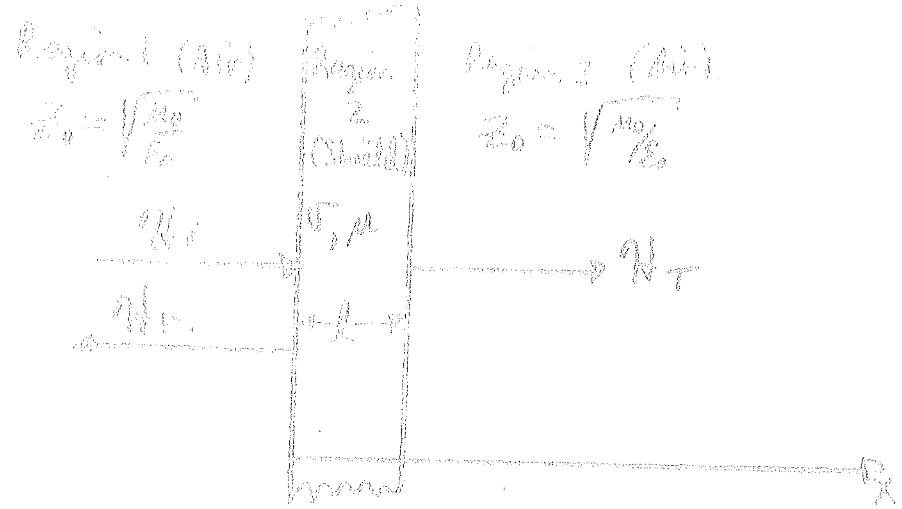
<u>DATA</u>:
Length of brick = 8 inches
Width of brick = 4 inches
Weight of brick = 5 KG.
Thermal conductivity of brick = 0.5 watt per cm per 1°C
Heat capacity of brick = 4.8 watt-sec per cm^3 per 1°C.

Regard the brick as "infinitely thick", and make the problem one-dimensional. You will need the fact that

$$L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}\,u_{-1}(t).$$

T(TEMP) ~ V(VOLTAGE)
q(H FLOW / AREA) ~ i(CURRENT)
$1/K$(CONDUC) ~ R
ρ_v(SP HT/MASS × DENSITY) ~ C

3.



Maxwell's equations for the electric field, $\mathcal{E}$, and magnetic field $\mathcal{H}$ are given by

$$\left.\begin{aligned} \frac{\partial \mathcal{E}}{\partial x} &= -\mu_0 \frac{\partial \mathcal{H}}{\partial t}, \\ \frac{\partial \mathcal{H}}{\partial x} &= -\varepsilon_0 \frac{\partial \mathcal{E}}{\partial t}, \end{aligned}\right\} \qquad \mu_0 = 4\pi\cdot 10^{-7}\ (\text{h/m}),\ \varepsilon_0 = \frac{1}{36\pi}\cdot 10^{-9}\ (\text{fd/m}).$$

in Regions 1 and 3

$$\frac{\partial \mathcal{E}}{\partial x} = -\mu \frac{\partial \mathcal{H}}{\partial t}, \quad \frac{\partial \mathcal{H}}{\partial x} = -\sigma \mathcal{E}$$

in Region 2. μ = magnetic permeability and σ = electrical conductivity of metal shield.

(a) Show that the <u>magnetic transfer function</u> $\frac{H_T(s)}{H_i(s)} = G(s)$

is given by $G(s) = \dfrac{1}{\cosh \gamma(s)\ell + \frac{1}{2}\left(\frac{Z_0}{Z_c(s)} + \frac{Z_c(s)}{Z_0}\right)\sinh \gamma(s)\ell}$, 1E2 * 1E5 X 1E-7

where $\gamma(s) = \sqrt{\mu\sigma s}$, $Z_c(s) = \sqrt{\frac{\mu s}{\sigma}}$.

(b) The poles of G(s) lie on the <u>negative real axis of the s-plane</u>. Determine the ten largest poles if ℓ = 1 cm, $\mu = 100\cdot 4\pi\cdot 10^{-7}$ h/m, $\frac{10^{-7}}{10^{-2}\text{cm}}$
$\sigma = 10^{5}\ (\Omega\cdot\text{cm})^{-1} = 10^{5}$ mhos/cm.

Hint: Show that the poles, $S_n = -\alpha_n^2$ satisfy

$$\cot \alpha_n \cdot \sqrt{\mu\sigma} = -\frac{1}{2}\left[Z_0/\alpha_n\sqrt{\mu/\sigma} - \frac{\alpha_n\sqrt{\frac{\mu}{\sigma}}}{Z_0} \right].$$

(c) An approximation to G(s) is given by

$$G(s) = \frac{G(0)}{(1 - s/s_1)(1 - s/s_2)\cdots(1 - s/s_{10})},$$

where $G(0) = G(s)|_{s=0} = \frac{1}{1 + \frac{1}{2} Z_0 \ell\sigma}$ and $\{s_n\}$ are given in (b).

Letting $s = j\omega$, plot the magnitude of the frequency response $|G(j\omega)|$ vs. ω. Discuss the results, interpreting $|G(j\omega)|$. How can the effectiveness of the shield be improved? Is the sh[illegible] a low-, band-, or high-pass filter? Can it be modeled with a l[illegible] parameter network?

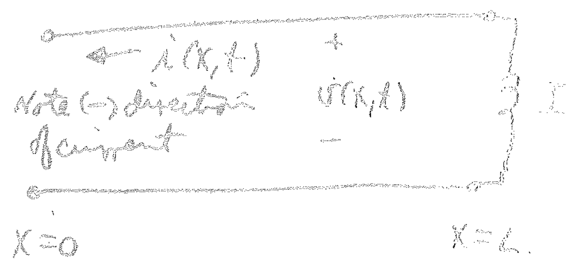
Solutions

1. **Torsional System**   **Analog**   **Electrical System**

$$\omega \sim -i \qquad \tau \sim v$$

$$\frac{\partial \tau}{\partial x} = J \frac{\partial \omega}{\partial t} \qquad\qquad \frac{\partial v}{\partial x} = -L \frac{\partial i}{\partial t}$$

$$\frac{\partial \omega}{\partial x} = \frac{1}{C} \frac{\partial \tau}{\partial t} \qquad\qquad \frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t}$$



bndry. conditions:

$$\omega(0,t) = 0$$

$$I \frac{d\omega(L,t)}{dt} = \tau(L,t)$$

Initial cond.

$$\omega(x,0) = 0$$

$$\varphi(x,0) = \varphi_0(x) = kx$$

$$\tau(x,0) = \tau_0(x) = C \frac{\partial \varphi_0}{\partial x} = KC$$

bndry. conditions

$$i(0,t) = 0$$

$$v(L,t) = -I \frac{d i(L,t)}{dt}$$

Initial cond:

$$i(x,0) = 0$$

$$v(x,0) = v_0(x) = KC \quad \text{(a constant)}$$

Problem is to determine $\omega(L,t)$ in the mechanical system or $i(L,t)$ in the electrical system.

Start with

$$\frac{d^2 V}{dx^2} - \left(\frac{s}{v_p}\right)^2 V = -\frac{s}{v_p^2} v_0(x) = -\frac{s}{v_p^2} KC$$

$$\therefore V(x,s) = \underbrace{A e^{sx/u_p} + B e^{-sx/u_p}}_{\text{complementary function}} + \frac{KC}{s} \quad \leftarrow \text{particular integral}$$

$$I(x,s) = -\frac{1}{sL}\frac{dV}{dx} = \frac{1}{Z_0}\left[A e^{sx/u_p} - B e^{-sx/u_p}\right]$$

B.C.: $I(0,s) = 0 \Rightarrow A = B.$

$$V(l,s) = -sI \cdot I(l,s) \Rightarrow A\left[e^{sl/u_p} + e^{-sl/u_p}\right] + \frac{KC}{s}$$

(mom. of inertia)

$$= -\frac{sI A}{Z_0}\left[e^{sl/u_p} - e^{-sl/u_p}\right].$$

$$\therefore A = B = -\frac{KC}{s}\cdot\frac{Z_0}{(Z_0+sI)e^{sl/u_p} + (Z_0-sI)e^{-sl/u_p}}$$

$$\Rightarrow I(l,s) = -\frac{KC}{s}\cdot\frac{\left(e^{sl/u_p} - e^{-sl/u_p}\right)}{e^{sl/u_p}(Z_0+sI) + e^{-sl/u_p}(Z_0-sI)}$$

$$= -\frac{KC}{s(Z_0+sI)}\cdot\frac{1-e^{-2sl/u_p}}{1-\rho_L e^{-2sl/u_p}}, \quad \rho_L = \frac{sI-Z_0}{sI+Z_0}$$

(note $sI - Z_0$)

$$= -\frac{KC}{s(Z_0+sI)}\left[1 - \frac{2Z_0}{sI+Z_0}e^{-2sl/u_p} - \frac{sI-Z_0}{sI+Z_0}\cdot\frac{2Z_0}{sI+Z_0}e^{-4sl/u_p} + \cdots\right]$$

Let us expand the first two terms into their partial fractions. The remaining ones behave similarly.

$$\frac{KC}{s(Z_0+sI)} = \frac{-KC/Z_0}{s} + \frac{KC/Z_0}{s+Z_0/I}.$$

$$\frac{KC\cdot 2Z_0}{s(Z_0+sI)^2} = \frac{2KC/Z_0}{s} - \frac{2KC/Z_0}{(s+Z_0/I)} - \frac{2KC/I}{(s+Z_0/I)^2}.$$

Thus $\mathcal{L}^{-1}\left\{\frac{KC}{s(z_0+sI)}\right\} = -\frac{KC}{z_0}u_{-1}(t) + \frac{KC}{z_0}e^{-\frac{z_0}{I}t}$

$$\mathcal{L}^{-1}\left\{\frac{2KCz_0}{s(z_0+sI)^2}\right\} = \frac{2KC}{z_0}u_{-1}(t) - \frac{2KC}{z_0}e^{-\frac{z_0}{I}t} - \frac{2KC}{I}t\,e^{-\frac{z_0}{I}t}$$

and the first part of $i(l,t)$ (which is equal to velocity because of the negative sign in our electrical analog) is

$$i(l,t) = \text{velocity of lumped mass of inertia} = -\frac{KC}{z_0}u_{-1}(t) + \frac{KC}{z_0}e^{[illegible]}$$

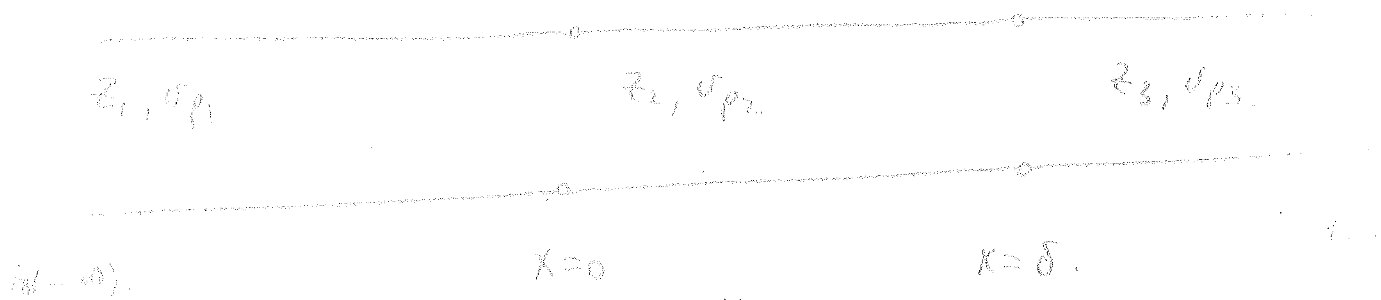
$$+\frac{2KC}{z_0}u_{-1}\left(t-\frac{2l}{v_p}\right) - \frac{2KC}{z_0}e^{-\frac{z_0}{I}\left(t-\frac{2l}{v_p}\right)}\cdot u_{-1}\left(t-\frac{2l}{v_p}\right)$$

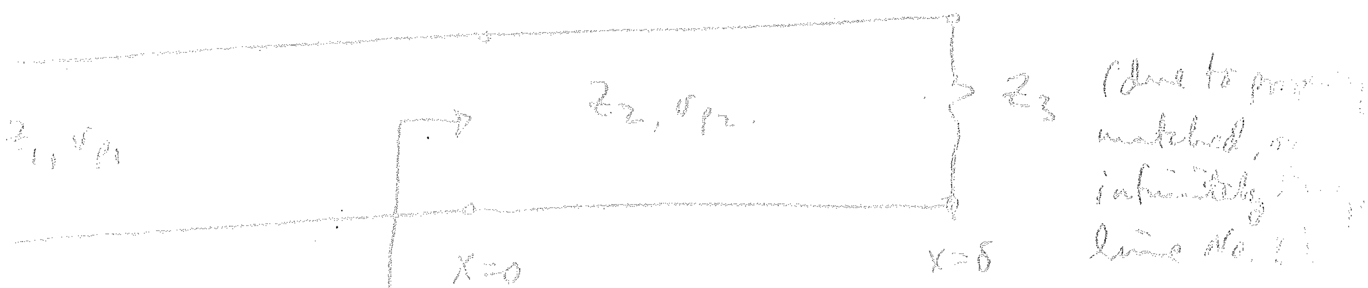
$$-\frac{2KC}{I}\left(t-\frac{2l}{v_p}\right)e^{-\frac{z_0}{I}\left(t-\frac{2l}{v_p}\right)}\cdot u_{-1}\left(t-\frac{2l}{v_p}\right) + \cdots\cdots$$

$$= -\frac{KC}{z_0}\left(1-e^{-z_0t/(I)}\right)u_{-1}(t) + \frac{2KC}{z_0}\left[1-e^{-\frac{z_0}{I}\left(t-\frac{2l}{v_p}\right)} - \frac{z_0}{I}\left(t-\frac{2l}{v_p}\right)e^{-\frac{z_0}{I}\left(t-\frac{2l}{v_p}\right)}\right]\cdot u_{-1}\left(t-\frac{2l}{v_p}\right)$$

$+ \cdots\cdots$

Electrical equivalent ckt.



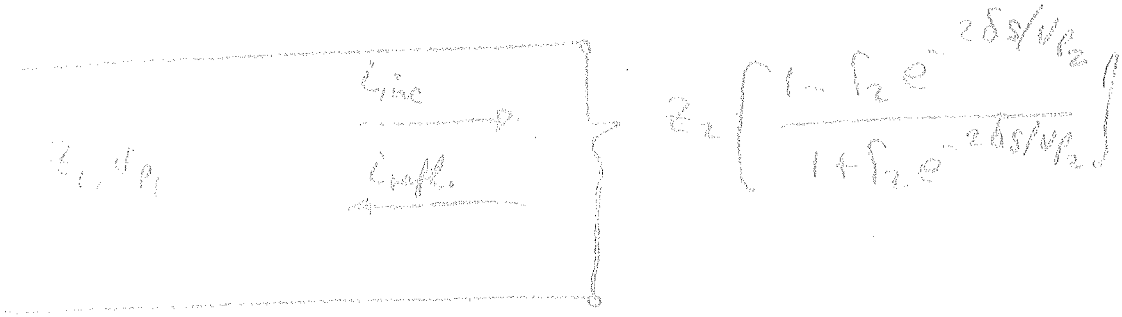


Looking into $x=0$ we see an impedance given by (1-38) for instance

$$Z_{in}(s) = Z_2\left(\frac{1-\Gamma_2 e^{-2\delta s/v_{p2}}}{1+\Gamma_2 e^{-2\delta s/v_{p2}}}\right)$$

, where we have replaced Γ_2 by $-\Gamma_2$ because of the way Γ_2 is defined.

Thus, the above circuit is equivalent to:



The current reflection coefficient is the negative of the voltage reflection coefficient. Thus

$$\frac{I_{ref}}{I_{inc}} = \frac{Z_1 - Z_{in}}{Z_1 + Z_{in}} = \frac{Z_1 - \dfrac{Z_2 - Z_2\Gamma_2 e^{-2\delta s/v_{P2}}}{1+\Gamma_2 e^{-2\delta s/v_{P2}}}}{Z_1 + \dfrac{Z_2 - Z_2\Gamma_2 e^{-2\delta s/v_{P2}}}{1+\Gamma_2 e^{-2\delta s/v_{P2}}}}$$

$$= \frac{Z_1 - Z_2 + (Z_1\Gamma_2 + Z_2\Gamma_2)e^{-2\delta s/v_{P2}}}{Z_1 + Z_2 + (Z_1\Gamma_2 - Z_2\Gamma_2)e^{-2\delta s/v_{P2}}}$$

$$= \frac{\left(\dfrac{Z_1 - Z_2}{Z_1 + Z_2}\right) + \Gamma_2 e^{-2\delta s/v_{P2}}}{1 + \left(\dfrac{Z_1 - Z_2}{Z_1 + Z_2}\right)\Gamma_2 e^{-2\delta s/v_{P2}}}$$

$$= \frac{\Gamma_1 + \Gamma_2 e^{-2\delta s/v_{P2}}}{1 + \Gamma_1\Gamma_2 e^{-2\delta s/v_{P2}}}$$

But by our analogy $I \sim -\text{Velocity} = -sU$.

$$\therefore \quad \boxed{\frac{U_{ref}}{U_{inc}}(s) = \frac{-I_{ref}/s}{-I_{inc}/s} = \frac{I_{ref}}{I_{inc}} = \frac{\Gamma_1 + \Gamma_2 e^{-2\delta s/v_{P2}}}{1 + \Gamma_1\Gamma_2 e^{-2\delta s/v_{P2}}}}$$

$$\frac{1}{U_{inc}}(s) = \left(\Gamma_1 + \Gamma_2 e^{-2\delta s/u_{p2}}\right)\left(1 - \Gamma_1\Gamma_2 e^{-2\delta s/u_{p2}}\right.$$

$$+ (\Gamma_1\Gamma_2)^2 e^{-4\delta s/u_{p2}} - (\Gamma_1\Gamma_2)^3 e^{-6\delta s/u_{p2}}$$

$$\left. + (\Gamma_1\Gamma_2)^4 e^{-8\delta s/u_{p2}} - (\Gamma_1\Gamma_2)^5 e^{-10\delta s/u_{p2}} + \cdots \right)$$

$$= \Gamma_1 \left[1 + \Gamma_4 e^{-2\delta s/u_{p2}} - \Gamma_3\Gamma_4 e^{-4\delta s/u_{p2}} + \Gamma_3^2\Gamma_4 e^{-6\delta s/u_{p2}}\right.$$

$$\left. - \Gamma_3^3\Gamma_4 e^{-8\delta s/u_{p2}} + \Gamma_3^4\Gamma_4 e^{-10\delta s/u_{p2}} + \cdots \right].$$

where $\Gamma_3 = \Gamma_1\Gamma_2$, $\Gamma_4 = \frac{\Gamma_2}{\Gamma_1} - \Gamma_1\Gamma_2$. Note that the magnitude of the ratio of the higher order terms is Γ_3.

$$\Gamma_1 = \frac{32\cdot10^6 - 1.45\cdot10^6}{32\cdot10^6 + 1.45\cdot10^6} = 0.91$$

$$\Gamma_2 = \frac{1.45\cdot10^6 - 8.6\cdot10^6}{1.45\cdot10^6 + 8.6\cdot10^6} = -0.71$$

$$\Gamma_3 = -(0.91)(0.71) = -0.65.$$

$$\Gamma_4 = -\frac{0.71}{0.91} + 0.65 = -0.13$$

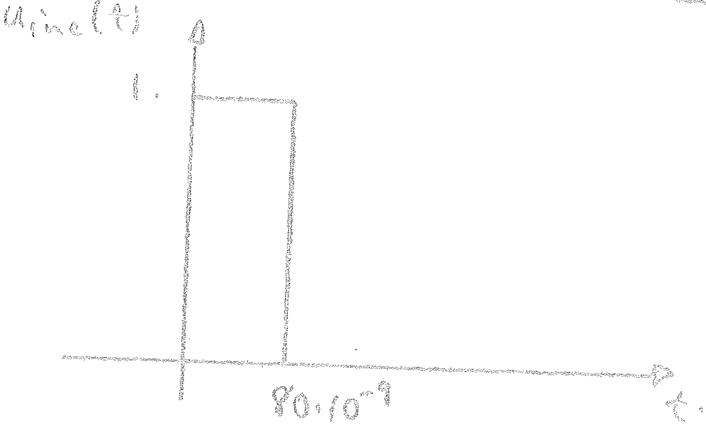
$\delta/u_{p2} = 2.76\cdot10^{-8}$ sec for $\delta = 90\ \mu$.

$= 3.85\cdot10^{-10}$ sec for $\delta = 0.5\ \mu$.

The principle contributions will be given by the first four terms (that's all I'll keep because I don't know how to case

[illegible]

$$u_{ref}(t) = 0.91\, u_{inc}(t) - 0.12\, u_{inc}(t - 55.2 \cdot 10^{-9})$$
$$- 0.076\, u_{inc}(t - 110.4 \cdot 10^{-9}) - 0.05\, u_{inc}(t - 165.6 \cdot 10^{-9})$$

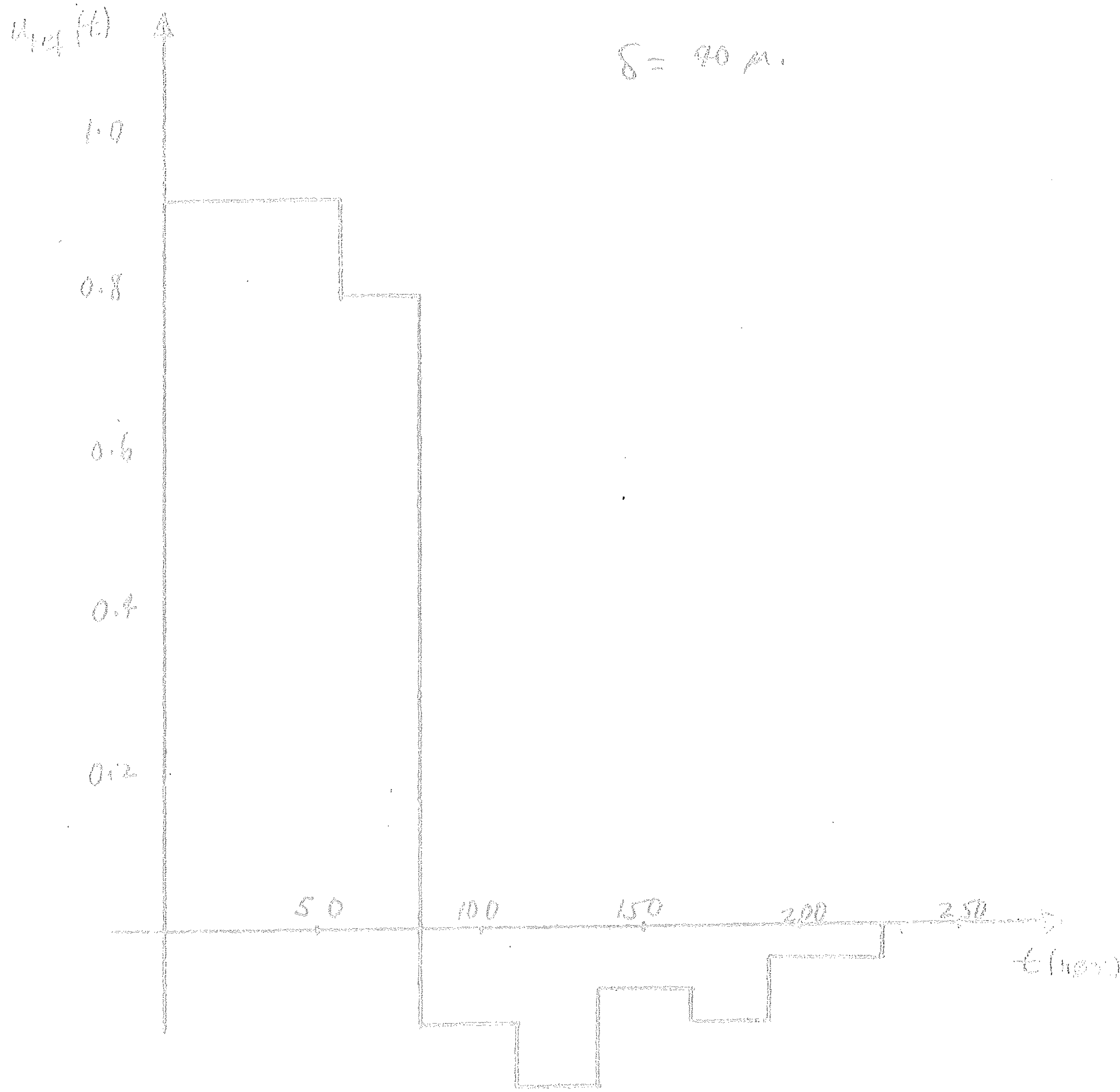


For $\delta = 0.5\,\mu$.

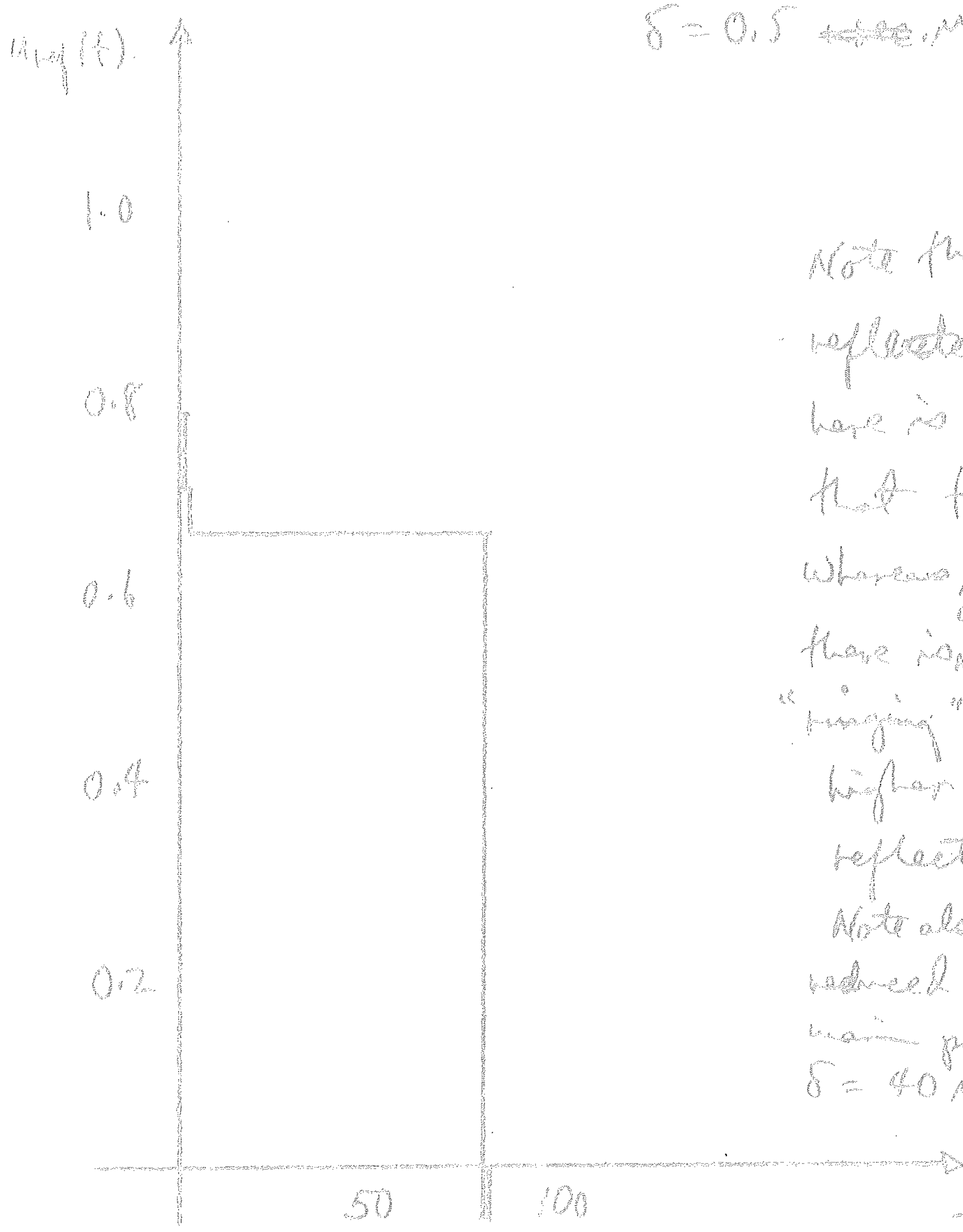
$$u_{ref}(t) = 0.91\, u_{inc}(t) - 0.12\, u_{inc}(t - 0.69 \cdot 10^{-9})$$
$$- 0.076\, u_{inc}(t - 1.38 \cdot 10^{-9}) - 0.05\, u_{inc}(t - 2.07 \cdot 10^{-9}).$$

Note that the only thing that differs in the two solutions is the time of arrival of the multiply reflected waves.

The results are sketched on the next two pages.


δ = 90 m.
1.0
0.8
0.6
0.4
0.2
50
100
150
200
250

δ = 0.5 m.



Note that the reflected pulse shape here is similar to that transmitted, whereas for δ = 40 m there is a considerable "ringing" tail to the higher order multiple reflections.

Note also the considerably reduced amplitude of the main pulse compared with δ = 40 m.

3. First determine $\mathcal{L}^{-1}\left\{\frac{1}{sP(s)}\right\}$, $P(s) = 5s^3 + 11s^2 +$ [illegible]

Roots of $P(s)$ are -1.48, $-0.36 \pm j0.08$, ≈ -0.36 (double root).

$$\therefore \frac{1}{s \cdot P(s)} = \frac{1}{s(s+1.48)(s+0.36)^2} = \frac{5.2}{s} - \frac{0.54}{s+1.48} - \frac{4.66}{s+0.36} - \frac{[illegible]}{(s+0.36)^2}$$

$$\therefore f(\tau) = \mathcal{L}^{-1}\left\{\frac{1}{sP(s)}\right\} = \left[5.2 - 0.54e^{-1.48t} - 4.66e^{-0.36t} - 2.49[illegible]\right] \cdot u_{-1}(t).$$

Now $C(\xi,\tau) = f(\tau) + u_{-1}(\tau-\xi)e^{-\alpha\xi}\left[1 - f(\tau-\xi)\right]$.

$$= e^{-\xi} \cdot u_{-1}(\tau-\xi) + \left[f(\tau) - e^{-\xi}f(\tau-\xi)u_{-1}(\tau-\xi)\right].$$

Calculate $f(\tau)$ for $\tau = 0, 0.2, 0.4, 0.6, 0.8, 1.0$:

$f(0) = 0$

$f(0.2) \approx 0$

$f(0.4) \approx 0$

$f(0.6) \approx 0.04$

$f(0.8) \approx 0.07$

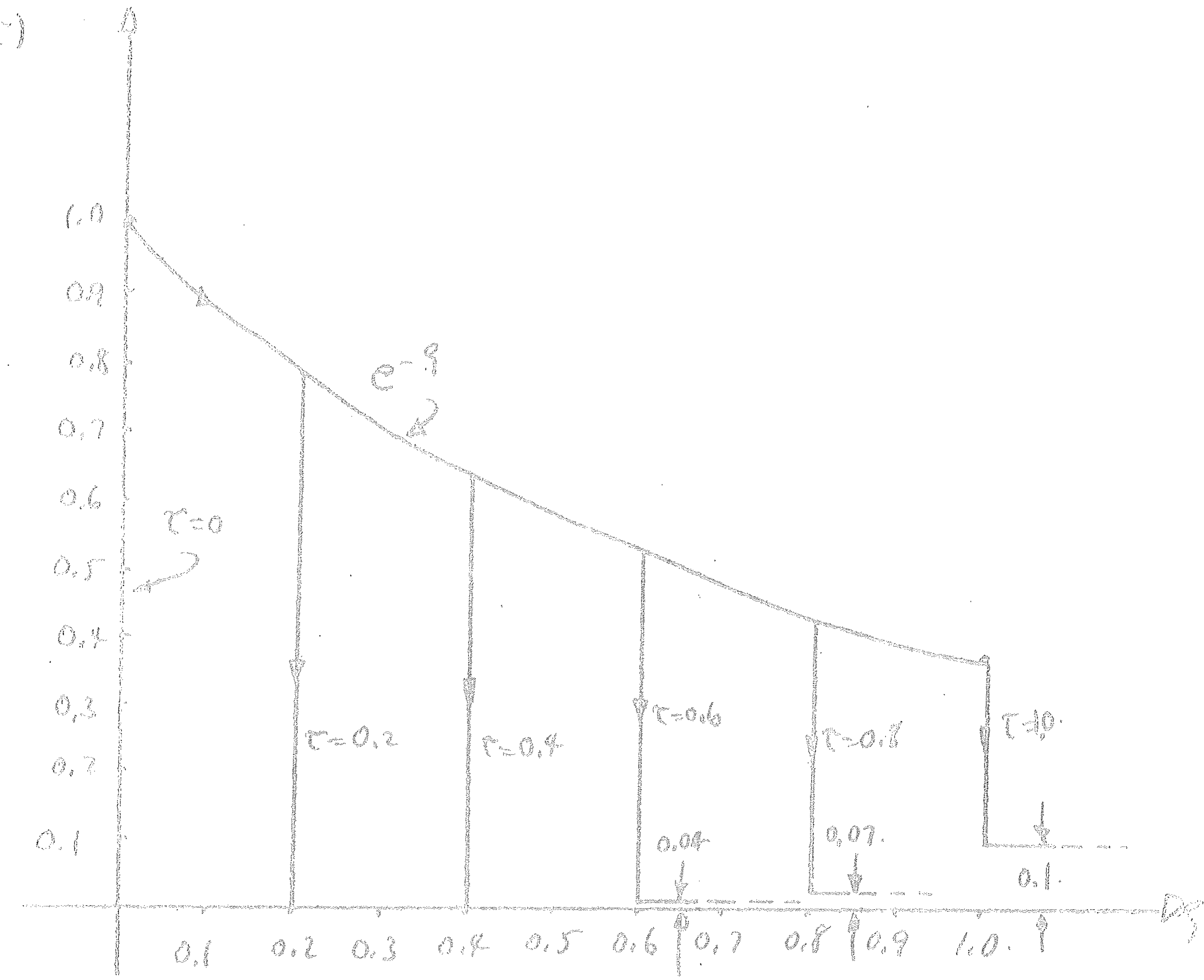
$f(1.0) \approx 0.10$

Because these values of $f(\tau)$ are so small, it follows that we can ignore the term $e^{-\xi}f(\tau-\xi)u_{-1}(\tau-\xi)$ because it will be smaller still due to the exponential decay $e^{-\xi}$ factor. Hence,

$$C(\xi,\tau) \approx e^{-\xi}u_{-1}(\tau-\xi) + f(\tau).$$

Of course, for $\xi > \tau$, the term $e^{-\xi}f(\tau-\xi)u_{-1}(\tau-\xi)$ drops out anyway because of the factor $u_{-1}(\tau-\xi)$.

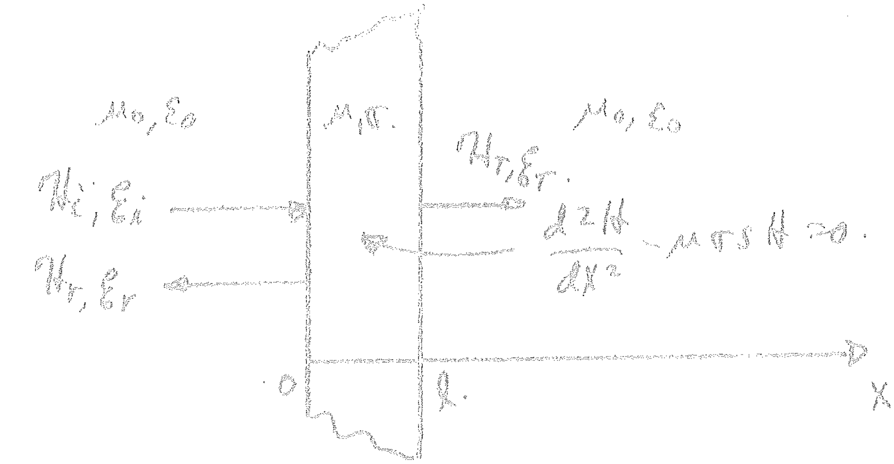
This result is sketched on the next page.



Note that the "switch-off" function $u_{-1}(\tau - \xi)$ accounts for the transport of drug by blood flow (convection). The exponential decay with distance ($e^{-\xi}$), of course, is due to the osmosis (or diffusion) of drug <u>across</u> the capillary wall into the intercellular tissue space.

Note also the "leakage" flow of drug that arrives at a point, ξ, prior to the convected amount. This leakage is quite small.

(a.)



within the shield

$$H = A_1 e^{-\gamma x} + A_2 e^{\gamma x}, \quad \gamma = (\mu\sigma s)^{1/2}$$

$$E = -\frac{1}{\sigma} \cdot \frac{dH}{dx} = Z_c \left[A_1 e^{-\gamma x} - A_2 e^{\gamma x} \right], \quad Z_c = \sqrt{\frac{\mu s}{\sigma}}$$

At the boundaries $\mathcal{H}$ and $\mathcal{E}$ must be continuous (just as for a transmission-line):

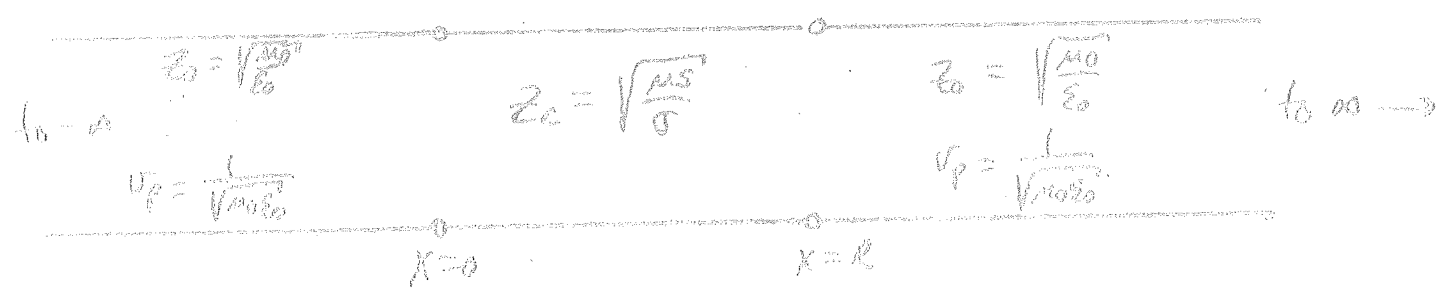
$X = 0$: $\quad H_i + H_r = H = A_1 + A_2.$

$$E_i + E_r = (H_i - H_r) \cdot Z_0 = Z_c (A_1 - A_2)$$

$X = l$: $\quad H_T = A_1 e^{-\gamma l} + A_2 e^{\gamma l}$

$$E_T = H_T Z_0 = Z_c \left[A_1 e^{-\gamma l} - A_2 e^{\gamma l} \right].$$

(Note that the equivalent transmission-lines are current ~ $\mathcal{H}$, voltage ~ $\mathcal{E}$.



[illegible]

$$V_T\left(1+\frac{Z_0}{Z_c}\right) = 2A_1 e^{-\gamma l} \Longrightarrow A_1 = \frac{V_T}{2}\left(1+\frac{Z_0}{Z_c}\right)e^{\gamma l}$$

$$V_T\left(1-\frac{Z_0}{Z_c}\right) = 2A_2 e^{\gamma l} \Longrightarrow A_2 = \frac{V_T}{2}\left(1-\frac{Z_0}{Z_c}\right)e^{-\gamma l}.$$

Eliminate V_T from the top two equations:

$$2V_i = A_1\left(1+\frac{Z_c}{Z_0}\right) + A_2\left(1-\frac{Z_c}{Z_0}\right)$$

$$= \left(1+\frac{Z_c}{Z_0}\right)\left(1+\frac{Z_0}{Z_c}\right)\frac{V_T}{2}e^{\gamma l} + \left(1-\frac{Z_c}{Z_0}\right)\left(1-\frac{Z_0}{Z_c}\right)\frac{V_T}{2}e^{-\gamma l}$$

$$= \frac{V_T}{2}\left(e^{\gamma l}+e^{-\gamma l}\right) + \frac{V_T}{2}\left(\frac{Z_c}{Z_0}+\frac{Z_0}{Z_c}\right)\left(e^{\gamma l}-e^{-\gamma l}\right)$$

$$+ \frac{V_T}{2}\left(e^{\gamma l}+e^{-\gamma l}\right).$$

$$\therefore \quad 2V_i = 2V_T \cosh\gamma l + V_T\left(\frac{Z_c}{Z_0}+\frac{Z_0}{Z_c}\right)\sinh\gamma l.$$

$$\therefore \quad (a) \quad \boxed{\frac{V_T}{V_i}(s) = G(s) = \frac{1}{\cosh\gamma(s)l + \frac{1}{2}\left(\frac{Z_c(s)}{Z_0}+\frac{Z_0}{Z_c(s)}\right)\sinh\gamma(s)l}}$$

The pole locations are given by setting the denominator of $G(s) = 0$:

$$\coth\gamma(s)l = -\frac{1}{2}\left(\frac{Z_c(s)}{Z_0}+\frac{Z_0}{Z_c(s)}\right).$$

Let us now prove that the roots of this equation all lie on the negative real axis.

Start by rewriting coth γl in exponential form:

$$\coth \gamma(s)l = \frac{e^{\gamma(s)l} + e^{-\gamma(s)l}}{e^{\gamma(s)l} - e^{-\gamma(s)l}} = -\frac{1}{2}\left[\frac{Z_c(s)}{Z_0} + \frac{Z_0}{Z_c(s)}\right].$$

This is converted into the form

$$e^{2\gamma(s)l} = \frac{Z_c^2 + Z_0^2 - 2Z_0Z_c}{Z_c^2 + Z_0^2 + 2Z_0Z_c}.$$

Next write for the roots (poles) $\sqrt{s_n} = \xi_n + j\omega_n$, so that

$$\gamma = \sqrt{\mu\sigma s} = \sqrt{\mu\sigma}\,(\xi + j\omega), \quad Z_c = \sqrt{\frac{\mu s}{\sigma}} = \sqrt{\frac{\mu}{\sigma}}\,(\xi + j\omega)$$

Then

$$e^{2\gamma l} = e^{2l\sqrt{\mu\sigma}\,\xi} \cdot e^{j2l\omega\sqrt{\mu\sigma}}$$

$$= \frac{\left(\frac{\mu}{\sigma}\right)(\xi^2 - \omega^2 + j2\omega\xi) + Z_0^2 - 2Z_0\sqrt{\frac{\mu}{\sigma}}\,(\xi + j\omega)}{\left(\frac{\mu}{\sigma}\right)(\xi^2 - \omega^2 + j2\omega\xi) + Z_0^2 + 2Z_0\sqrt{\frac{\mu}{\sigma}}\,(\xi + j\omega)}$$

$$= \frac{\left[Z_0^2 + \left(\frac{\mu}{\sigma}\right)(\xi^2 - \omega^2) - 2Z_0\sqrt{\frac{\mu}{\sigma}}\,\xi\right] + j\left[2\omega\xi\left(\frac{\mu}{\sigma}\right) - 2Z_0\sqrt{\frac{\mu}{\sigma}}\,\omega\right]}{\left[Z_0^2 + \left(\frac{\mu}{\sigma}\right)(\xi^2 - \omega^2) + 2Z_0\sqrt{\frac{\mu}{\sigma}}\,\xi\right] + j\left[2\omega\xi\left(\frac{\mu}{\sigma}\right) + 2Z_0\sqrt{\frac{\mu}{\sigma}}\,\omega\right]}$$

Equating magnitudes squared of each side:

$$e^{4\sqrt{\mu\sigma}\,l\xi} = \frac{\left[Z_0^2 + \left(\frac{\mu}{\sigma}\right)(\xi^2 - \omega^2) - 2Z_0\sqrt{\frac{\mu}{\sigma}}\,\xi\right]^2 + \left[2\omega\xi\left(\frac{\mu}{\sigma}\right) - 2Z_0\sqrt{\frac{\mu}{\sigma}}\,\omega\right]^2}{\left[Z_0^2 + \left(\frac{\mu}{\sigma}\right)(\xi^2 - \omega^2) + 2Z_0\sqrt{\frac{\mu}{\sigma}}\,\xi\right]^2 + \left[2\omega\xi\left(\frac{\mu}{\sigma}\right) + 2Z_0\sqrt{\frac{\mu}{\sigma}}\,\omega\right]^2}$$

When $\xi > 0$, the left-hand side is > 1, and the right-hand side is < 1.

When $\xi < 0$, the left-hand side is < 1, and the right-hand side is > 1.

Each statement follows from a comparison of each bracketed term in the numerator with the one directly below it in the denominator.

Thus, $\xi = 0$ is required if the equation is to be satisfied. But $\sqrt{s_n} = j\omega_n$, then, or $s_n = -\omega_n^2$, i.e., a negative real no.

To determine ω_n, we have $\coth j\omega_n l\sqrt{\mu\sigma} = -j\cot \omega_n l\sqrt{\mu\sigma}$

$$= -\frac{1}{2}\left(\frac{j\sqrt{\frac{\mu}{\sigma}}\,\omega_n}{z_0} + \frac{z_0}{j\sqrt{\frac{\mu}{\sigma}}\,\omega_n}\right) = \frac{j}{2}\left(\frac{z_0}{\omega_n\sqrt{\frac{\mu}{\sigma}}} - \frac{\omega_n\sqrt{\frac{\mu}{\sigma}}}{z_0}\right)$$

or

$$\boxed{\cot \omega_n l\sqrt{\mu\sigma} = -\frac{1}{2}\left(\frac{z_0}{\omega_n\sqrt{\frac{\mu}{\sigma}}} - \frac{\omega_n\sqrt{\frac{\mu}{\sigma}}}{z_0}\right)}$$

Now for the values of the parameters listed in the problem.

$$\frac{z_0}{\omega_n\sqrt{\frac{\mu}{\sigma}}} \sim \frac{1.2\cdot 10^8}{\omega_n}, \quad \frac{\omega_n\sqrt{\frac{\mu}{\sigma}}}{z_0} \sim 10^{-8}\omega_n. \quad \text{Hence,}$$

we can drop the second term on the right-hand side and write

$$\cot \omega_n l\sqrt{\mu\sigma} = -\frac{1}{2}\left(\frac{z_0}{\omega_n\sqrt{\frac{\mu}{\sigma}}}\right)$$

or $$\tan \omega_n l\sqrt{\mu\sigma} = -\frac{2\,\omega_n\sqrt{\frac{\mu}{\sigma}}}{z_0} \sim \frac{-2\omega_n}{1.2}\,10^{-8}$$

But the right hand side is clearly practically zero for the first 10 values of ω_n, so that the roots will closely satisfy the single transcendental equation

$$\tan \omega_n l\sqrt{\mu\sigma} = 0 \text{ or } \omega_n l\sqrt{\mu\sigma} = n\pi, \quad n = 1, 2, \ldots, 10$$

$$\therefore \quad S_n = -\omega_n^2 = -\left(\frac{n\pi}{l\sqrt{\mu\sigma}}\right)^2 = -\frac{n^2\pi^2}{(\mu\sigma)l^2}$$

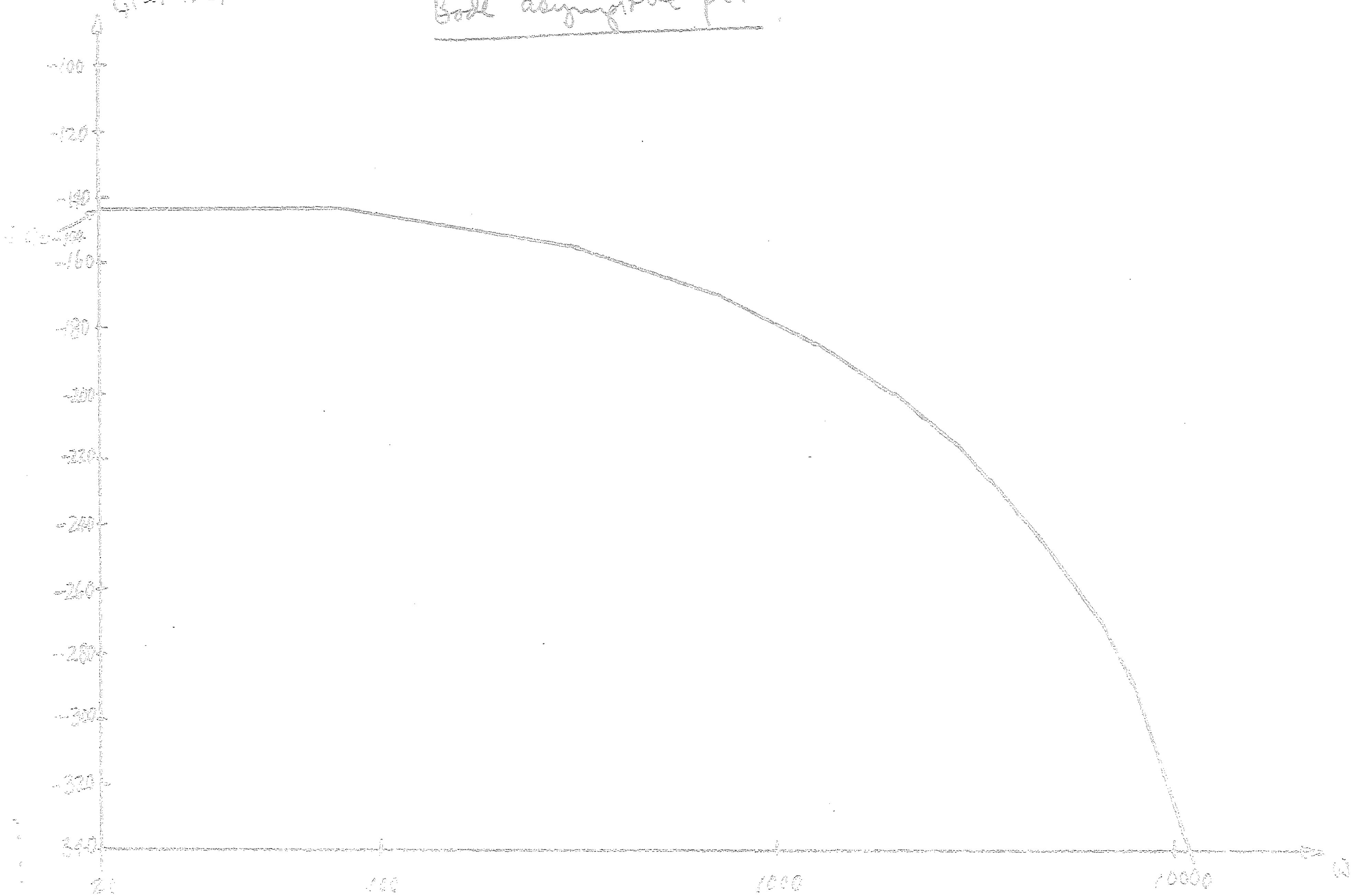
Now $\mu = 4\pi \cdot 10^{-7}\ \frac{h}{cm}$, $\sigma = 10^5\ \frac{mhos}{cm}$, $l = 1cm$

$$\therefore (b) \quad \boxed{S_n = -\frac{n^2\pi^2}{0.1256} = -8\pi^2 n^2 = -80\,n^2, \quad n = 1, 2, \ldots, 10}$$

The frequency response is sketched on the next page. Clearly the system is a low-pass filter. By increasing l, σ or μ we improve the shielding effect.

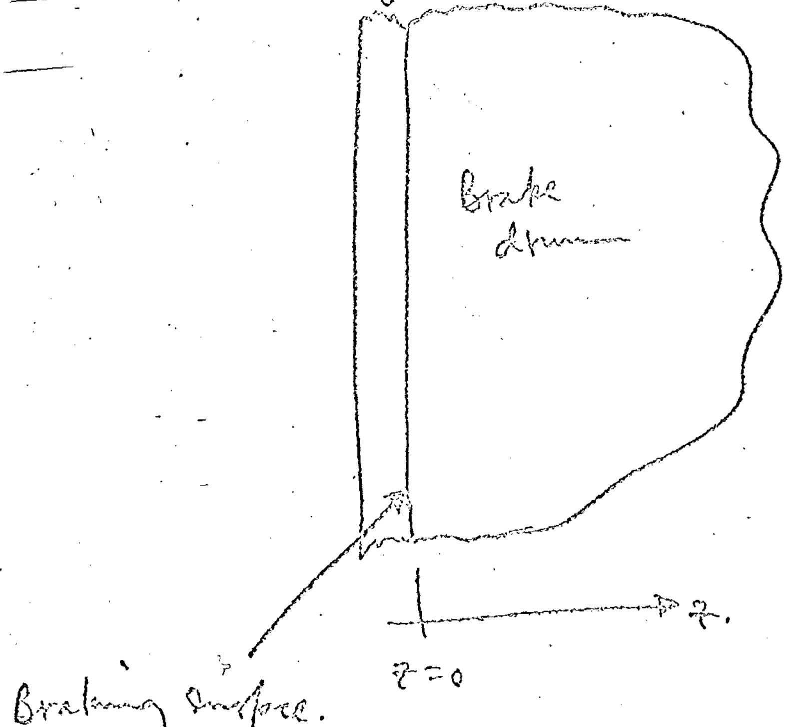
Note that an RC-filter could be used to model this system.

Bode asymptotic plot



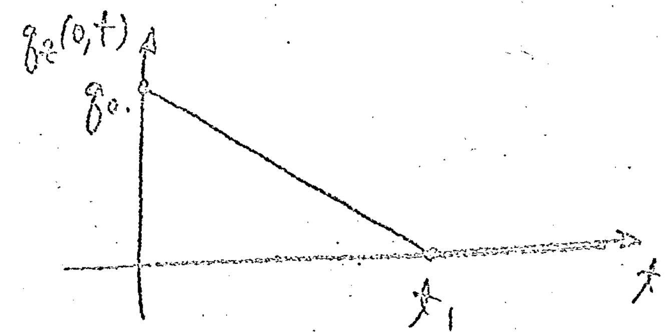
Solution of related braking problem.

(2).



The rate of energy dissipation (power) per drum at $t = 0$, when the brakes are applied is q_0 watts, while at $t = t_1$, when the car is at rest $q = 0$. Since we assumed that the car was slowing linearly we have for $q_z(0,t)$ (the input power)

$$q_z(0,t) = q_0 \frac{(t_1 - t)}{t_1}$$
$$= 0, \quad t > t_1$$



$$T = -\frac{1}{sC}\cdot\frac{dQ}{dz} = \frac{Q_0(s)}{sC}\cdot(CR_T s)^{1/2}\cdot e^{-(CR_T s)^{1/2} z}$$

$$= \left(\frac{R_T}{sC}\right)^{1/2} Q_0(s)\, e^{-(CR_T s)^{1/2} s}.$$

Thus $\left(\frac{R_T}{sC}\right)^{1/2}$ is the characteristic thermal impedance of the medium. We are interested in the conditions at $z=0$;

$$T(s,0) = \left(\frac{R_T}{sC}\right)^{1/2} Q_0(s) = \left(\frac{R_T}{sC}\right)^{1/2}\left\{\frac{q_0}{s} - \frac{q_0}{t_1}\cdot\frac{1}{s^2}\left(1-e^{-st_1}\right)\right\}$$

A previous bit of information was that $\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}\,u_{-1}(t)$

Hence, $\mathcal{L}^{-1}\left[\frac{1}{s\sqrt{s}}\right] = \int_0^t \frac{1}{\sqrt{\pi t}}\,dt = \frac{2}{\sqrt{\pi}}\, t^{1/2}\cdot u_{-1}(t)$

and $\mathcal{L}^{-1}\left[\frac{1}{s^2\sqrt{s}}\right] = \int_0^t \frac{2}{\sqrt{\pi}}\, t^{1/2}\,dt = \frac{4}{3\sqrt{\pi}}\, t^{3/2}\, u_{-1}(t)$

$$\mathcal{L}^{-1}\left[\frac{e^{-st_1}}{s^2\sqrt{s}}\right] = \frac{4}{3\sqrt{\pi}}\,(t-t_1)^{3/2}\, u_{-1}(t-t_1).$$

The Laplace transform of $q_z(0,t)$ is:

$$Q_0(s) = \mathcal{L}[q_z(0,t)] = \frac{q_0}{s} - \frac{q_0}{t_1}\cdot\frac{1}{s^2}\left(1-e^{-st_1}\right)$$

The equation of dynamics for q_z is:

$$\frac{\partial^2 q}{\partial z^2} = C R_T \frac{\partial q}{\partial t} \quad , \text{ where } R_T = \frac{1}{\sigma_T}.$$

In Laplace transform notation

$$\frac{d^2 Q}{dz^2} = C R_T s Q ,$$

So that the solution for the "outgoing" heat wave is

$$Q(s,z) = Q_0(s)\, e^{-(C R_T s)^{1/2} z}$$

From the equation

$$\frac{\partial q}{\partial z} = -C\frac{\partial T}{\partial t}.$$

which becomes in $\mathcal{L}$-transform notation ($\mathcal{T}(z,s) = \mathcal{L}[T(z,t)$

$$\frac{dQ}{dz} = -sC\mathcal{T} ,$$

we get

Thus

$$T(0,t) = 2\left(\frac{t R_T}{c\pi}\right)^{1/2} g_0\left(1 - \frac{2}{3}\frac{t}{t_1}\right)u_{-1}(t) + \frac{4}{3}\left(\frac{R_T}{c\pi}\right)^{1/2}(t-t_1)^{3/2}u_{-1}(t-t_1)$$

Because we are interested in the temperature at the time the car stops, we may ignore the second term which contributes a non-zero value *after* the car stops.

We must calculate g_0 and t_1.

$$t_1 = \frac{\text{distance}}{\text{average speed}} = \frac{120\,(ft)}{44\,\frac{(ft)}{sec.}} = \boxed{2.73 \text{ sec.}}$$

$$\text{Kinetic energy at 60 m.p.h.} = \frac{1}{2}\frac{W v^2}{g} = \frac{1}{2}\cdot\frac{4000}{32}\times(88)^2$$

$$= 4.84 \times 10^5 \text{ ft-lb.}$$

The distance traveled by a point on a brake-drum surface is 60 ft, since the drum radius is one-half that of the tire tread.

The work done in stopping the car is $4.84 \cdot 10^5$ ft-lbf, so that the constant retarding force is

$$f = \frac{4.84 \times 10^5 \text{ ft-lbf}}{60 \text{ ft}} \approx 8000 \text{ lbf}.$$

At 60 mph., the rate of heat supply to one drum is

$$q_0 = \frac{8000}{4} \text{ lbs} \cdot 88 \frac{\text{ft}}{\text{sec}} \cdot \frac{1}{\text{area of drum}} \quad \frac{\text{foot-lbs/sec.}}{(\text{cmeter})^2}.$$

$$= 2000 \times 88 \times \frac{1}{(1.5)(15\pi)(2.54)^2}$$

$$= 387 \frac{\text{ft-lbs/sec}}{(\text{cm})^2}.$$

$$= (1.356)(387) \quad \frac{\text{watts}}{(\text{cm})^2}$$

$$= \boxed{523 \quad \frac{\text{watts}}{(\text{cm})^2}}$$

Substituting in $t = t_1$, the values for R_T and C, given, as well as q_0 as just derived gives us (again, neglecting the second term in $T(0,t)$ because it vanishes at $t = t_1$)

(1) $T(0, t_1) = 210°\ C$ — Temperature at time auto stops.

To find max. temperature, differentiate

$$2\left(\frac{t\,k_T}{c\pi}\right)^{1/2} q_0\left(1 - \frac{2}{3}\cdot\frac{t}{t_1}\right)$$

with respect to t, and set the derivative equal to zero and then solve for t. This yields $t = \frac{1}{2} t_1$, i.e., the temperature reaches a maximum ~~temperature~~ half-way through the braking. After that time the heat developed diminishes, i.e., the rate of generation of heat is less than the rate of diffusion into the brake metal.

The maximum temperature is obtained by substituting $t = \frac{1}{2} t_1 = 1.37$ into the above expression. We get

$$T_{max} = 210\sqrt{2} \simeq 300°C$$

175/200

BOB MARKS

1)

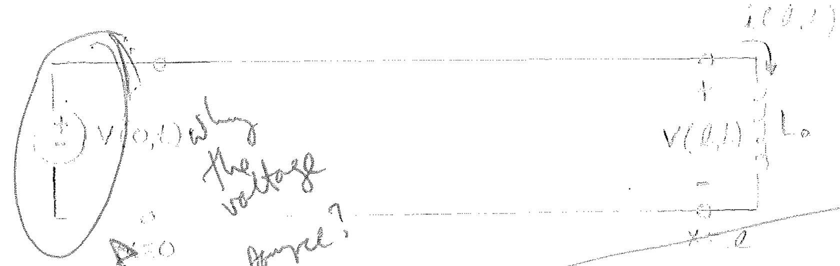
$\rho(x,0) = kx$

$$\frac{\delta \rho}{\delta x} = J\frac{\delta \omega}{\delta t} \;\; , \;\; \frac{\delta V}{\delta x} = -L\frac{\delta i}{\delta t}$$

$$\frac{\delta \omega}{\delta x} = \frac{1}{C_0}\frac{\delta \rho}{\delta t} \;\; , \;\; \frac{\delta i}{\delta x} = -C\frac{\delta V}{\delta t}$$

ANALOGIES: $\rho \sim V$, $\omega \sim i$, $J \sim L$, $\frac{1}{C_0} \sim C$, $I \sim L_0$, $k \sim Q$

∴ SYSTEM MAY BE MODELED AS LOSSLESS TRANSMISSION LINE TERMINATED BY AN INDUCTANCE:



Why the voltage source?

BOUNDRY CONDITIONS:

$\omega(0,t) = 0 \Rightarrow i(0,t) = 0 \Rightarrow I(0,s) = 0$

$V(\ell,s) = sL_0 I(\ell,s)$ ✓

Doesn't agree with your circuit, but you don't use the voltage source anyway.

What is C?

INITIAL CONDITIONS:

$\rho(x,0) = C_0 \frac{\delta \rho(x,0)}{\delta x} = C_0 k \Rightarrow V(x,0) = \frac{Q}{C}$ ✓ ?

$\omega(x,0) = 0 \Rightarrow i(x,0) = 0$ ✓

NOW: $\frac{\delta^2 V}{\delta x^2} = \frac{1}{V_p^2}\frac{\delta^2 V}{\delta t^2} \Rightarrow V_p = \frac{1}{\sqrt{LC}}$

$$\frac{d^2 V(x,s)}{dx^2} = \left(\frac{s}{V_p}\right)^2 V(x,s) - \frac{s}{V_p^2}V(x,0) - \frac{\delta V(x,0)}{V_p^2\,\delta t}$$

$$= \left(\frac{s}{V_p}\right)^2 V(x,s) - \frac{sQ}{C\,V_p^2}$$ ✓

SIMILARLY: $\frac{\delta^2 i}{\delta x^2} = \frac{1}{V_p^2}\frac{\delta^2 i}{\delta t^2}$

$$\frac{d^2 I(x,s)}{dx^2} = \left(\frac{s}{V_p}\right)^2 I(x,s) - s\,i(x,0) - \frac{\delta i(x,0)}{\delta t}$$

$$= \left(\frac{s}{V_p}\right)^2 I(x,s)$$ ✓

$$\therefore\ I(x,s) = \frac{1}{Z_0}\left(A e^{\frac{s}{V_p}\ell} - B e^{-\frac{s}{V_p}\ell}\right)$$

$$V(x,s) = A e^{\frac{s}{V_p}\ell} + B e^{-\frac{s}{V_p}\ell} + \frac{Q}{sC}$$

EMPLOYING BOUNDRY CONDITIONS:

$$I(0,s) = 0 = A - B \Rightarrow A = B$$

$$V(\ell,s) = -s L_0 I(\ell,s)$$

$$2A\cosh\frac{s}{V_p}\ell + \frac{Q}{sC} = \frac{-2sL_0A}{Z_0}\sinh\frac{s}{V_p}\ell$$

$$2A\left[\cosh\frac{s}{V_p}\ell + \frac{sL_0}{Z_0}\sinh\frac{s}{V_p}\ell\right] = \frac{-Q}{sC}$$

$$A\left[\left(1+\frac{sL_0}{Z_0}\right)e^{\frac{s}{V_p}\ell} + \left(1-\frac{sL_0}{Z_0}\right)e^{-\frac{s}{V_p}\ell}\right] = \frac{-Q}{sC}$$

$$\frac{A}{Z_0}\left[(Z_0+sL_0)e^{\frac{s}{V_p}\ell} + (Z_0-sL_0)e^{-\frac{s}{V_p}\ell}\right] = \frac{-Q}{sC}$$

$$\Rightarrow A = \frac{-QZ_0}{sC\left[(Z_0+sL_0)e^{\frac{s}{V_p}\ell} + (Z_0-sL_0)e^{-\frac{s}{V_p}\ell}\right]}$$

$$\therefore\ I(\ell,s) = \frac{-Q\left[e^{\frac{s}{V_p}\ell} - e^{-\frac{s}{V_p}\ell}\right]}{sC\left[(Z_0+sL_0)e^{\frac{s}{V_p}\ell} + (Z_0-sL_0)e^{-\frac{s}{V_p}\ell}\right]}$$

$$= \frac{-Q\left[1 - e^{-\frac{2s}{V_p}\ell}\right]}{sC\left[(Z_0+sL_0) + (Z_0-sL_0)e^{-2s\ell/V_p}\right]}$$

$$= \frac{-Q\left[1 - e^{-2s\ell/V_p}\right]}{sC(Z_0+sL_0)\left[1 - \frac{sL_0-Z_0}{Z_0+sL_0}e^{-2s\ell/V_p}\right]}$$

$$\Gamma_L = \frac{sL_0 - Z_0}{Z_0 + sL_0}\ ;\ sL = Z_L$$

EXPANDING THE DENOMINATOR:

$$I(\ell,s) = \frac{-Q}{sC(Z_0+sL_0)}\left(1 - e^{-2s\ell/V_p}\right)\left[1 + \Gamma_L e^{-2s\ell/V_p} + \Gamma_L^2 e^{-4s\ell/V_p} + \Gamma_L^3 e^{-6s\ell/V_p} + \Gamma_L^4 e^{-8s\ell/V_p} + \sum_{n=0}^{\infty}\Gamma_L^n e^{-2ns\ell/V_p}\right]$$

$$= \frac{-Q}{sCL_0(s+Z_0/L_0)}\left[1 + (\Gamma_L - 1)e^{-2s\ell/V_p} + (\Gamma_L^2 - \Gamma_L)e^{-4s\ell/V_p} + (\Gamma_L^3 - \Gamma_L^2)e^{-6s\ell/V_p} + \sum_{n=4}^{\infty}(\Gamma^n - \Gamma^{n-1})e^{-2ns\ell/V_p}\right]$$

$$I(\ell,s) = \frac{-Q}{sCL_0(s+Z_0/L_0)}\left[1+\sum_{n=1}^{\infty}\Gamma_L^{n-1}(\Gamma_L-1)e^{-2ns\ell/v_p}\right]$$

LET $\frac{-Q}{CL_0(s+Z_0/L_0)} = I_0$?

$$\Rightarrow I(\ell,s) = I_0\left[\frac{1}{s}+(\Gamma_L-1)\sum_{n=1}^{\infty}\Gamma^{n-1}\frac{e^{-2ns\ell/v_p}}{s}\right]$$

$$i(\ell,t) = I_0\left[\mu(t)+(\Gamma_L-1)\sum_{n=1}^{\infty}\Gamma^{n-1}\mu\left(t-\frac{2ns\ell}{v_p}\right)\right]$$

$$\Rightarrow \omega(\ell,t) = I_0\left[\mu(t)+(\Gamma_L-1)\sum_{n=1}^{\infty}\mu\left(t-\frac{2ns\ell}{v_p}\right)\Gamma^{n-1}\right]$$

THE RESULTANT WAVEFORM WOULD BE A PULSE CHAIN WEIGHTED BY $\Gamma_L^{n-1}(\Gamma_L-1)I_0$. $\Gamma_L<0 \Rightarrow$ CHAIN WON'T BLOW UP, FOR REFLECTED PULSES WILL TEND TO CANCEL.

$$\Rightarrow \Theta_\ell(t) = \omega_0\left[r(t)+(\Gamma_L-1)\sum_{n=1}^{\infty}r\left(t-\frac{2ns\ell}{v_p}\right)\Gamma^{n-1}\right]+k\ell$$

$\ni r(t) =$ UNIT RAMP FUNCTION

Good work

2) $v_i(t) = \mu(t) - \mu(t-\tau)$

$\Rightarrow V_i(s) = \frac{1}{s}(1 - e^{-\tau s})$

$$\frac{V_r(s)}{V_i(s)} = \frac{\Gamma_1 + \Gamma_2 e^{-2s\delta/v_p}}{1 + \Gamma_1\Gamma_2 e^{-2s\delta/v_p}}$$ — Derive this.

(-10)

$$\therefore V_r(s) = \left(\frac{1-e^{-\tau s}}{s}\right)\left(\Gamma_1 + \Gamma_2 e^{\frac{-2s\delta}{v_p}}\right)\left(1 - \Gamma_1\Gamma_2 e^{-\frac{2s\delta}{v_p}} + (\Gamma_1\Gamma_2)^2 e^{-\frac{4s\delta}{v_p}} - (\Gamma_1\Gamma_2)^3 e^{-\frac{6s\delta}{v_p}} + (\Gamma_1\Gamma_2)^4 e^{-\frac{8s\delta}{v_p}} - (\Gamma_1\Gamma_2)^5 e^{\frac{-10s\delta}{v_p}} + \sum_{n=6}^{\infty}(-1)^n(\Gamma_1\Gamma_2)^n e^{-\frac{2ns\delta}{v_p}}\right)$$

$$V_r(s) = \frac{1}{s}(1 - e^{-\tau s})\left[\sum_{n=1}^{\infty}(-1)^{n+1} n (\Gamma_1\Gamma_2)^n e^{-\frac{2ns\delta}{v_p}}\right]$$

$$= \frac{1}{s}\left[\sum_{n=1}^{\infty}(-1)^{n+1} n (\Gamma_1\Gamma_2)^n \left(e^{-2\frac{sn\delta}{v_p}} - e^{-s(\tau + \frac{2n\delta}{v_p})}\right)\right]$$

$$v_r(t) = \sum_{n=1}^{\infty}(-1)^{n+1} n (\Gamma_1\Gamma_2)^n \left[\mu\left(t - \frac{2n\delta}{v_p}\right) - \mu\left(t - \left(\tau + \frac{2n\delta}{v_p}\right)\right)\right]$$

$Z_1 = 32 \times 10^6 \frac{Kg}{s \cdot m^2}$

$Z_2 = 1.45 \times 10^6 \frac{Kg}{s \cdot m^2}$

$Z_3 = 8.6 \times 10^6 \frac{Kg}{s \cdot m^2}$

$\Gamma_1 = \frac{Z_1 - Z_2}{Z_1 + Z_2}$

$\Gamma_2 = \frac{Z_2 - Z_3}{Z_2 + Z_3}$

$V_p = 1450 \frac{m}{sec}$ ($V_p = V_{p_2}$ = PHASE VELOCITY IN WATER)

$\tau = 80 \times 10^{-9}$ SEC

$\delta_1 = 40 \times 10^{-6}$ m

$\delta_2 = .5 \times 10^{-6}$ m

$\frac{2\delta}{v_p}$ = TIME FOR PULSE TO MAKE A ROUND TRIP IN THE WATER

$\Gamma_1\Gamma_2 < 0 \Rightarrow$ FIRST REFLECTED PULSE IS INVERTED

$v_r(t)\Big|_n$ = PULSE OF WIDTH τ, STARTING AT $\frac{2n\delta}{v_p}$ AND WEIGHTED BY $(-1)^{n+1} n (\Gamma_1\Gamma_2)^n$

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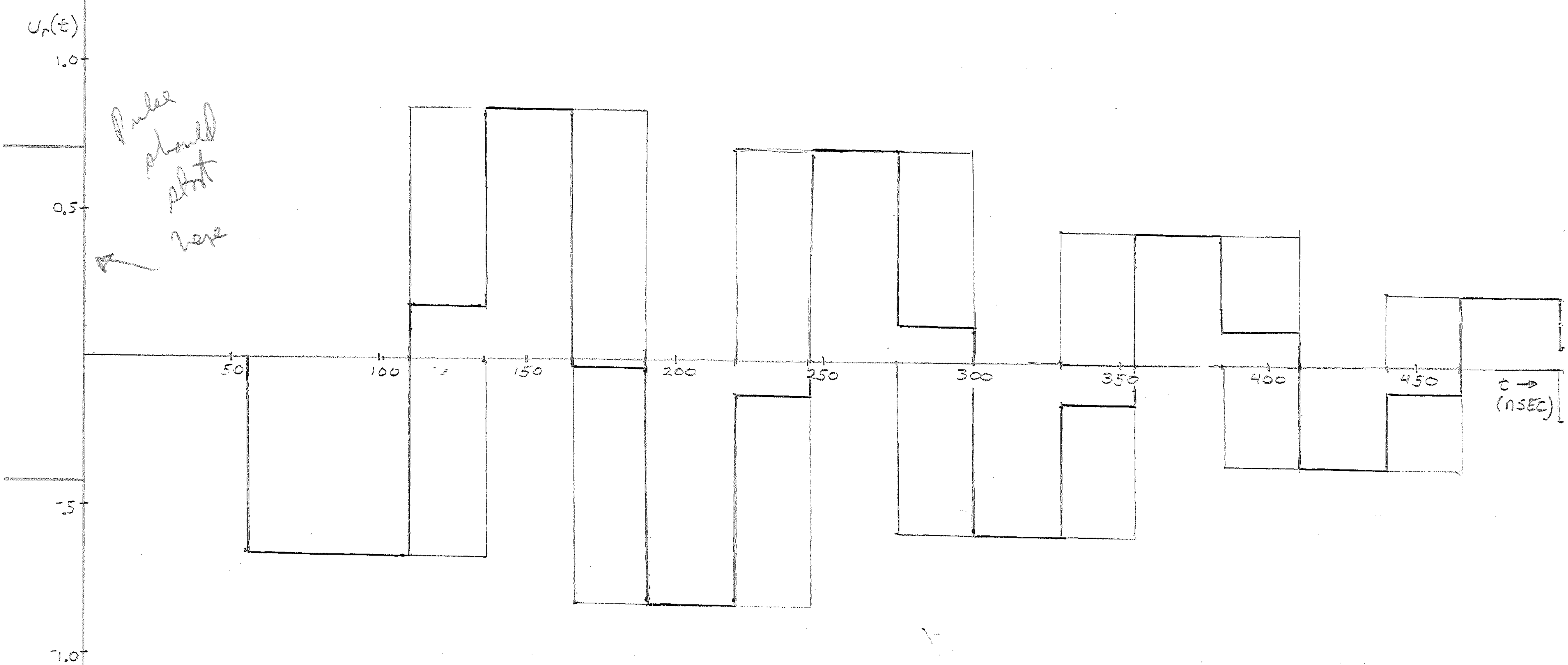
ROBERT J. MARKS II
TRAVELING WAVES I-PROB 2

REFLECTION COEFFICIENT 1= 0.9133034
REFLECTION COEFFICIENT 2=-0.7114428

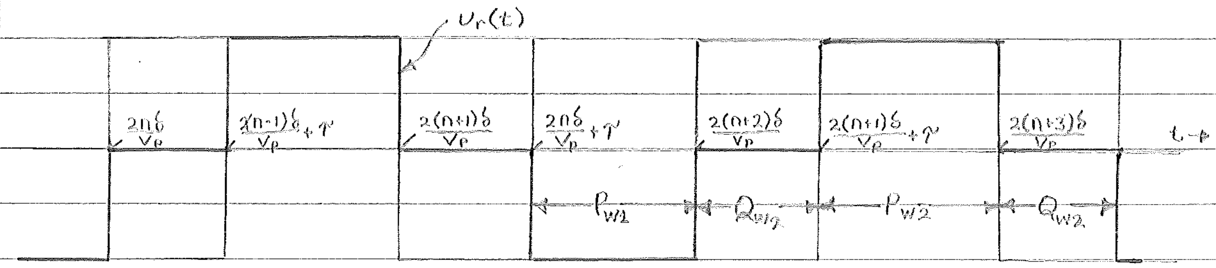

CASE 1
 DEL= 0.40000079E-04      ROUND TRIP TIME= 0.55172513E-07

FOR N=  1, UR(T)= 0.64976310E 00 FOR T= 0.55172513E-07 TO  0.13517274E-06 SEC
FOR N=  2, UR(T)= 0.84438407E 00 FOR T= 0.11034502E-06 TO  0.19034524E-06 SEC
FOR N=  3, UR(T)= 0.82297444E 00 FOR T= 0.16551754E-06 TO  0.24551775E-06 SEC
FOR N=  4, UR(T)= 0.71298444E 00 FOR T= 0.22069005E-06 TO  0.30069026E-06 SEC
FOR N=  5, UR(T)= 0.57908868E 00 FOR T= 0.27586259E-06 TO  0.35586276E-06 SEC
FOR N=  6, UR(T)= 0.45152455E 00 FOR T= 0.33103509E-06 TO  0.41103527E-06 SEC
FOR N=  7, UR(T)= 0.34228128E 00 FOR T= 0.38620760E-06 TO  0.46620778E-06 SEC
FOR N=  8, UR(T)= 0.25417339E 00 FOR T= 0.44138011E-06 TO  0.52138034E-06 SEC
FOR N=  9, UR(T)= 0.18579655E 00 FOR T= 0.49655261E-06 TO  0.57655279E-06 SEC
FOR N= 10, UR(T)= 0.13413748E 00 FOR T= 0.55172518E-06 TO  0.63172535E-06 SEC
FOR N= 11, UR(T)= 0.95873341E-01 FOR T= 0.60689774E-06 TO  0.68689792E-06 SEC
FOR N= 12, UR(T)= 0.67958131E-01 FOR T= 0.66207019E-06 TO  0.74207037E-06 SEC
FOR N= 13, UR(T)= 0.47836408E-01 FOR T= 0.71724275E-06 TO  0.79724293E-06 SEC
FOR N= 14, UR(T)= 0.33473283E-01 FOR T= 0.77241520E-06 TO  0.85241538E-06 SEC
FOR N= 15, UR(T)= 0.23303248E-01 FOR T= 0.82758777E-06 TO  0.90758794E-06 SEC
FOR N= 16, UR(T)= 0.16151029E-01 FOR T= 0.88276022E-06 TO  0.96276039E-06 SEC
FOR N= 17, UR(T)= 0.11150239E-01 FOR T= 0.93793278E-06 TO  0.10179330E-05 SEC
FOR N= 18, UR(T)= 0.76711904E-02 FOR T= 0.99310523E-06 TO  0.10731052E-05 SEC
FOR N= 19, UR(T)= 0.52613699E-02 FOR T= 0.10482779E-05 TO  0.11282779E-05 SEC
FOR N= 20, UR(T)= 0.35985726E-02 FOR T= 0.11034503E-05 TO  0.11834504E-05 SEC
FOR N= 21, UR(T)= 0.24551306E-02 FOR T= 0.11586228E-05 TO  0.12386228E-05 SEC
FOR N= 22, UR(T)= 0.16712176E-02 FOR T= 0.12137954E-05 TO  0.12937955E-05 SEC
FOR N= 23, UR(T)= 0.11352542E-02 FOR T= 0.12689679E-05 TO  0.13489680E-05 SEC
FOR N= 24, UR(T)= 0.76971785E-03 FOR T= 0.13241403E-05 TO  0.14041404E-05 SEC
FOR N= 25, UR(T)= 0.52097323E-03 FOR T= 0.13793128E-05 TO  0.14593129E-05 SEC
FOR N= 26, UR(T)= 0.35204947E-03 FOR T= 0.14344855E-05 TO  0.15144855E-05 SEC
FOR N= 27, UR(T)= 0.23754677E-03 FOR T= 0.14896579E-05 TO  0.15696580E-05 SEC
FOR N= 28, UR(T)= 0.16006577E-03 FOR T= 0.15448304E-05 TO  0.16248304E-05 SEC
FOR N= 29, UR(T)= 0.10771927E-03 FOR T= 0.16000030E-05 TO  0.16800031E-05 SEC
FOR N= 30, UR(T)= 0.72405528E-04 FOR T= 0.16551755E-05 TO  0.17351756E-05 SEC
FOR N= 31, UR(T)= 0.48614645E-04 FOR T= 0.17103479E-05 TO  0.17903480E-05 SEC
FOR N= 32, UR(T)= 0.32606971E-04 FOR T= 0.17655204E-05 TO  0.18455205E-05 SEC
FOR N= 33, UR(T)= 0.21848889E-04 FOR T= 0.18206931E-05 TO  0.19006931E-05 SEC
FOR N= 34, UR(T)= 0.14626801E-04 FOR T= 0.18758655E-05 TO  0.19558656E-05 SEC
FOR N= 35, UR(T)= 0.97834836E-05 FOR T= 0.19310382E-05 TO  0.20110383E-05 SEC
FOR N= 36, UR(T)= 0.65385729E-05 FOR T= 0.19862104E-05 TO  0.20662105E-05 SEC
FOR N= 37, UR(T)= 0.43665377E-05 FOR T= 0.20413831E-05 TO  0.21213832E-05 SEC
FOR N= 38, UR(T)= 0.29138964E-05 FOR T= 0.20965558E-05 TO  0.21765558E-05 SEC
FOR N= 39, UR(T)= 0.19431672E-05 FOR T= 0.21517280E-05 TO  0.22317281E-05 SEC
FOR N= 40, UR(T)= 0.12949724E-05 FOR T= 0.22069007E-05 TO  0.22869007E-05 SEC
FOR N= 41, UR(T)= 0.86246086E-06 FOR T= 0.22620733E-05 TO  0.23420734E-05 SEC
FOR N= 42, UR(T)= 0.57406339E-06 FOR T= 0.23172456E-05 TO  0.23972456E-05 SEC
FOR N= 43, UR(T)= 0.38188625E-06 FOR T= 0.23724183E-05 TO  0.24524183E-05 SEC
FOR N= 44, UR(T)= 0.25390619E-06 FOR T= 0.24275909E-05 TO  0.25075910E-05 SEC
FOR N= 45, UR(T)= 0.16872837E-06 FOR T= 0.24827631E-05 TO  0.25627632E-05 SEC
FOR N= 46, UR(T)= 0.11206975E-06 FOR T= 0.25379358E-05 TO  0.26179359E-05 SEC
FOR N= 47, UR(T)= 0.74401811E-07 FOR T= 0.25931085E-05 TO  0.26731086E-05 SEC
FOR N= 48, UR(T)= 0.49372133E-07 FOR T= 0.26482807E-05 TO  0.27282808E-05 SEC
FOR N= 49, UR(T)= 0.32748523E-07 FOR T= 0.27034534E-05 TO  0.27834535E-05 SEC
FOR N= 50, UR(T)= 0.21713042E-07 FOR T= 0.27586256E-05 TO  0.28386257E-05 SEC
```

CASE 1: $\delta = 40$ MICRONS



CONSIDER THE IDEAL CASE WHERE EACH REFLECTED PULSE IS OF EQUAL AMPLITUDE:



$$P_{W1} = \frac{2(n+2)\delta}{V_p} - \left(\frac{2n\delta}{V_p} + \tau\right)$$
$$= \frac{4\delta}{V_p} - \tau$$

KNOWING τ, V_p, AND HAVING MEASURED P_{W1}:

$$\underline{\delta = \frac{V_p}{4}\left(P_{W1} + \tau\right)}$$ (NOTE: EXPRESSION ALSO HOLDS FOR $P_{W2} = P_{W1}$)

SIMILARLY:
$$Q_{W1} = \frac{2(n+1)\delta}{V_p} + \tau - \frac{2(n+2)\delta}{V_p}$$
$$= \frac{-2\delta}{V_p} + \tau$$

AND

$$\underline{\delta = \frac{V_p}{2}\left(\tau - Q_{W1}\right)}$$ (NOTE: EXPRESSION ALSO HOLDS FOR $Q_{W2} = Q_{W1}$)

THUS, BY MEASURING ANY POSITIVE OR NEGATIVE PULSE WIDTH, OR BY MEASURING THE DURATION OF A MINIMAL VALUE (IDEALY 0), THE FORMER CASE SHOULD BE USED FOR $P_W > Q_W$, AND VISA VICE-VERSA TO MINIMIZE ERROR. THIS MODEL HOLDS ONLY IF TWO ADJACENT POSITIVE OR NEGATIVE PULSES DO NOT OVERLAP (IF $\frac{2(n+2)\delta}{V_p} > \frac{2n\delta}{V_p} + \tau$), AND TWO ADJACENT PULSES DO OVERLAP $\left(\frac{2(n+1)\delta}{V_p} < \frac{2n\delta}{V_p} + \tau\right)$

FOR $U_1(t)$ GRAPH; $P_{W1} > Q_{W1}$

$P_{W1} = 30 \times 10^{-9}$ SEC

$$\Rightarrow \delta = \frac{V_P}{4}(P_W + \tau)$$

$$= \frac{(1.45 \times 10^3 m/s)}{4}(30 \times 10^{-9} + 80 \times 10^{-9}) SEC$$

$$\doteq 40 \times 10^{-6} M$$

$\doteq$ 40 MICRONS, THE DESIRED RESULT

CHECKING WITH Q_{W1} YIELDS:

$Q_{W1} = 25 \times 10^{-9}$ SEC

$$\Rightarrow \delta = \frac{V_P}{2}(\tau - Q_W)$$

$$= \frac{1.45 \times 10^3}{2}(80 \times 10^{-9} - 25 \times 10^{-9}) M$$

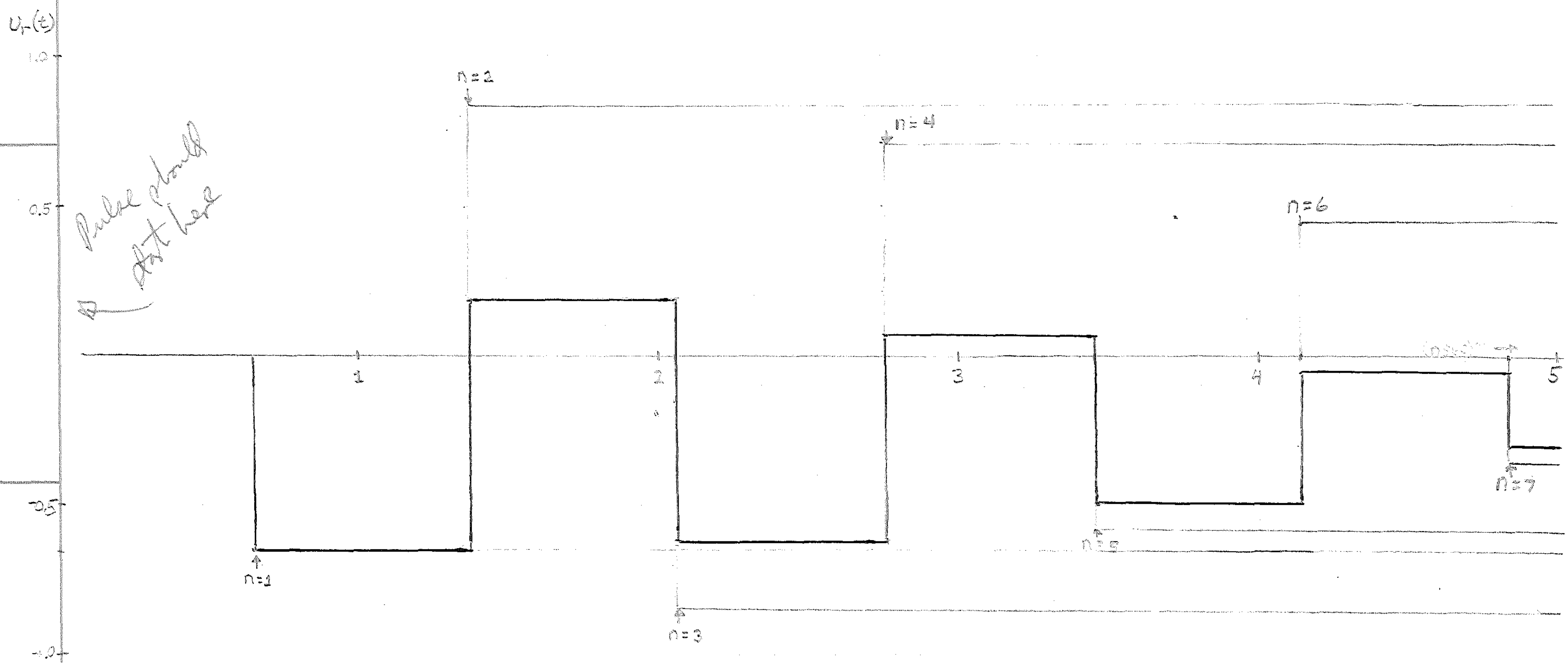
$\doteq$ 40 MICRONS

CASE 2

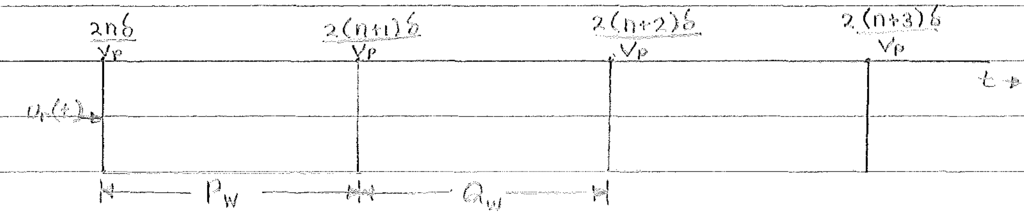
DEL= 0.50000096E-06   ROUND TRIP TIME= 0.68965644E-09

```
FOR N=  1, UR(T)= 0.64976310E 00 FOR T= 0.68965644E-09 TO  0.80689886E-07 SEC
FOR N=  2, UR(T)= 0.84438407E 00 FOR T= 0.13793128E-08 TO  0.81379539E-07 SEC
FOR N=  3, UR(T)= 0.82297444E 00 FOR T= 0.20689694E-08 TO  0.82069192E-07 SEC
FOR N=  4, UR(T)= 0.71298444E 00 FOR T= 0.27586257E-08 TO  0.82758859E-07 SEC
FOR N=  5, UR(T)= 0.57908868E 00 FOR T= 0.34482821E-08 TO  0.83448512E-07 SEC
FOR N=  6, UR(T)= 0.45152455E 00 FOR T= 0.41379388E-08 TO  0.84138165E-07 SEC
FOR N=  7, UR(T)= 0.34228128E 00 FOR T= 0.48275952E-08 TO  0.84827817E-07 SEC
FOR N=  8, UR(T)= 0.25417339E 00 FOR T= 0.55172515E-08 TO  0.85517484E-07 SEC
FOR N=  9, UR(T)= 0.18579655E 00 FOR T= 0.62069078E-08 TO  0.86207137E-07 SEC
FOR N= 10, UR(T)= 0.13413748E 00 FOR T= 0.68965642E-08 TO  0.86896790E-07 SEC
FOR N= 11, UR(T)= 0.95873341E-01 FOR T= 0.75862214E-08 TO  0.87586443E-07 SEC
FOR N= 12, UR(T)= 0.67958131E-01 FOR T= 0.82758777E-08 TO  0.88276110E-07 SEC
FOR N= 13, UR(T)= 0.47836408E-01 FOR T= 0.89655341E-08 TO  0.88965762E-07 SEC
FOR N= 14, UR(T)= 0.33473283E-01 FOR T= 0.96551904E-08 TO  0.89655415E-07 SEC
FOR N= 15, UR(T)= 0.23303248E-01 FOR T= 0.10344846E-07 TO  0.90345068E-07 SEC
FOR N= 16, UR(T)= 0.16151029E-01 FOR T= 0.11034503E-07 TO  0.91034735E-07 SEC
FOR N= 17, UR(T)= 0.11150239E-01 FOR T= 0.11724159E-07 TO  0.91724388E-07 SEC
FOR N= 18, UR(T)= 0.76711904E-02 FOR T= 0.12413815E-07 TO  0.92414041E-07 SEC
FOR N= 19, UR(T)= 0.52613699E-02 FOR T= 0.13103472E-07 TO  0.93103693E-07 SEC
FOR N= 20, UR(T)= 0.35985726E-02 FOR T= 0.13793128E-07 TO  0.93793360E-07 SEC
FOR N= 21, UR(T)= 0.24551306E-02 FOR T= 0.14482784E-07 TO  0.94483013E-07 SEC
FOR N= 22, UR(T)= 0.16712176E-02 FOR T= 0.15172442E-07 TO  0.95172666E-07 SEC
FOR N= 23, UR(T)= 0.11352542E-02 FOR T= 0.15862099E-07 TO  0.95862319E-07 SEC
FOR N= 24, UR(T)= 0.76971785E-03 FOR T= 0.16551755E-07 TO  0.96551986E-07 SEC
FOR N= 25, UR(T)= 0.52097323E-03 FOR T= 0.17241411E-07 TO  0.97241638E-07 SEC
FOR N= 26, UR(T)= 0.35204947E-03 FOR T= 0.17931068E-07 TO  0.97931291E-07 SEC
FOR N= 27, UR(T)= 0.23754677E-03 FOR T= 0.18620724E-07 TO  0.98620944E-07 SEC
FOR N= 28, UR(T)= 0.16006577E-03 FOR T= 0.19310380E-07 TO  0.99310611E-07 SEC
FOR N= 29, UR(T)= 0.10771927E-03 FOR T= 0.20000037E-07 TO  0.10000026E-06 SEC
FOR N= 30, UR(T)= 0.72405528E-04 FOR T= 0.20689693E-07 TO  0.10068991E-06 SEC
FOR N= 31, UR(T)= 0.48614645E-04 FOR T= 0.21379349E-07 TO  0.10137956E-06 SEC
FOR N= 32, UR(T)= 0.32606971E-04 FOR T= 0.22069006E-07 TO  0.10206923E-06 SEC
FOR N= 33, UR(T)= 0.21848889E-04 FOR T= 0.22758662E-07 TO  0.10275888E-06 SEC
FOR N= 34, UR(T)= 0.14626801E-04 FOR T= 0.23448318E-07 TO  0.10344854E-06 SEC
FOR N= 35, UR(T)= 0.97834836E-05 FOR T= 0.24137975E-07 TO  0.10413819E-06 SEC
FOR N= 36, UR(T)= 0.65385729E-05 FOR T= 0.24827631E-07 TO  0.10482786E-06 SEC
FOR N= 37, UR(T)= 0.43665377E-05 FOR T= 0.25517287E-07 TO  0.10551751E-06 SEC
FOR N= 38, UR(T)= 0.29138964E-05 FOR T= 0.26206944E-07 TO  0.10620716E-06 SEC
FOR N= 39, UR(T)= 0.19431672E-05 FOR T= 0.26896600E-07 TO  0.10689682E-06 SEC
FOR N= 40, UR(T)= 0.12949724E-05 FOR T= 0.27586256E-07 TO  0.10758648E-06 SEC
FOR N= 41, UR(T)= 0.86246086E-06 FOR T= 0.28275913E-07 TO  0.10827614E-06 SEC
FOR N= 42, UR(T)= 0.57406339E-06 FOR T= 0.28965569E-07 TO  0.10896579E-06 SEC
FOR N= 43, UR(T)= 0.38188625E-06 FOR T= 0.29655225E-07 TO  0.10965544E-06 SEC
FOR N= 44, UR(T)= 0.25390619E-06 FOR T= 0.30344885E-07 TO  0.11034511E-06 SEC
FOR N= 45, UR(T)= 0.16872837E-06 FOR T= 0.31034538E-07 TO  0.11103476E-06 SEC
FOR N= 46, UR(T)= 0.11206975E-06 FOR T= 0.31724198E-07 TO  0.11172441E-06 SEC
FOR N= 47, UR(T)= 0.74401811E-07 FOR T= 0.32413851E-07 TO  0.11241407E-06 SEC
FOR N= 48, UR(T)= 0.49372133E-07 FOR T= 0.33103511E-07 TO  0.11310373E-06 SEC
FOR N= 49, UR(T)= 0.32748523E-07 FOR T= 0.33793163E-07 TO  0.11379339E-06 SEC
FOR N= 50, UR(T)= 0.21713042E-07 FOR T= 0.34482823E-07 TO  0.11448304E-06 SEC
```

CASE 2; $\delta = 1/2$ MICRON



AGAIN, CONSIDER THE IDEAL CASE, WHERE AMPLITUDES OF REFLECTED PULSES ARE EQUAL, YET VARY OPPOSITE IN SIGN:



THIS MODEL HOLDS UNTIL $\frac{2n\delta}{V_p} \geq \frac{2\delta}{V_p} + \tau$, AT WHICH POINT THE EFFECTS OF τ NEED BE CONSIDERED. FOR THIS PARTICULAR PROBLEM:

$\tau = 80 \times 10^{-9}$ SEC

$\frac{2\delta}{V_p}$ = ROUND TRIP TIME = $.69 \times 10^{-9}$ SEC

$\Rightarrow n = 127$ BEFORE THE EFFECTS OF τ NEED BE NOTED.

BY THIS TIME, THE PULSE CHAIN AMPLITUDE WOULD BE ALMOST IMMEASURABLY SMALL IN THE NON-IDEAL CASE

NOW:

$$P_W = Q_W = \frac{2(n+1)\delta}{V_p} - \frac{2n\delta}{V_p} = \frac{2\delta}{V_p}$$

$$\Rightarrow \delta = \frac{V_p}{2} P_W = \frac{V_p}{2} Q_W$$

HENCE, KNOWING V_p, AND BY MEASURING P_W OR Q_W, δ MAY BE COMPUTED.

FROM GRAPH: $P_W = Q_W = .69 \times 10^{-9}$ SEC

$$\delta = \frac{V_p}{2} P_W$$
$$= \frac{1.45 \times 10^3 \text{ m/s}}{2} (.69\,10^{-9})\text{s}$$
$$= .5 \text{ MICRONS, THE DESIRED RESULT}$$

DISCUSSION:

THE INTERESTING FACET OF THIS PROCESS IS THAT SMALL DISTANCES (δ) MAY BE MEASURED WITH PULSES LONG IN DURATION COMPARED WITH THE ROUND TRIP TIME OF THE PULSE IN THE MEDIUM. THIS IS ACCOMPLISHED BY HAVING A POSITIVE REFLECTION COEFFICIENT AT $X=0$ AND A NEGATIVE REFLECTION COEFFICIENT AT $X=\delta$. HENCE, THE RESULTANT REFLECTED PULSES TEND TO CANCEL OUT EACH OTHER IN SUCH A MANNER THAT δ MAY BE COMPUTED FROM THE RESULTANT WAVEFORM.

Interesting observation

IN CASE 1 ($\delta = 40$ MICRONS), $\frac{2(n+2)\delta}{V_p} > \frac{2n\delta}{V_p} + \tau$, SO THAT TWO ADJACENT POSITIVE OR NEGATIVE PULSES DID NOT OVERLAP. ALSO $\frac{2(n+1)\delta}{V_p} < \frac{2n\delta}{V_p} + \tau$, SO THAT TWO ADJACENT PULSES, ALWAYS OF OPPOSITE SIGN, NEARLY CANCEL OUT IN THE RESULTANT WAVEFORM. THE NON-CANCELED PULSE (EITHER POSITIVE OR NEGATIVE) HAS A DURATION OF $\frac{4\delta}{V_p} - \tau$, FROM WHICH THE VALUE OF δ MAY BE COMPUTED.

IN CASE 2, $\frac{2\delta}{V_p} \ll \tau$, THUS REFLECTED PULSES CANCEL OUT IN SUCH A MANNER THAT THE DURATION OF A PULSE IN THE RESULANT WAVEFORM $= \frac{2\delta}{V_p} =$ ROUND TRIP TIME IN MEDIUM OF LENGH δ. AGAIN, THE VALUE OF δ MAY BE COMPUTED.

Good work

3) $P(s) = \frac{Ks^3}{\alpha^2} + \frac{(1+2k)}{\alpha} s^2 + (1+k)s + 1$

$k = 5,\ \alpha = 1$

$\Rightarrow P(s) = 5s^3 + 11s^2 + 6s + 1$

A CUBIC, WHICH YEILDS ROOTS:

$s_1 = -.3596 - j(.07593) \approx -.3596$

$s_2 = -.3596 + j(.07593) = -.3596$

$s_3 = -1.481$

$\Rightarrow f(s) = [s(s-s_1)(s-s_2)(s-s_3)]^{-1}$

(LETTING s_1 DIFFER FROM s_2 BY A SMALL INCREMENT FOR EASE IN TAKING INVERSE TRANSFORMS: $s_1 - s_2 = 10^{-6}$

Why? You can always take the transform of a double root.

$f(s) = \frac{A_0}{s} + \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \frac{A_3}{s-s_3}$

$A_0 = \left.\frac{1}{(s-s_1)(s-s_2)(s-s_3)}\right|_{s=0} = -5.223$

$A_1 = \left.\frac{1}{s(s-s_2)(s-s_3)}\right|_{s=s_1} = 2.480 \times 10^{8}$

$A_2 = \left.\frac{1}{s(s-s_1)(s-s_3)}\right|_{s=s_2} = -2.480 \times 10^{8}$?

$A_3 = \left.\frac{1}{s(s-s_1)(s-s_2)}\right|_{s=s_3} = -.5371$

THEN:

$f(\tau) = (A_0 + A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t})\,\mu(t)$

AND

$C(\theta,\tau) = \Big(A_0 + A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 \tau} + \mu(\tau-\theta)\, e^{-\theta}\left[1 - A_0 - A_1 e^{s_1(\tau-\theta)} - A_2 e^{s_2(\tau-\theta)} - A_3 e^{s_3(\tau-\theta)}\right]\Big)\,\mu(\tau)$

```
  88888888      00000000      66666666      99999999
 8888888888    0000000000    6666666666    9999999999
 88      88    00      00    66      66    99      99
 88      88    00      00    66            99      99
 88      88    00      00    66            99      99
  88888888     00      00    666666666     9999999999
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 88      88    00      00    66      66    99      99
 8888888888    0000000000    6666666666    9999999999
  88888888      00000000      66666666      99999999

 // JOB T        8069        ROBERT J. MARKS II
 // NOTE  PLEASE DO NOT FOLD OUTPUT-8069 (ROBERT J. MARKS)
 // LOAD POLY


    TRAVELING WAVES I-MIDTERM

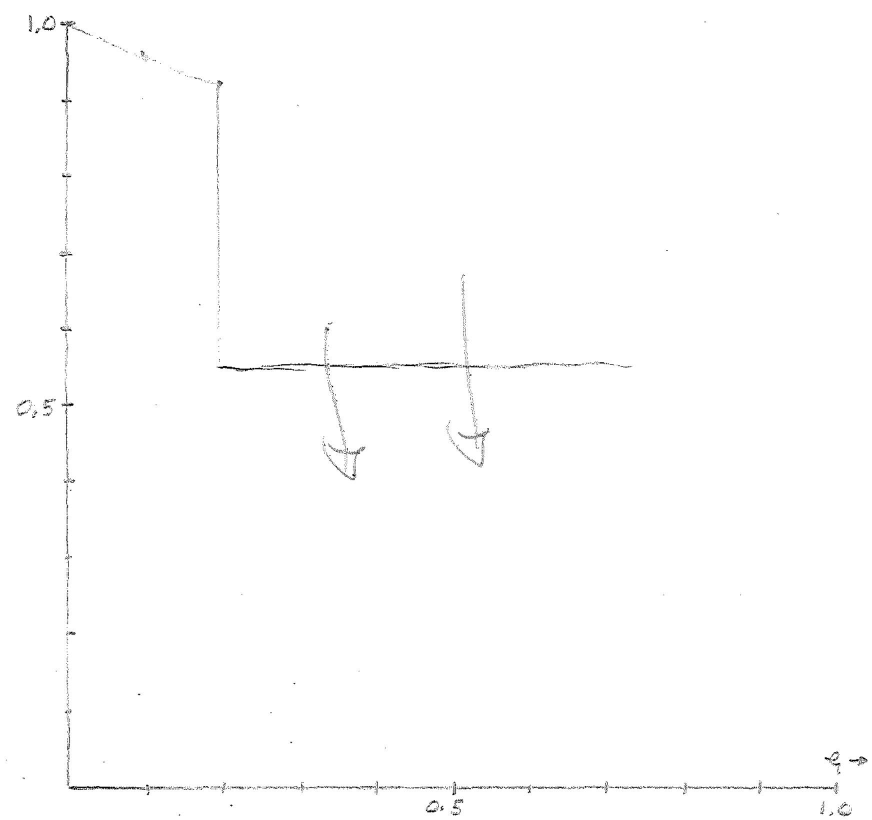
   DEGREE = 3                                      COEFFICIENTS
                                                --------------------
                                                A( 1)=  0.500000E 01
                                                A( 2)=  0.110000E 02
                                                A( 3)=  0.600000E 01
                                                A( 4)=  0.100000E 01

                                                   COMPLEX ROOTS
                         -------------------------------------------------------------
                         ROOTR( 1)= -0.359568E 00      ROOTI( 1)= -0.759393E-01
                         ROOTR( 2)= -0.359568E 00      ROOTI( 2)=  0.759393E-01
                         ROOTR( 3)= -0.148086E 01      ROOTI( 3)=  0.000000E 00
    EXECUTION TIME 0003
```

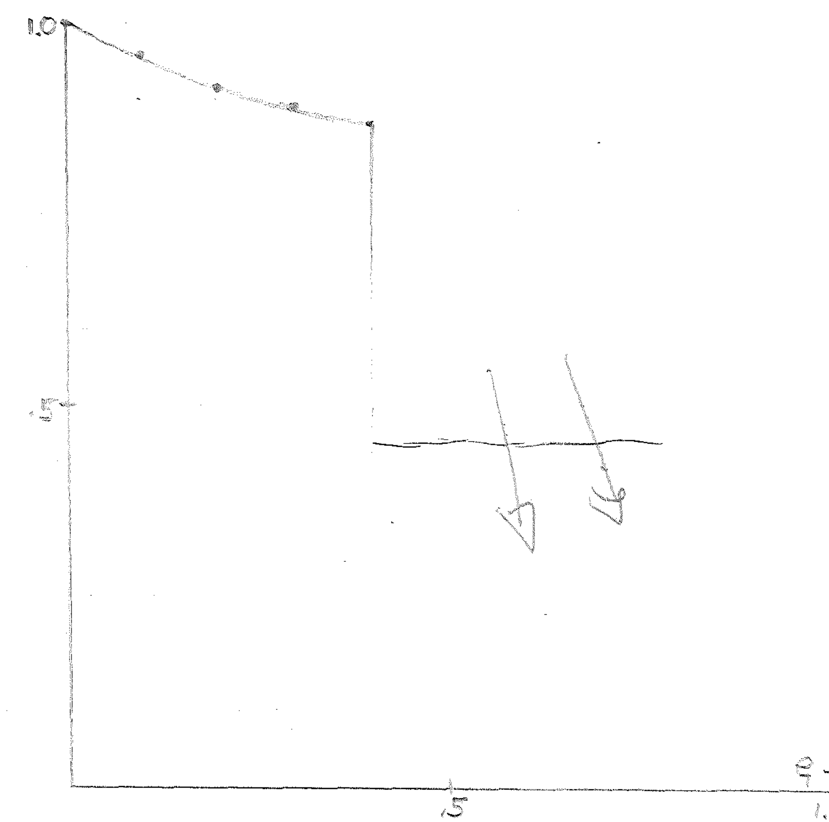
DATA (FROM MONROE)

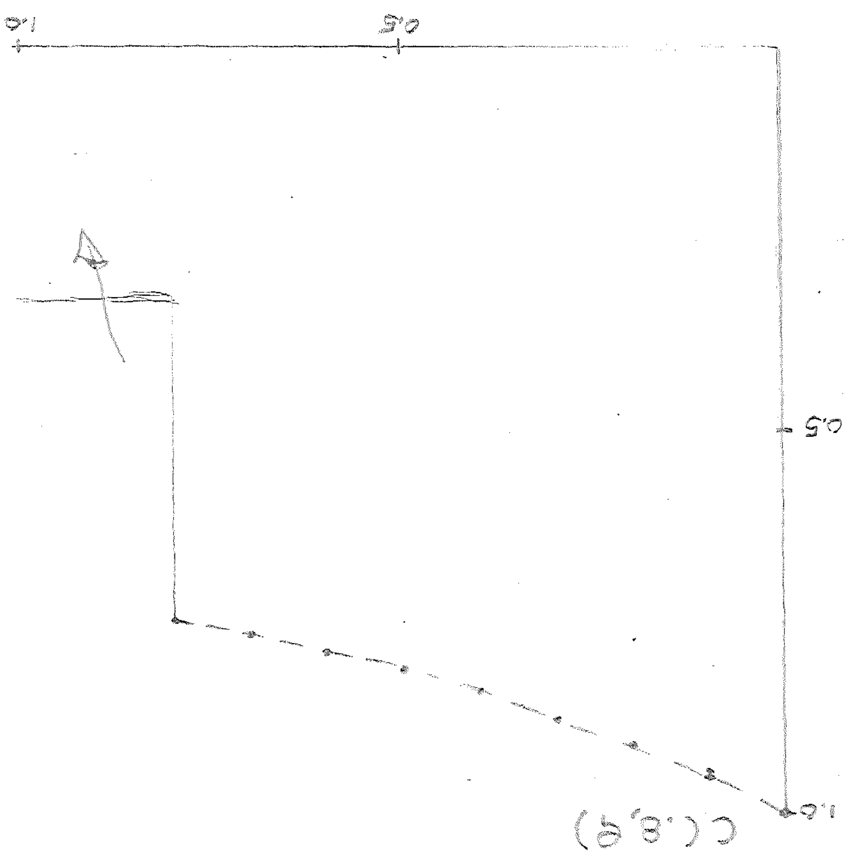
q	C(q,.2)	C(q,.4)	C(q,.6)	C(q,.8)	C(q,1.0)
0.00	1.000	1.000	1.000	1.000	1.000
0.10	0.960	0.958	0.956	0.954	0.953
0.20	0.927	0.921	0.917	0.914	0.912
0.30	0.553	0.889	0.883	0.878	0.874
0.40	0.553	0.862	0.853	0.846	0.841
0.50	0.553	†0.453	0.826	0.818	0.811
0.60	0.553	0.453	0.804	0.793	0.785
0.70			0.371	0.772	0.762
0.80			0.371	0.753	0.742
0.90				0.304	0.725
1.00					0.709

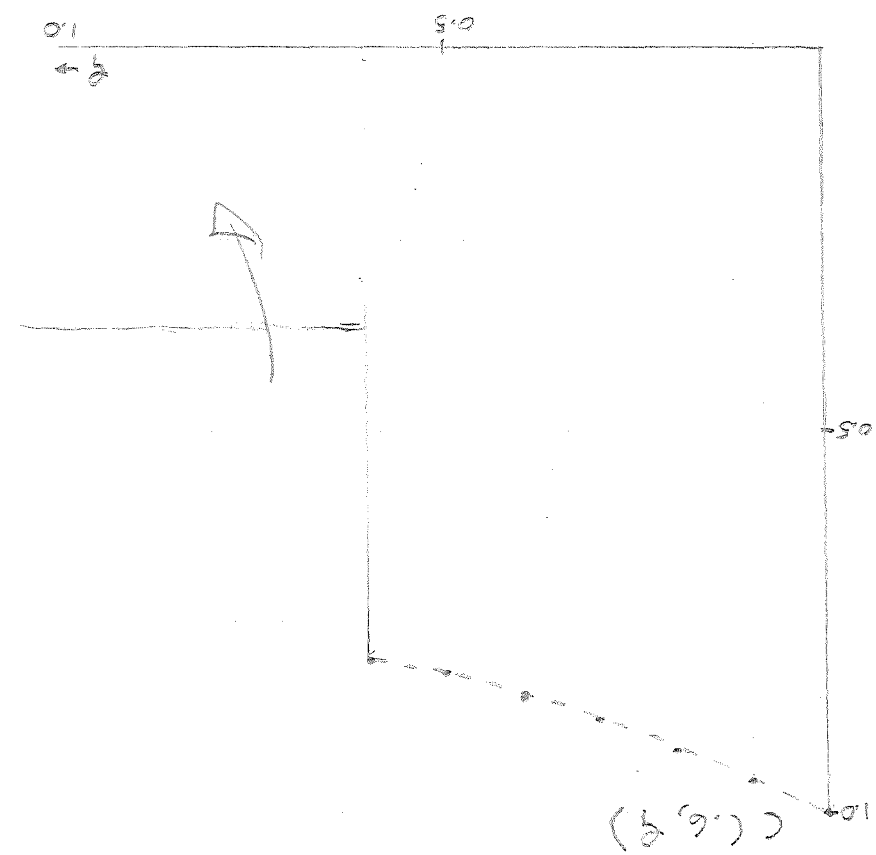
C(.2, q)



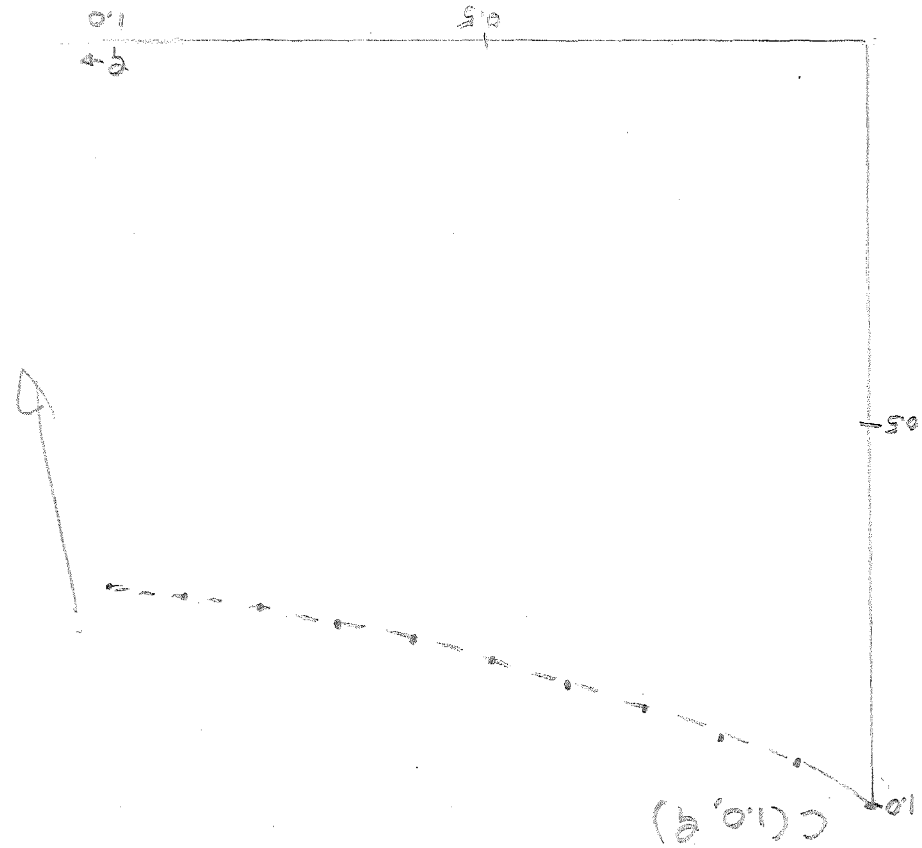
C(.4, q)







PADMASTER® / Made in U.S.A.



DISCUSSION:

IN DERIVATION, $e^{-(s+\alpha)}$ WAS SET $\ll 1$ TO EXPRESS SHORT TERM RESPONSE.

$$e^{-(s+\alpha)} \ll 1 \qquad (s, \alpha > 0)$$

$\Rightarrow \alpha$ BE MADE VERY LARGE

$\alpha = \frac{2hl}{FU} \Rightarrow l$ MUST BE MADE VERY LARGE

THIS SIMULATES SHORT TIME RESPONSE, IN THAT DRUG AT $t = 0^+$ WILL REGARD l AS NEAR INFINATE, HAVING NOT "LEARNED" OF ITS VALUE.

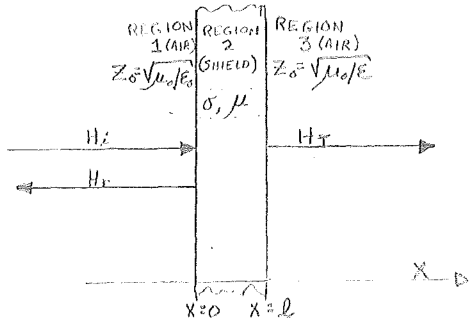
ON $C(\tau, \varrho)$ GRAPHS, THE DRUG ALL BUT CEASES TO SEEP AT $\tau = \varrho$. WHEN $\tau = \varrho$, $x = \mu t$, AND THE DRUG CONCENTRATION HAS REACHED EQUALIBRIUM IN THE TISSUE SPACE AND BLOOD, AT WHICH POINT OSMOSIS CEASES. ?

Point out where "seepage" occurs on the graphs.

Actually, we have not computed the drug concentration in the tissue space, so we cannot say precisely that osmosis ceases. It is not likely, however, that it ceases in the short times at which we are looking.

-10

5) a)



IN REGIONS 1 AND 3

$$\frac{\delta \mathcal{E}}{\delta x} = -\mu_0 \frac{\delta H}{\delta t} \sim \frac{\delta V}{\delta x} = -L_0 \frac{\delta i}{\delta t}$$

$$\frac{\delta H}{\delta x} = -\varepsilon_0 \frac{\delta \mathcal{E}}{\delta x} \sim \frac{\delta i}{\delta x} = -C_0 \frac{\delta V}{\delta t}$$

IN REGION 2

$$\frac{\delta \mathcal{E}}{\delta x} = -\mu \frac{\delta H}{\delta t} \sim \frac{\delta V}{\delta x} = -L \frac{\delta i}{\delta t}$$

$$\frac{\delta H}{\delta x} = -\sigma \mathcal{E} \sim \frac{\delta i}{\delta t} = -GV$$

#(1,3) REGIONS MAY BE MODELED BY LOSSLESS TRANSMISSION LINES. #2 REGION MAY BE MODELED AS TRANSMISSION LINE WITH ONLY DISTRIBUTED INDUCTANCE AND DISTRIBUTED SHUNT CONDUCTANCE. ✓

ANALOGIES

$$\mathcal{E} \sim V,\ H \sim i,\ \mu_0 \sim L_0,\ \mu \sim L,\ \varepsilon_0 \sim C_0,\ \sigma \sim G$$

IN REGION 2

$$\frac{d\mathbb{V}(x,s)}{dx} = -sLI(x,s) - \cancel{i(x,0)}^{\,0}$$

$$\Rightarrow \frac{d^2\mathbb{V}(x,s)}{dx} = -sL\frac{dI(x,s)}{dx}$$

$$\frac{dI(x,s)}{dx} = -G\mathbb{V}(x,s)$$

$$\therefore \frac{d^2\mathbb{V}(x,s)}{dx} = sLG\,\mathbb{V}(x,s)$$

$$= \gamma^2(s)\mathbb{V}(x,s) \ni \gamma(s) = \sqrt{sLG}$$

SOLUTION OF DIFFERENTIAL:

$$\mathbb{V}(x,s) = Ae^{-\gamma(s)x} + Be^{\gamma(s)x}$$

$$I(x,s) = \frac{1}{Z_c(s)}\left[Ae^{-\gamma(s)x} - Be^{\gamma(s)x}\right]$$

$$Z_c^2(s) = \frac{\cancel{R}^{\,0} + sL}{G + \cancel{sC}_{\,0}} \Rightarrow Z_c(s) = \sqrt{\frac{sL}{G}}$$

$V_i(s)$ AND $I_i(s)$ = INCIDENT VOLTAGE AND CURRENT @ X=0 (LAPLACE TRANSFORMS THEREOF)

$V_r(s)$ AND $I_r(s)$ = REFLECTED VOLTAGE AND CURRENT @ X=0 (LAPLACE TRANSFORMS THEREOF)

$V_t(s)$ AND $I_t(s)$ = TRANSMITTED VOLTAGE AND CURRENT @ X=0 (LAPLACE TRANSFORMS THEREOF)

BOUNDRY CONDITIONS:

CONTINUITY AT X=0 ⇒

$$V: \quad A + B = V_i(s) + V_r(s)$$

$$I: \quad \frac{1}{Z_c(s)}(A - B) = I_i(s) + I_r(s)$$

CONTINUITY AT X=ℓ ⇒

$$V: \quad A e^{-\gamma(s)\ell} + B e^{\gamma(s)\ell} = V_t(s)$$

$$I: \quad \frac{1}{Z_c(s)}\left(A e^{-\gamma(s)\ell} - B e^{\gamma(s)\ell}\right) = I_t(s)$$

ALSO $Z_0 = \frac{V_i}{I_i} = \frac{V_t}{I_t} = \frac{V_r}{I_r} \quad \left(= \sqrt{L_0/C_0}\right)$

THEN

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/Z_c(s) & -1/Z_c(s) & -1 & 1 & 0 & 0 \\ 0 & 0 & e^{-\gamma(s)\ell} & +e^{\gamma(s)\ell} & 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{Z_c(s)}e^{-\gamma(s)\ell} & \frac{1}{-Z_c(s)}e^{\gamma(s)\ell} & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & Z_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & Z_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & Z_0 \end{bmatrix} \begin{bmatrix} V_i(s) \\ V_r(s) \\ A \\ B \\ I_i(s) \\ I_r(s) \\ V_t(s) \\ I_t(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

THROUGH THE USE OF DETERMINANTS AND OTHER ALGEBRAIC TRICKERY:

Could we see a step or two.

$$\frac{I_t(s)}{I_i(s)} = \left[\cosh \gamma(s)\ell + \frac{1}{2}\left(\frac{Z_0}{Z_c(s)} + \frac{Z_c(s)}{Z_0}\right)\sinh \gamma(s)\ell\right]^{-1}$$

$$\gamma(s) = \sqrt{sLG}\,; \quad Z_0 = \sqrt{L_0/C_0}\,; \quad Z_c(s) = \sqrt{\frac{sL}{G}}$$

BY ANALOGY:

$$G(s) = \frac{H_t(s)}{I_i(s)} = \left[\cosh \gamma(s)\ell + \frac{1}{2}\left(\frac{Z_0}{Z_c(s)} + \frac{Z_c(s)}{Z_0}\right)\sinh \gamma(s)\ell\right]^{-1}$$

$$\gamma(s) = \sqrt{\mu\sigma s}\,; \quad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,; \quad Z_c(s) = \sqrt{\frac{\mu s}{\sigma}}$$

b) CHARACTERISTIC EQUATION:

$$\cosh \gamma(s) l + \frac{1}{2}\left(\frac{Z_0}{Z_c(s)} + \frac{Z_c(s)}{Z_0}\right) \sinh \gamma(s) l = 0$$

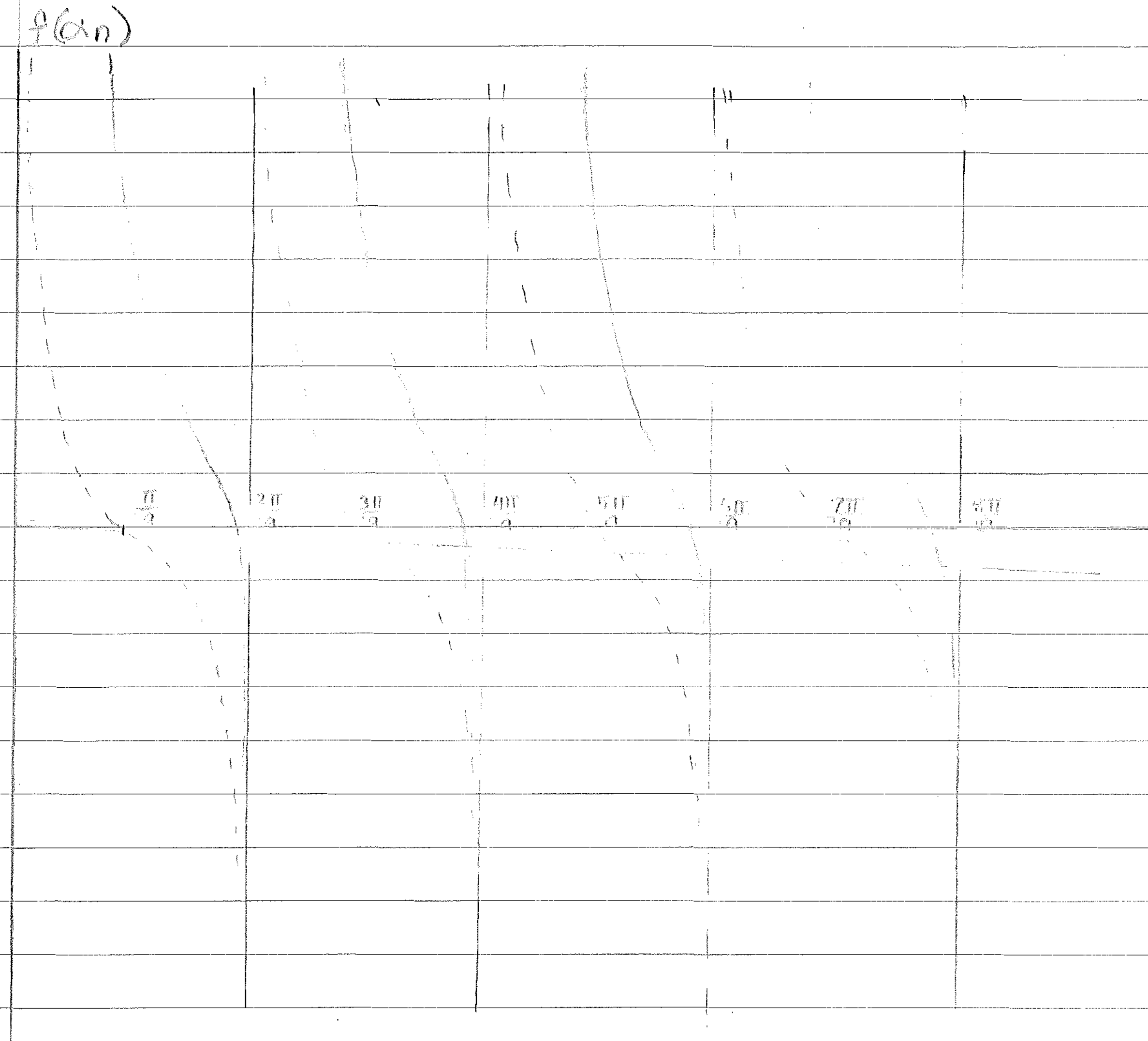
$$\Rightarrow \coth \gamma(s) l = -\frac{1}{2}\left(\frac{Z_0}{Z_c(s)} + \frac{Z_c(s)}{Z_0}\right); \quad Z_c(s) = \sqrt{\frac{\mu s}{\sigma}}$$

$$\coth x = j \cot jx \; ; \; -\alpha_n^2 = s_n$$

$$\therefore \cot \alpha_n \sqrt{\mu \sigma} = -\frac{1}{2}\left[\frac{Z_0}{\alpha_n \sqrt{\mu/\sigma}} - \frac{\alpha_n \sqrt{\mu/\sigma}}{Z_0}\right]$$

$$\Rightarrow \cot a\alpha_n + \frac{b}{\alpha_n} - c\alpha_n = 0$$

$$a = \sqrt{\mu \sigma} = \sqrt{(4\pi \times 10^{-7}) \times 10^{5}} = .354$$
$$Z_0 = \sqrt{\mu_0 / \varepsilon_0} = 120\pi = 377$$
$$\sqrt{\mu/\sigma} = 2\sqrt{\pi} \times 10^{-6} = 3.54 \times 10^{-6}$$
$$b = \frac{Z_0}{2\sqrt{\mu/\sigma}} = 5.32 \times 10^{7}$$
$$c = \frac{\sqrt{\mu/\sigma}}{2 Z_0} = 4.69 \times 10^{-9}$$



DISREGARDING MINIMAL $-c\alpha_n$ TERM ($c<10^{-9}$)
AND RECOGNIZING THAT b/α_n TERM
WILL BULGE COTANGENT CURVE TOWARD
$\frac{2n\pi}{2a}$ ASYMPTOTE, ROOTS FOR TRANSCENDENTAL
EQUATION $\cot a\alpha_n + b/\alpha_n - c\alpha_n = f(\alpha_n)$ MAY
BE ESTIMATED AT $\frac{2n\pi}{a}$ FOR $1<n<10$.

$$\therefore \quad \alpha_n = \frac{2n\pi}{a} \quad n=1,2,\ldots,10$$
$$s_n = -\alpha_n^2 = -\left(\frac{2n\pi}{a}\right)^2 \quad n=1,2,\ldots,10$$

NOTE: PREVIOUS TO CLASS, IT WAS NOTED THAT
THE COTANGENT HAS A PERIOD OF π INSTEAD
OF 2π, WHICH MESSES UP REST OF PROBLEM.
(SIGH!). AT ANY RATE, WHEREVER A FACTOR OF
2π EXISTS, π SHOULD BE SUBSTITUTED HENCEFORTH)

c) $G(0) = (1 + \frac{1}{2} Z_0 \ell \sigma)^{-1} = .531 \times 10^{-7}$

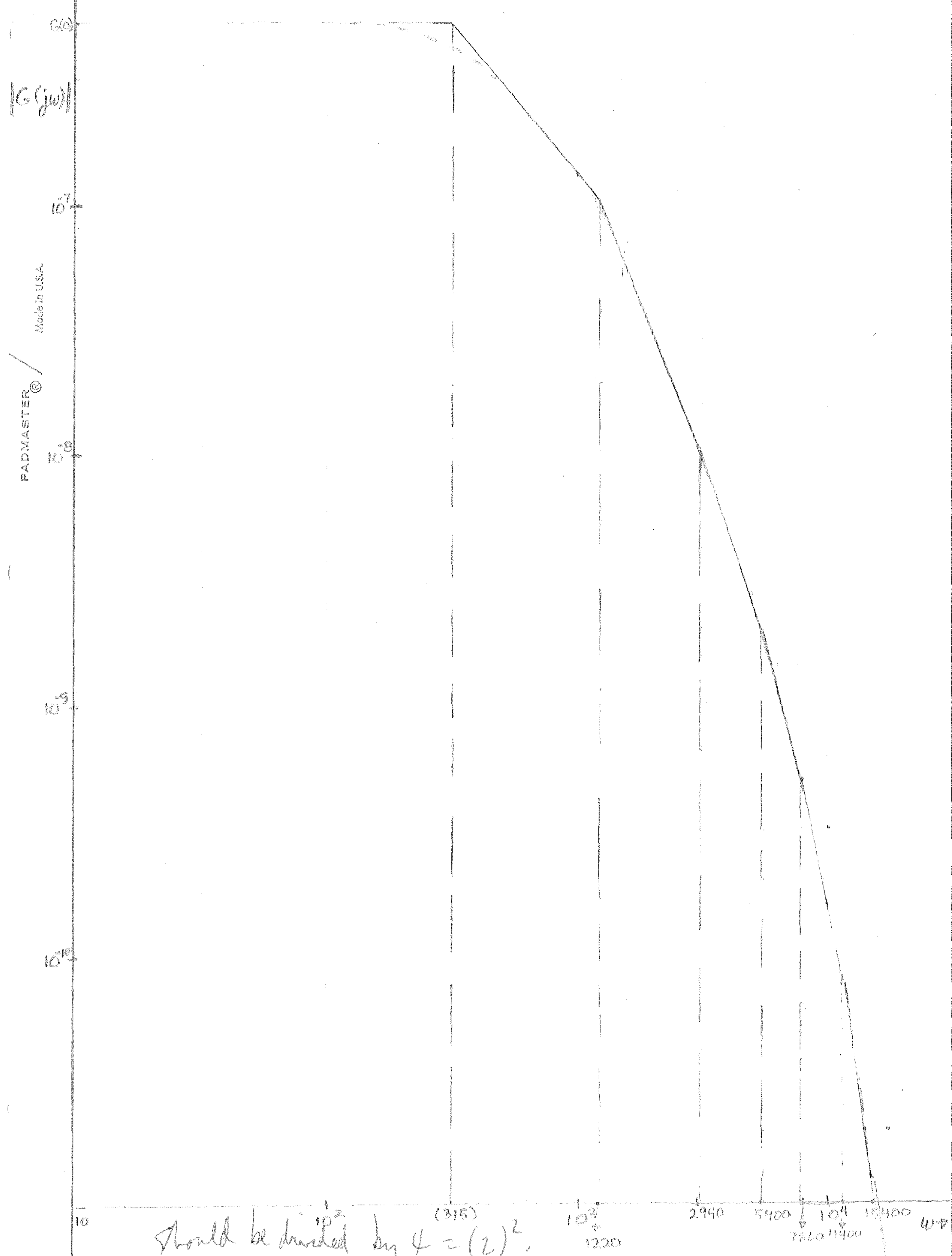
$$\Rightarrow G(s) = G(0) \left[\prod_{n=1}^{10} (1 - s/s_n) \right]^{-1}$$

$$= G(0) \left[\prod_{n=1}^{10} \left(1 + \frac{j\omega}{(\frac{2n\pi}{a})^2}\right) \right]^{-1}$$

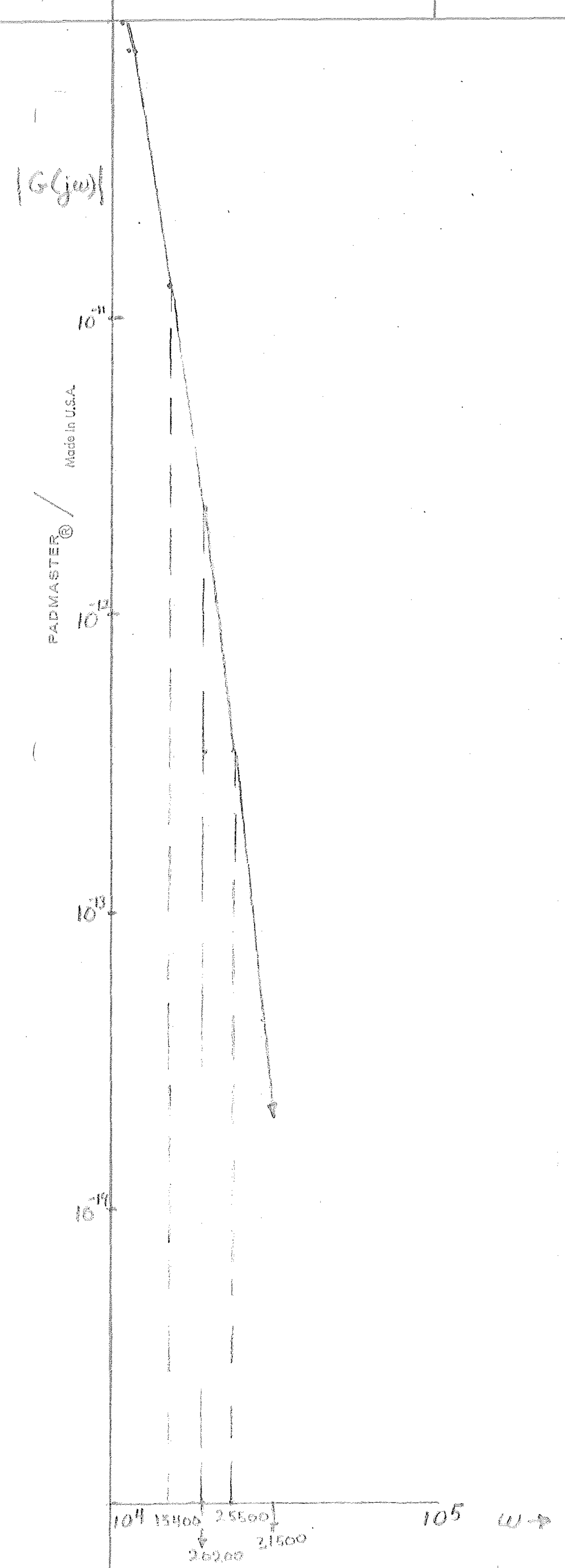
$$= \frac{G(0)}{(1 + j\omega/(\frac{2\pi}{a})^2)(1 + j\omega/(4\pi/a)^2)(1 + j\omega/(6\pi/a)^2) \cdots (1 + j\omega/(20\pi/a)^2)}$$

n	$\frac{2\pi n}{a}$	$\left(\frac{2\pi n}{a}\right)^2$
1	17.75	315
2	35.50	1220
3	54.25	2940
4	71.00	5400
5	88.75	7860
6	106.50	11400
7	124.25	15400
8	142.00	20200
9	159.75	25500
10	177.50	31500

USING BODE PLOT →



Should be divided by 4 = (2)^2.


|G(jω)|
10-11
10-12
10-13
10-14
10^4
15400
25500
10^5
ω→
20200
21500

DISCUSSION:

THE SHIELD ACTS AS A LOW PASS FILTER, BLOCKING HARMFUL HIGH-FREQUENCY ELECTRO-MAGNETIC RADIATION, NOTICEABLE ATTENUATION BEGINNING AT ABOUT .3 kHz, AND RAPIDLY INCREASING, UNTIL AT 30 kHz, THE ORIGINAL LOW-FREQUENCY ATTENUATION IS CUT ABOUT 7 DECADES

ONE OBVIOUS WAY TO IMPROVE SHIELDING WOULD BE TO DECREASE $G(0) = [1 + \frac{1}{2} l Z_0 \sigma]^{-1}$, WHICH WOULD IMPLY INCREASING THE SHIELD'S THICKNESS, AND/OR INCREASING ITS CONDUCTIVITY. ✓

TO LOWER THE FIRST NOTICEABLE POINT OF ATTENUATION, THE FACTOR $\frac{2\pi}{a}$ NEED BE DECREASED, OR INCREASE $a = \sqrt{\mu\sigma}$ ($\Rightarrow$ INCREASE μ OR σ OF SHIELD). THIS WOULD ALSO DECREASE THE INTERVALS BETWEEN $\left(\frac{2\pi n}{a}\right)^2$ AND $\left(\frac{2\pi(n+1)}{a}\right)^2$, THUS SHARPENING CUTOFF. ✓

Good work

BOB MARKS

175
200

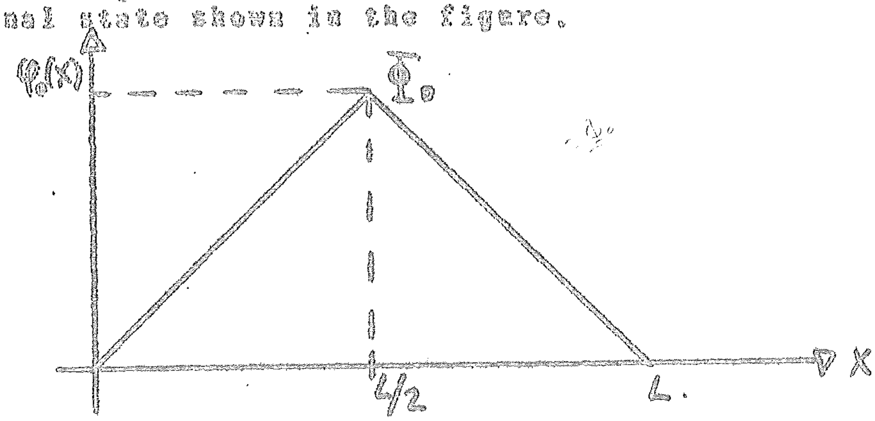
TRAVELING WAVES: FINAL EXAM

Fall Quarter, Nov. 1971

The three foundations of learning:
Seeing much, suffering much,
and studying much.

Catherall.

1. A rod of length L, fixed at both ends, is in the initial torsional state shown in the figure.



a. Determine the Laplace transform, $\Phi(x,s)$.

b. Determine and plot $\varphi(x,t)$ for $t = 0, T/8, T/4, 3T/8, T/2$, where T is the period of vibration.

c. Compare this result with those of problems (1-5) and (1-6). Be concise but accurate.

d. Sketch the torque distribution along the rod at the same instants of time listed in part b. Interpret the results in terms of traveling waves.

4. A brick sliding across a table with a velocity of 10 m/sec. comes to rest at a uniform rate (starting at t=0) in a distance of 1 meter.

(a) Find the rise in temperature of the brick-table interface at the time the brick stops.

(b) Find the maximum temperature of the brick-table interface, and calculate the time at which it occurs.

DATA: Length of brick = 8 inches.
Width of brick = 4 inches.
Weight of brick = 5 KG.
Thermal conductivity of brick = 0.5 $\frac{\text{watt}}{\text{Cm-}^\circ\text{C}}$
Heat capacity of brick = 4.8 $\frac{\text{watt-sec}}{\text{Cm}^3\text{-}^\circ\text{C}}$

Regard the brick as infinitely thick, and make the problem one-dimensional. You will need the fact that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}\,u_{-1}(t).$$

Reference: N. W. McClachlan, Complex Variable Theory and Transform Calculus, Cambridge University Press, 1953, pp. 307-310.

[illegible] theory of magneto-elastic vibrations (see [illegible] Jacingh, "Thermal Noise in Spin-Phonon Systems", IEEE Transactions on Sonics and Ultrasonics, Vol. SU-16, No. 3, July, 1969, pp. 147-156), the displacement, u, of an elastic system is related to the components, S_x, S_y, of the magnetic dipole moment of "atomic spin" by the equations

$$\text{(i)} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial z^2} + Ac^2 \frac{\partial S_x}{\partial z}$$

$$\text{(ii)} \qquad \frac{\partial S_x}{\partial t} = -\omega_0 S_y, \quad \frac{\partial S_y}{\partial t} = \omega_0 S_x + \omega_0 A \frac{\partial u}{\partial z},$$

where A is a positive constant, less than unity, involving "magnetostrictive" coupling coefficients, and a d.c. magnetic field. ω_0 is the resonant frequency of precession of the atomic spins, and $c^2 = \mu/\rho$ where μ is an elastic constant and ρ the mass density.

(a) Take the Fourier transform of (i) and (ii), then eliminate S_x and S_y to show that the Fourier transform, $U(z,\omega)$ of $u(z,t)$ satisfies

$$\frac{d^2 U}{dz^2} = -\frac{\omega^2}{c^2}\left(\frac{\omega^2 - \omega_0^2}{\omega^2 - \omega_1^2}\right) U,$$

where $\omega_1^2 = \omega_0^2(1 - A^2)$.

(b) Sketch $c\beta(\omega)$, $c\dfrac{\partial \beta}{\partial \omega}$ and $v_g(\omega)/c$ where $v_g(\omega)$ is the group velocity, versus ω. Make these sketches similar to Figure 2-9. Identify cut-off and propagating frequency ranges.

(c) The incremental equivalent circuit of the system consists of a shunt capacitance per unit length of value $1/\mu$ and a series impedance per unit length $Z(\omega)$. Show that

$$Z(\omega) = j\omega\rho\left(\frac{\omega^2 - \omega_0^2}{\omega^2 - \omega_1^2}\right).$$

Draw an equivalent LC-circuit for $z(\omega)$.

What does this "look like" at high frequencies $(\omega \gg \omega_0)$.

(d) Show that the characteristic impedance $Z_c(\omega)$ is given by

$$Z_c(\omega) = Z_0\left(\frac{\omega^2 - \omega_0^2}{\omega^2 - \omega_1^2}\right)^{1/2}, \qquad Z_0 = \sqrt{\mu\rho}.$$

Discuss this result in light of your sketches in (b).

(e) Using the $c\beta(\omega) - \omega$ dispersion curve and space-time ray theory, as described in Section 2-10, discuss the possibility of using this magneto-elastic system as a pulse-compression filter. Describe (qualitatively, using simple sketches) the appropriate frequency vs. time characteristics of the FM-pulse. You need not write down any specific formulas. (READ 2-10)

THE SMITH CHART

(Chap. 4)

4\. A transmission-line of characteristic impedance $Z_0 = 50\ \Omega$ is terminated in a load $50 + j100\ \Omega$, at the frequency of interest.

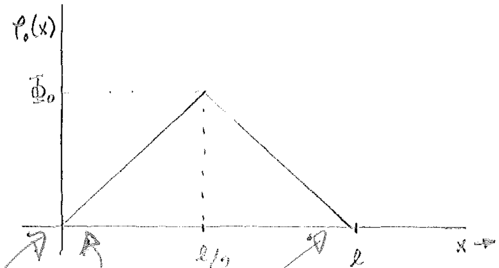
(a) Using the incident voltage wave phasor at the load as the reference, sketch phasor diagrams showing phase angles and relative magnitudes of the incident and reflected voltage wave phasors at the load (i), at a point on the line corresponding to a voltage minimum (ii), and at a point on the line corresponding to a voltage maximum (iii). Label the points on the line.

(b) Using the results of (a) determine the SWR.

(c) If the line is 2.10 wavelengths long, design a single-susceptance tuner to match the load to the source. Indicate both the value of the lumped element and the length of an equivalent short circuited stub.

(d) Design a double-stub tuner to match the line.

1 a)



$$* \mathcal{L}\{\rho_0(x)\} = \Phi(k,0) = \frac{2\Phi_0}{\ell}\frac{1}{k^2}\left[1 - e^{-\frac{k\ell}{2}}\right]^2$$

Is this true?

BOUNDRY CONDITIONS:

$$\rho(0,t) = \frac{\delta \rho(0,t)}{\delta x} = \Phi(0,s) = \frac{d\Phi(0,s)}{dx} = 0$$

FROM PROBLEM 1-5:

$$\frac{d^2\Phi(x,s)}{dx^2} = \frac{1}{V_p^2}\left[s^2\Phi(x,s) - s\rho_0(x) - \frac{\delta\rho_0(x)}{\delta t}\right] \qquad \omega_0(x) = 0$$

TAKING SPACIAL LAPLACE:

$$k^2\Phi(k,s) - k\Phi(0,s) - \frac{\delta\Phi(0,s)}{\delta x} = \frac{1}{V_p^2}\left[s^2\Phi(k,s) - \frac{2\Phi_0 s}{\ell}\frac{1}{k^2}\left(1 - e^{-\frac{k\ell}{2}}\right)^2\right]$$

$$\Rightarrow \Phi(k,s) = \frac{-2s\Phi_0}{\ell k^2 V_p^2}\left[1 - e^{-\frac{k\ell}{2}}\right]^2\left(k^2 - \left(\frac{s}{V_p}\right)^2\right)$$

$$= \frac{-2s\Phi_0}{\ell V_p^2}\left[\frac{1}{k^2(k^2 + (\frac{js}{V_p})^2)} - \frac{2e^{-k\ell/2}}{k^2(k^2 + (\frac{js}{V_p})^2)} + \frac{e^{-k\ell}}{k^2(k^2 + (\frac{js}{V_p})^2)}\right]$$

Spatial Laplace is not suitable for a two point boundary value problem.

$$\mathcal{L}^{-1}\{e^{-x_0 k}F(k)\} = f(x - x_0)\mu(x - x_0)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{k^2(k^2 + (\frac{js}{V_p})^2)}\right\} = \frac{1}{(\frac{js}{V_p})^3}\cdot\left[\frac{js}{V_p}x - \sin\frac{js}{V_p}x\right]\mu(x)$$

$$\Rightarrow \Phi(x,s) = \mathcal{L}^{-1}\{\Phi(k,s)\}$$

$$= \frac{-j2s\Phi_0}{\ell V_p^2}\left(\frac{V_p}{s}\right)^3\left[\left(\frac{js}{V_p}x - \sin\frac{js}{V_p}x\right)\mu(x) - \left(\frac{j2s}{V_p}(x - \tfrac{\ell}{2}) - 2\sin\frac{js}{V_p}(x - \tfrac{\ell}{2})\right)\mu(x - \tfrac{\ell}{2}) + \left(\frac{js}{V_p}(x - \ell) - \sin\frac{js}{V_p}(x - \ell)\right)\mu(x - \ell)\right]$$

$$= \frac{2\Phi_0 V_p}{\ell s^2}\left[\left(\frac{s}{V_p}x + j\sin j\frac{sx}{V_p}\right)\mu(x) - \left(\frac{2s}{V_p}(x - \tfrac{\ell}{2}) + j2\sin j\frac{s}{V_p}(x - \tfrac{\ell}{2})\right)\mu(x - \tfrac{\ell}{2}) + \left(\frac{s}{V_p}(x - \ell) + j\sin j\frac{s}{V_p}(x - \ell)\right)\mu(x - \ell)\right]$$

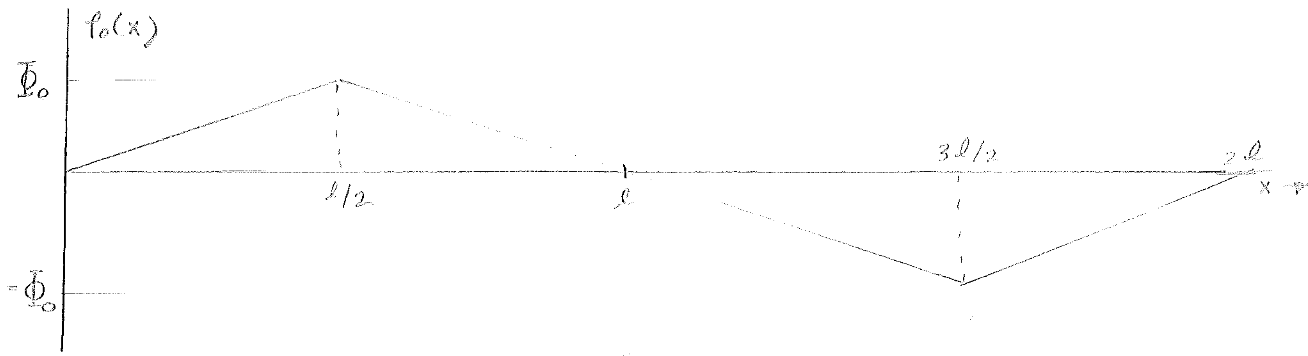
$$-\sinh u = +j\sin ju$$

$$\therefore \Phi(x,s) = \frac{2\Phi_0 V_p}{\ell s^2}\left[\left(\frac{s}{V_p}x - \sinh\frac{sx}{V_p}\right)\mu(x) - \left(\frac{2s}{V_p}(x - \tfrac{\ell}{2}) - 2\sinh\frac{s(x - \ell/2)}{V_p}\right)\mu(x - \tfrac{\ell}{2}) + \left(\frac{s}{V_p}(x - \ell) - \sinh\frac{s(x - \ell)}{V_p}\right)\mu(x - \ell)\right]$$

Why is your approach so similar to Prinzipes'? Please see me about this!

* COOPER-McGILLEM "METHODS OF SIGNAL AND SYSTEMS ANALYSIS" [illegible]

b)



$$\rho_0(x) = -2\Phi_0\left[\frac{x}{\ell} - 1\right] \qquad \ell/2 < x < 3\ell/2$$

$$= 2\Phi_0\left[\frac{x}{\ell} - 2\right] \qquad 3\ell/2 < x < 5\ell/2$$

ALSO $\rho_0(x) = \rho_0(x + 2\ell)$

$\rho_0(x) = -\rho_0(-x)$

$$b_n = \frac{1}{\ell}\int_\alpha^{\alpha+2\ell} \rho_0(x) \sin\frac{n\pi x}{\ell}\,dx \Rightarrow \alpha = \ell/2$$

NOW

$$\int_{\ell/2}^{3\ell/2} \rho_0(x)\sin\frac{n\pi x}{2}dx = -2\Phi_0\int_{\ell/2}^{3\ell/2}\left(\frac{x}{\ell} - 1\right)\sin\frac{n\pi x}{\ell}dx$$

$$= -2\Phi_0\left[\frac{1}{\ell}\int_{\ell/2}^{3\ell/2} x\sin\frac{n\pi x}{\ell} - \int_{\ell/2}^{3\ell/2}\sin\frac{n\pi x}{\ell}dx\right]$$

$$= -2\Phi_0\left[\frac{\ell}{n^2\pi^2}\sin\frac{n\pi x}{\ell} - \frac{\ell x}{n\pi}\cos\frac{n\pi x}{\ell} + \frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right]_{\ell/2}^{3\ell/2}$$

$$= -2\Phi_0\left[\frac{\ell}{n^2\pi^2}\sin\frac{n\pi x}{\ell} + \frac{\ell}{n\pi}(1 - x)\cos\frac{n\pi x}{\ell}\right]_{\ell/2}^{3\ell/2}$$

$$= -2\Phi_0\left[\frac{\ell}{n^2\pi^2}\left(\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right) + \frac{\ell}{n\pi}\left(\frac{2 - 3\ell}{2}\cos\frac{3n\pi}{2} - \frac{2 - \ell}{2}\cos\frac{n\pi}{2}\right)\right]$$

$$\int_{3\ell/2}^{5\ell/2} \rho_0(x)\sin\frac{n\pi x}{2}dx = 2\Phi_0\int_{3\ell/2}^{5\ell/2}\left[\frac{x}{\ell} - 2\right]\sin\frac{n\pi x}{\ell}dx$$

$$= 2\Phi_0\left[\frac{1}{\ell}\int_{3\ell/2}^{5\ell/2} x\sin\frac{n\pi x}{\ell}dx - 2\int_{3\ell/2}^{5\ell/2}\sin\frac{n\pi x}{\ell}dx\right]$$

$$= 2\Phi_0\left[\frac{\ell}{n^2\pi^2}\sin\frac{n\pi x}{\ell} - \frac{\ell x}{n\pi}\cos\frac{n\pi x}{\ell} + \frac{2\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right]_{3\ell/2}^{5\ell/2}$$

$$= 2\Phi_0\left[\frac{\ell}{n^2\pi^2}\sin\frac{n\pi x}{\ell} + \frac{\ell}{n\pi}(2 - x)\cos\frac{n\pi x}{\ell}\right]_{3\ell/2}^{5\ell/2}$$

$$= 2\Phi_0\left[\frac{\ell}{n^2\pi^2}\left(\sin\frac{5n\pi}{2} - \sin\frac{3n\pi}{2}\right) + \frac{\ell}{n\pi}\left(\frac{4 - 5\ell}{2}\cos\frac{5n\pi}{2} - \frac{4 - 3\ell}{2}\cos\frac{3n\pi}{2}\right)\right]$$

$$\therefore b_n = 2\Phi_0\Big[\frac{1}{n^2\pi^2}\Big(\sin\frac{5n\pi}{2} - \sin\frac{3n\pi}{2}\Big) + \frac{1}{n\pi}\Big(\frac{4-5\ell}{2}\cos\frac{5n\pi}{2} - \frac{4-3\ell}{2}\cos\frac{3n\pi}{2}\Big)$$
$$- \frac{1}{n^2\pi^2}\Big(\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\Big) - \frac{1}{n\pi}\Big(\frac{2-3\ell}{2}\cos\frac{3n\pi}{2} - \frac{2-\ell}{2}\cos\frac{n\pi}{2}\Big)\Big]$$
$$= 2\Phi_0\Big[\frac{1}{n^2\pi^2}\Big(\sin\frac{5n\pi}{2} - 2\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\Big)$$
$$+ \frac{1}{n\pi}\Big(\frac{4-5\ell}{2}\cos\frac{5n\pi}{2} - 3(1-\ell)\cos\frac{3n\pi}{2} + \frac{2-\ell}{2}\cos\frac{n\pi}{2}\Big)\Big]$$

NOW: $\sin\frac{5n\pi}{2} = -\sin\frac{3n\pi}{2} = \sin\frac{n\pi}{2}$

$\cos\frac{5n\pi}{2} = \cos\frac{3n\pi}{2} = \cos\frac{n\pi}{2}$

$$\Rightarrow b_n = 2\Phi_0\Big[\frac{1}{n^2\pi^2}\Big(\sin\frac{n\pi}{2} + 2\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\Big)\Big]$$
$$= \frac{8\Phi_0}{(n\pi)^2}\sin\frac{n\pi}{2}$$

FOR ODD n; $\sin\frac{n_o\pi}{2} = 1$, FOR $n_o = 1, 5, 9, 13, \ldots, (4m+1)$

$= -1$, FOR $n_o = 3, 7, 11, \ldots, (4m+3)$

$m = 0, 1, 2, 3, \ldots$

FOR EVEN n; $\sin\frac{n_e\pi}{2} = 0$

$$\therefore b_n = \frac{8\Phi_0}{(n_o\pi)^2}\ ;\ n_o = 1, 5, 9, 13, \ldots, (4m+1), \ldots$$
$$= \frac{-8\Phi}{(n_o\pi)^2}\ ;\ n_o = 3, 7, 11, 15, \ldots, (4m+3), \ldots$$
$$= \frac{8\Phi_0}{(n\pi)^2}\sin\frac{n\pi}{2}$$

$$\therefore \rho_o(x) = \Big[\frac{8\Phi_0}{\pi^2}\sum_{m=1}^{\infty}\frac{\sin\frac{n_o\pi}{2}}{n_o^2}\sin\frac{n\pi x}{\ell}\Big]\Big[\mu(x) - \mu(x-\ell)\Big]$$

$n_o = 2m - 1$

$$\frac{\delta^2 \varphi(x,t)}{\delta x^2} = \frac{1}{V_p^2}\frac{\delta^2 \varphi(x,t)}{\delta t^2}$$

$$\frac{d^2 \Phi(x,s)}{dx^2} = \left(\frac{s}{V_p}\right)^2 \Phi(x,s) - \frac{s}{V_p^2}\varphi_o(x,t) - \frac{1}{V_p^2}\frac{\delta \varphi_o(x,t)}{\delta t} \quad (\varphi_o(x,t)=0)$$

$$= \left(\frac{s}{V_p}\right)^2 \Phi(x,s) - \frac{8 s \Phi_o}{(V_p \pi)^2}\sum_{m=1}^{\infty} \frac{\sin \frac{n_o \pi}{2}}{n_o^2} \sin \frac{n_o \pi x}{\ell} \Rightarrow n_o = 2m-1$$

LET $\Phi(x,s) = \sum_{m=1}^{\infty} C_s \sin \frac{n_o \pi x}{\ell}$

$$\Rightarrow \frac{d^2 \Phi(x,s)}{dx^2} = \sum_{m=1}^{\infty} C_s \left(\frac{n_o \pi}{\ell}\right)^2 \sin \frac{n_o \pi x}{\ell}$$

$$\therefore \sum_{m=1}^{\infty} C_s \left(\frac{n_o \pi}{\ell}\right)^2 \sin \frac{n_o \pi x}{\ell} + \sum_{m=1}^{\infty} C_s \left(\frac{s}{V_p}\right)^2 \sin \frac{n_o \pi x}{\ell} = \frac{8 s \Phi_o}{(\pi V_p)^2}\sum_{m=1}^{\infty} \frac{\sin \frac{n_o \pi}{2}}{n_o^2} \sin \frac{n_o \pi x}{\ell}$$

ASSUMING m^{TH} COMPONENT IN EACH SUMMATION ARGUMENT IS EQUAL:

$$C_s\left[\left(\frac{n_o \pi}{\ell}\right)^2 + \left(\frac{s}{V_p}\right)^2\right] = \frac{8 \Phi_o s}{(\pi V_p)^2} \frac{\sin \frac{n_o \pi}{2}}{n_o^2}$$

$$\Rightarrow C_s = \frac{8 \Phi_o \sin \frac{n_o \pi}{2}}{(n_o \pi V_p)^2} \frac{s}{\left(\frac{n_o \pi}{\ell}\right)^2 + \left(\frac{s}{V_p}\right)^2}$$

$$= \frac{8 \Phi_o \sin \frac{n_o \pi}{2}}{(n_o \pi)^2} \frac{s}{s^2 + \left(\frac{n_o \pi V_p}{\ell}\right)^2}$$

$$\therefore \Phi(x,s) = \sum_{m=1}^{\infty} \frac{8 \Phi_o \sin \frac{n_o \pi}{2}}{(n_o \pi)^2}$$

$$\Rightarrow \varphi(x,t) = \sum_{m=1}^{\infty} \frac{8 \Phi_o \sin \frac{n_o \pi}{2}}{(n_o \pi)^2} \cos \frac{n_o \pi V_p t}{\ell} \sin \frac{n_o \pi x}{\ell}; \quad n_o = 2m-1$$

$$\gamma(x,t) = \sum_{m=1}^{\infty} \frac{8 \Phi_o C \sin \frac{n_o \pi}{2}}{(n_o \pi)} \cos \frac{n_o \pi V_p t}{\ell} \cos n_o \pi x/\ell$$

NORMALIZING PARAMETERS:

$$x_n = x/\ell$$

$$\frac{\varphi(x_n,t)}{\Phi_0} = \varphi_n(x_n,t)$$

$$\frac{\tau(x_n,t)}{c\,\Phi_0} = \tau_n(x_n,t)$$

$$\therefore \quad \varphi_n(x_n,t) = \sum_{m=1}^{\infty} \frac{8 \sin n_0\pi/2}{(n_0\pi)^2} \cdot \cos \frac{n_0\pi v_p}{\ell} t \, \sin n_0\pi x_n$$

$$\tau_n(x_n,t) = \sum_{m=1}^{\infty} \frac{8 \sin n_0\pi/2}{n_0\pi} \cdot \cos \frac{n_0\pi v_p}{\ell} t \, \cos n_0\pi x_n$$

EVALUATION @ $t = 0, \frac{T}{8}, \frac{T}{4}, \frac{3T}{8}, \frac{T}{2}$

FUNDEMENTAL FREQUENCY = $\frac{\pi v_p}{\ell} \Rightarrow f_0 = \frac{v_p}{2\ell} \Rightarrow T = \frac{1}{f_0} = \frac{2\ell}{v_p}$

$\Rightarrow$ EVALUATION FOR $\frac{k}{8}T = \frac{k\ell}{4v_p}$; $k = 0, 1, 2, 3, 4$

EVALUATE:

$$\left.\begin{aligned} \varphi_n(x_n,t) &= \sum_{m=1}^{\infty} \frac{8 \sin n_0\pi/2}{(n_0\pi)^2} \cos \frac{k n_0\pi}{4} \sin n_0\pi x_n \\ \tau_n(x_n,t) &= \sum_{m=1}^{\infty} \frac{8 \sin n_0\pi/2}{n_0\pi} \cos \frac{k n_0\pi}{4} \cos n_0\pi x_n \end{aligned}\right\} k = 0, 1, 2, 3, 4$$

MACE

OPERATING SYSTEM

VERSION 4.1

JOB ORIGIN - EXPORT/IMPORT

```
X          X  RRRRRRRRRRR   RRRRRRRRRRR             JJ  MM        MM          44            44
XX        XX  RRRRRRRRRRRR  RRRRRRRRRRRR            JJ  MMM      MMM         444           444
XX        XX  RR        RR  RR        RR            JJ  MMMM    MMMM        4444          4444
 XX      XX   RR        RR  RR        RR            JJ  MM MM  MM MM       44 44         44 44
  XX    XX    RR        RR  RR        RR            JJ  MM  MMMM  MM      44  44        44  44
   XX  XX     RR        RR  RR        RR            JJ  MM   MM   MM     44   44       44   44
    XXXX      RR        RR  RR        RR            JJ  MM        MM    44    44      44    44
     XX       RRRRRRRRRRRR  RRRRRRRRRRRR            JJ  MM        MM   44     44     44     44
     XX       RRRRRRRRRRR   RRRRRRRRRRR             JJ  MM        MM  444444444444  444444444444
    XXXX      RR  RR        RR  RR                  JJ  MM        MM  444444444444  444444444444
   XX  XX     RR   RR       RR   RR                 JJ  MM        MM          44            44
  XX    XX    RR    RR      RR    RR                JJ  MM        MM          44            44
 XX      XX   RR     RR     RR     RR               JJ  MM        MM          44            44
XX        XX  RR      RR    RR      RR    JJ        JJ  MM        MM          44            44
XX        XX  RR       RR   RR       RR   JJJJJJJJJJJJ  MM        MM          44            44
X          X  RR        RR  RR        RR   JJJJJJJJJJ   MM        MM          44            44
```

```
====================================================================================================
====================================================================================================
====================================================================================================
====================================================================================================
====================================================================================================
====================================================================================================
```

```
                 PROGRAM TWSTRK(OUTPUT,TAPE5=OUTPUT)
000002           DIMENSION TORSN(5,51),TORQUE(5,51),X(51),
                +T1(51),T2(51),T3(51),T4(51),T5(51),Q1(51),Q2(51),Q3(51),Q4(51),
                +Q5(51)
000002           REAL N,K
000002           PI=4.*ATAN(1.)
000006           DO 90 I=1,5
000007           K=(I-1)
000011           DO 90 NX=1,51
000015           XN=(NX-1)
000016           X(NX)=XN/50.
000017           TORSN(I,NX)=0.0
000020           TORQUE(I,NX)=0.0
000021           DO 90 M=1,30
000023           N=M+M-1
000025           TORSNN=8.*SIN(N*PI/2.)*COS(K*N*PI/4.)*SIN(N*PI*X(NX))/(N*N*PI*PI)
000052           TORSN(I,NX)=TORSN(I,NX)+TORSNN
000056           TORKN=8.*SIN(N*PI/2.)*COS(K*N*PI/4.)*COS(N*PI*X(NX))/(N*PI)
000103           TORQUE(I,NX)=TORQUE(I,NX)+TORKN
000107        90 CONTINUE
000115           WRITE(5,16)
000121           WRITE(5,15)
000125           DO 91 J=1,51
000127           WRITE(5,14)X(J),(TORSN(L,J),L=1,5)
000140        91 CONTINUE
000142           WRITE(5,17)
000146           WRITE(5,15)
000152           DO 92 J=1,51
000154           WRITE(5,14)X(J),(TORQUE(L,J),L=1,5)
000165        92 CONTINUE
000167           DO 93 I=1,51
000204           T1(I)=TORSN(1,I)
000205           T2(I)=TORSN(2,I)
000207           T3(I)=TORSN(3,I)
000210           T4(I)=TORSN(4,I)
000212           T5(I)=TORSN(5,I)
000213           Q1(I)=TORQUE(1,I)
000215           Q2(I)=TORQUE(2,I)
000216           Q3(I)=TORQUE(3,I)
000220           Q4(I)=TORQUE(4,I)
000221           Q5(I)=TORQUE(5,I)
000223        93 CONTINUE
000227           WRITE(5,13)
000232           CALL SETPLT(1,X,T1,51,1H*,3,3HXT1)
000241           WRITE(5,13)
000245           CALL SETPLT(1,X,T2,51,1H*,3,3HXT2)
000254           WRITE(5,13)
000260           CALL SETPLT(1,X,T3,51,1H*,3,3HXT3)
000267           WRITE(5,13)
000273           CALL SETPLT(1,X,T4,51,1H*,3,3HXT4)
000302           WRITE(5,13)
000306           CALL SETPLT(1,X,T5,51,1H*,3,3HXT5)
000315           WRITE(5,13)
000321           CALL SETPLT(1,X,Q1,51,1H*,3,3HXQ1)
000330           WRITE(5,13)
000334           CALL SETPLT(1,X,Q2,51,1H*,3,3HXQ2)
000343           WRITE(5,13)
000347           CALL SETPLT(1,X,Q3,51,1H*,3,3HXQ3)
```

```
000356          WRITE(5,13)
000362          CALL SETPLT(1,X,Q4,51,1H*,3,3HXQ4)
000371          WRITE(5,13)
000375          CALL SETPLT(1,X,Q5,51,1H*,3,3HXQ5)
000404          WRITE(5,13)
000410          STOP
000412       13 FORMAT('7')
000412       14 FORMAT(5X,F5.3,5(1X,G9.2))
000412       15 FORMAT(' ROBERT J. MARKS II'/' TRAVELING WAVES FINAL'//,7X'X'5X'0'
               +,9X'T/8',7X'T/4',7X'3T/8',6X'T/2')
000412       16 FORMAT('7TORSIONAL DISPLACEMENT')
000412       17 FORMAT('7TORQUE DISTRIBUTION')
000412          END

PROGRAM LENGTH INCLUDING I/O BUFFERS
003712

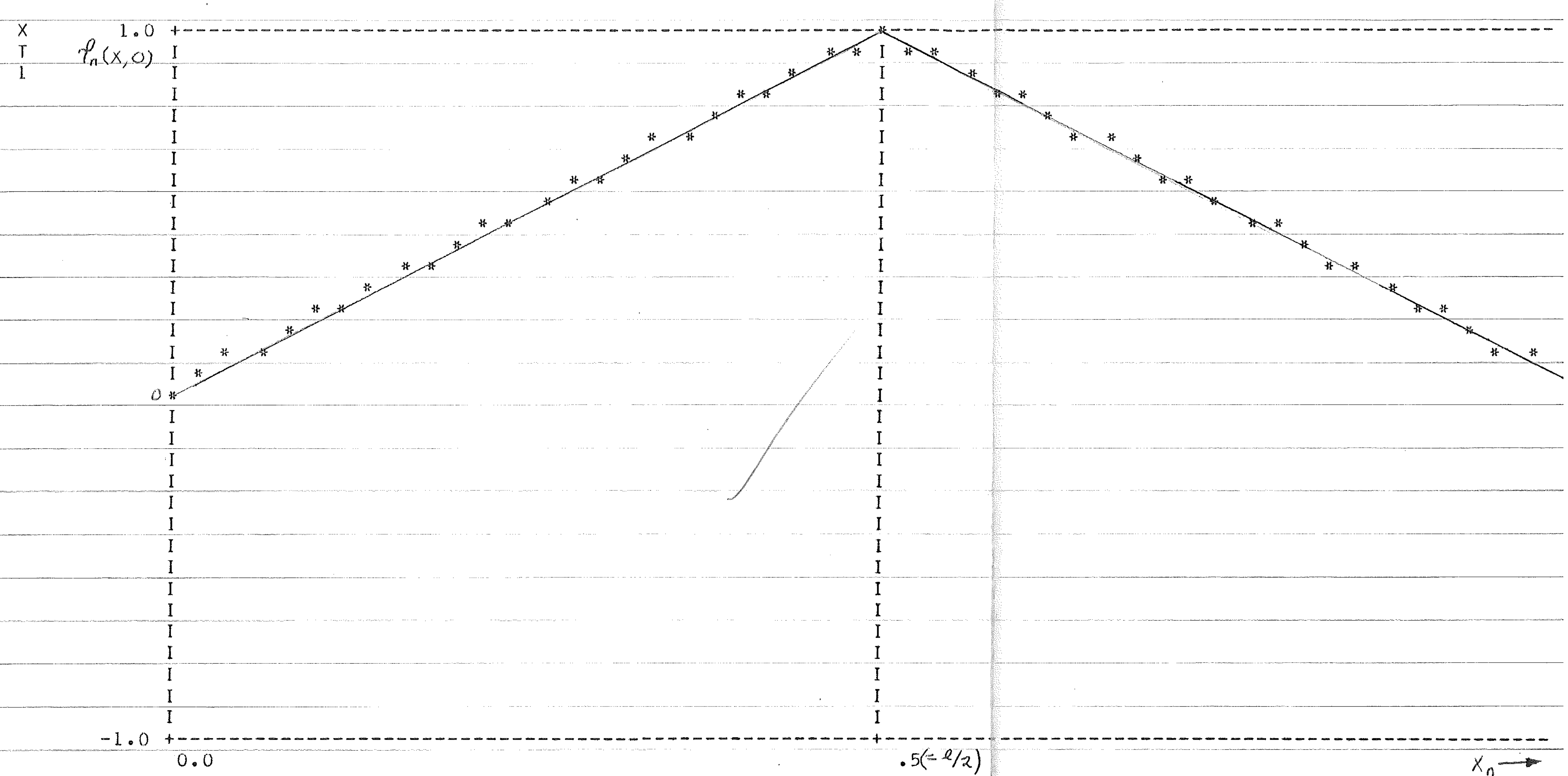
UNUSED COMPILER SPACE
025400
```

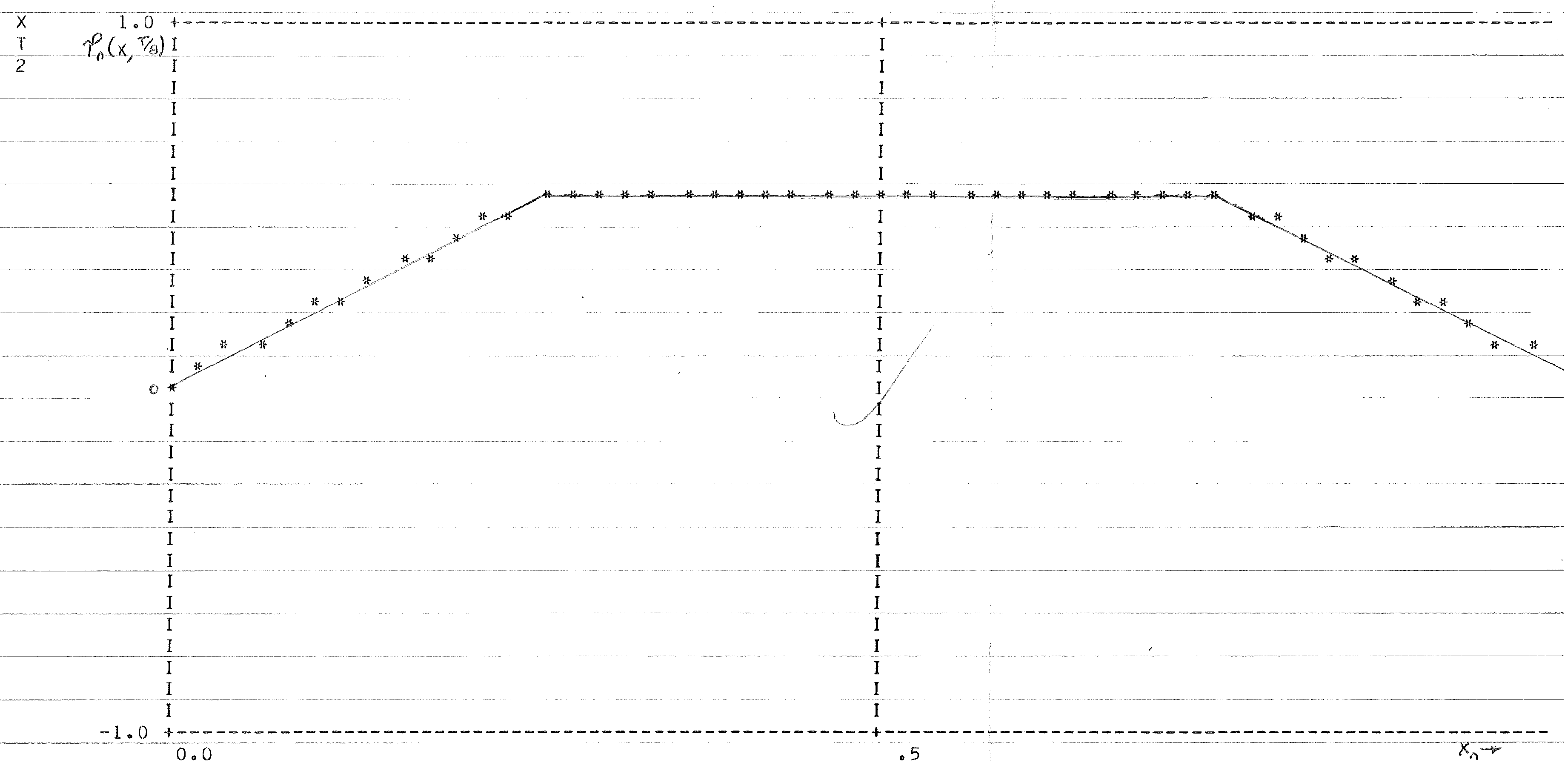
TORSIONAL DISPLACEMENT
ROBERT J. MARKS II
TRAVELING WAVES FINAL

X	0	T/8	T/4	3T/8	T/2
0.000	0.	0.	0.	0.	0.
.020	4.01E-02	3.99E-02	-3.59E-15	-3.99E-02	-4.01E-02
.040	7.99E-02	8.02E-02	-3.25E-15	-8.02E-02	-7.99E-02
.060	.12	.12	-3.43E-15	-.12	-.12
.080	.16	.16	-4.03E-15	-.16	-.16
.100	.20	.20	-3.65E-15	-.20	-.20
.120	.24	.24	-4.18E-15	-.24	-.24
.140	.28	.28	-4.45E-15	-.28	-.28
.160	.32	.32	-4.87E-15	-.32	-.32
.180	.36	.36	-4.92E-15	-.36	-.36
.200	.40	.40	-4.73E-15	-.40	-.40
.220	.44	.44	-4.99E-15	-.44	-.44
.240	.48	.48	-4.69E-15	-.48	-.48
.260	.52	.50	-5.01E-15	-.50	-.52
.280	.56	.50	-5.72E-15	-.50	-.56
.300	.60	.50	-5.94E-15	-.50	-.60
.320	.64	.50	-5.89E-15	-.50	-.64
.340	.68	.50	-5.77E-15	-.50	-.68
.360	.72	.50	-5.65E-15	-.50	-.72
.380	.76	.50	-6.02E-15	-.50	-.76
.400	.80	.50	-4.99E-15	-.50	-.80
.420	.84	.50	-5.71E-15	-.50	-.84
.440	.88	.50	-6.46E-15	-.50	-.88
.460	.92	.50	-6.97E-15	-.50	-.92
.480	.96	.50	-7.52E-15	-.50	-.96
.500	.99	.50	-7.44E-15	-.50	-.99
.520	.96	.50	-7.52E-15	-.50	-.96
.540	.92	.50	-6.97E-15	-.50	-.92
.560	.88	.50	-6.46E-15	-.50	-.88
.580	.84	.50	-5.71E-15	-.50	-.84
.600	.80	.50	-4.99E-15	-.50	-.80
.620	.76	.50	-6.02E-15	-.50	-.76
.640	.72	.50	-5.65E-15	-.50	-.72
.660	.68	.50	-5.77E-15	-.50	-.68
.680	.64	.50	-5.89E-15	-.50	-.64
.700	.60	.50	-5.94E-15	-.50	-.60
.720	.56	.50	-5.72E-15	-.50	-.56
.740	.52	.50	-5.01E-15	-.50	-.52
.760	.48	.48	-4.69E-15	-.48	-.48
.780	.44	.44	-4.99E-15	-.44	-.44
.800	.40	.40	-4.73E-15	-.40	-.40
.820	.36	.36	-4.92E-15	-.36	-.36
.840	.32	.32	-4.87E-15	-.32	-.32
.860	.28	.28	-4.45E-15	-.28	-.28
.880	.24	.24	-4.18E-15	-.24	-.24
.900	.20	.20	-3.65E-15	-.20	-.20
.920	.16	.16	-4.03E-15	-.16	-.16
.940	.12	.12	-3.43E-15	-.12	-.12
.960	7.99E-02	8.02E-02	-3.25E-15	-8.02E-02	-7.99E-02
.980	4.01E-02	3.99E-02	-3.59E-15	-3.99E-02	-4.01E-02
1.000	-1.49E-14	-9.66E-15	1.41E-27	9.66E-15	1.49E-14

TORQUE DISTRIBUTION
ROBERT J. MARKS II
TRAVELING WAVES FINAL

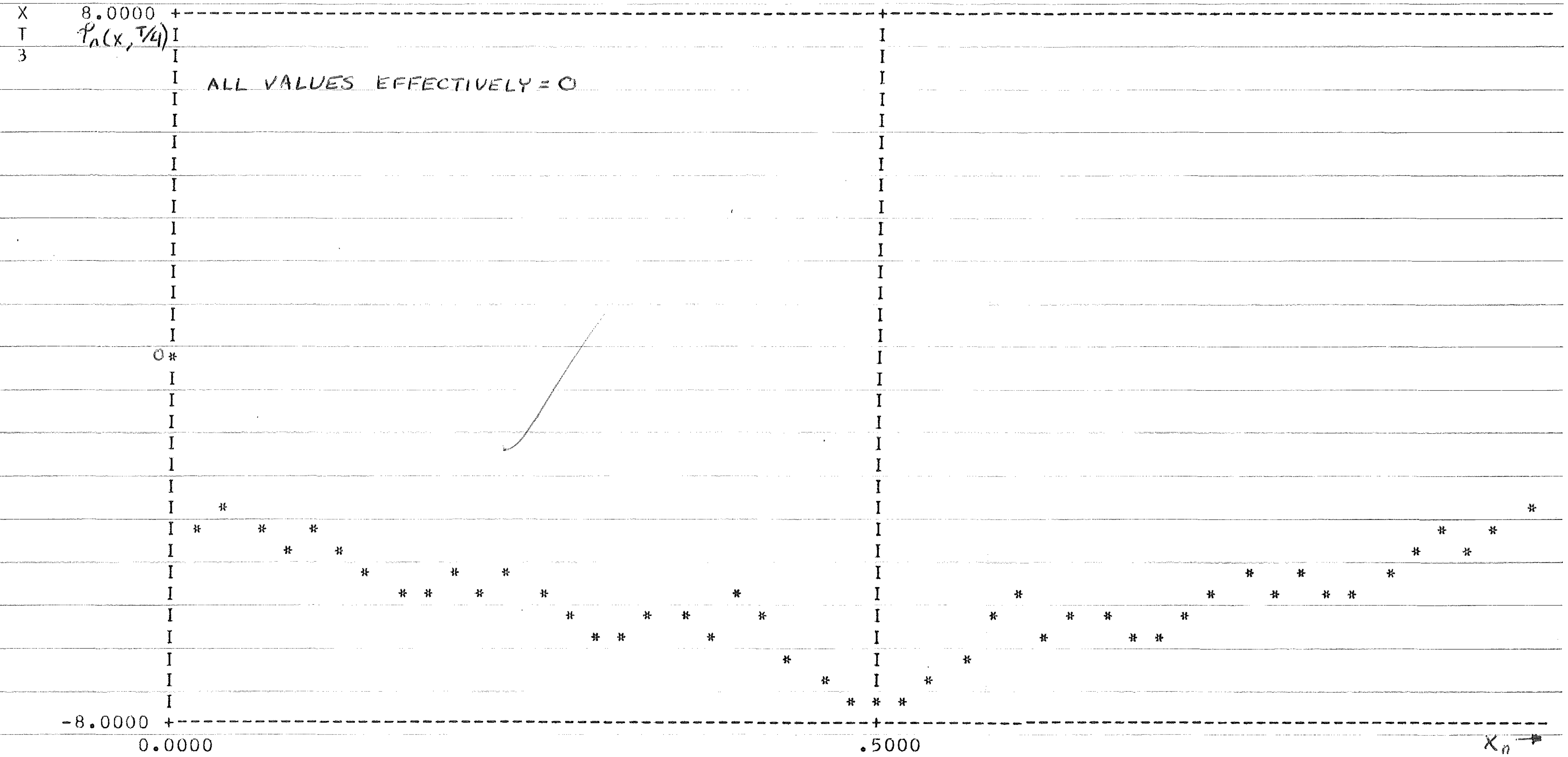
X	0	T/8	T/4	3T/8	T/2
0.000	2.0	2.0	-3.61E-13	-2.0	-2.0
.020	2.0	2.0	4.08E-14	-2.0	-2.0
.040	2.0	2.0	-4.64E-14	-2.0	-2.0
.060	2.0	2.0	1.01E-14	-2.0	-2.0
.080	2.0	2.0	-2.16E-14	-2.0	-2.0
.100	2.0	2.0	2.94E-15	-2.0	-2.0
.120	2.0	2.0	-1.36E-14	-2.0	-2.0
.140	2.0	2.0	-3.56E-14	-2.0	-2.0
.160	2.0	2.0	3.65E-15	-2.0	-2.0
.180	2.0	2.0	-3.52E-15	-2.0	-2.0
.200	2.0	2.1	6.28E-16	-2.1	-2.0
.220	2.0	1.9	2.91E-15	-1.9	-2.0
.240	2.0	2.0	-2.22E-15	-2.0	-2.0
.260	2.0	1.63E-02	-1.60E-14	-1.63E-02	-2.0
.280	2.0	6.80E-02	-5.16E-14	-6.80E-02	-2.0
.300	2.0	-5.57E-02	2.90E-14	5.57E-02	-2.0
.320	2.0	3.22E-02	-2.82E-14	-3.22E-02	-2.0
.340	2.0	-1.03E-02	4.47E-14	1.03E-02	-2.0
.360	2.0	-4.73E-03	-4.79E-14	4.73E-03	-2.0
.380	2.0	1.15E-02	4.70E-14	-1.15E-02	-2.0
.400	1.9	-1.14E-02	5.42E-15	1.14E-02	-1.9
.420	2.1	7.28E-03	-4.59E-14	-7.28E-03	-2.1
.440	2.0	-2.44E-03	-3.18E-14	2.44E-03	-2.0
.460	1.9	-6.79E-04	-2.58E-14	6.79E-04	-1.9
.480	2.3	1.23E-03	-1.28E-14	-1.23E-03	-2.3
.500	-3.61E-13	-1.97E-14	-2.50E-27	1.97E-14	3.61E-13
.520	-2.3	-1.23E-03	1.28E-14	1.23E-03	2.3
.540	-1.9	6.79E-04	2.58E-14	-6.79E-04	1.9
.560	-2.0	2.44E-03	3.18E-14	-2.44E-03	2.0
.580	-2.1	-7.28E-03	4.59E-14	7.28E-03	2.1
.600	-1.9	1.14E-02	-5.42E-15	-1.14E-02	1.9
.620	-2.0	-1.15E-02	-4.70E-14	1.15E-02	2.0
.640	-2.0	4.73E-03	4.79E-14	-4.73E-03	2.0
.660	-2.0	1.03E-02	-4.47E-14	-1.03E-02	2.0
.680	-2.0	-3.22E-02	2.82E-14	3.22E-02	2.0
.700	-2.0	5.57E-02	-2.90E-14	-5.57E-02	2.0
.720	-2.0	-6.80E-02	5.16E-14	6.80E-02	2.0
.740	-2.0	-1.63E-02	1.60E-14	1.63E-02	2.0
.760	-2.0	-2.0	2.22E-15	2.0	2.0
.780	-2.0	-1.9	-2.91E-15	1.9	2.0
.800	-2.0	-2.1	-6.28E-16	2.1	2.0
.820	-2.0	-2.0	3.52E-15	2.0	2.0
.840	-2.0	-2.0	-3.65E-15	2.0	2.0
.860	-2.0	-2.0	3.56E-14	2.0	2.0
.880	-2.0	-2.0	1.36E-14	2.0	2.0
.900	-2.0	-2.0	-2.94E-15	2.0	2.0
.920	-2.0	-2.0	2.16E-14	2.0	2.0
.940	-2.0	-2.0	-1.01E-14	2.0	2.0
.960	-2.0	-2.0	4.64E-14	2.0	2.0
.980	-2.0	-2.0	-4.08E-14	2.0	2.0
1.000	-2.0	-2.0	3.61E-13	2.0	2.0

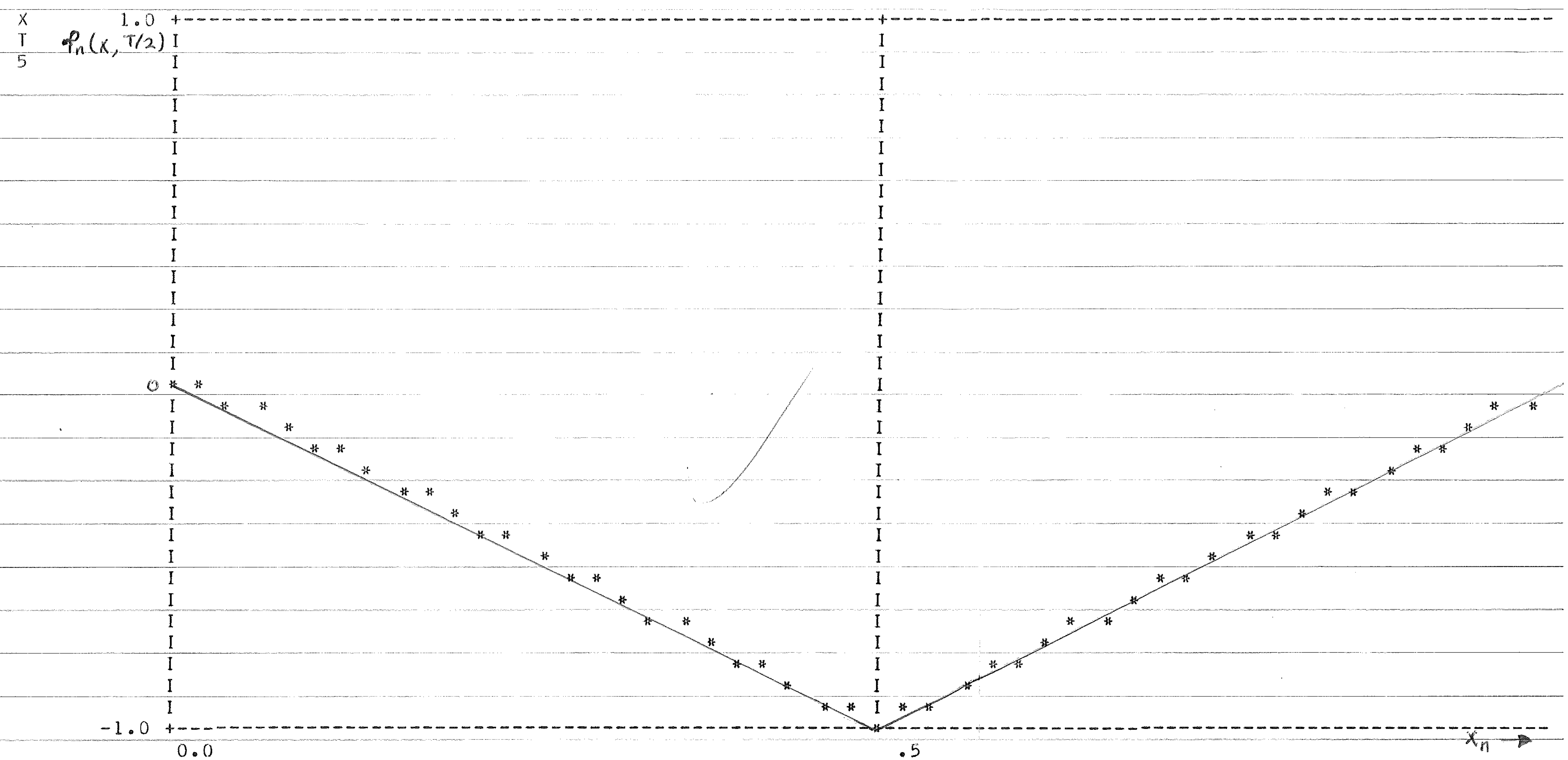

X
T
L
1.0
$\varphi_n(x,0)$
0
-1.0
0.0
.5(= ℓ/2)
X_n

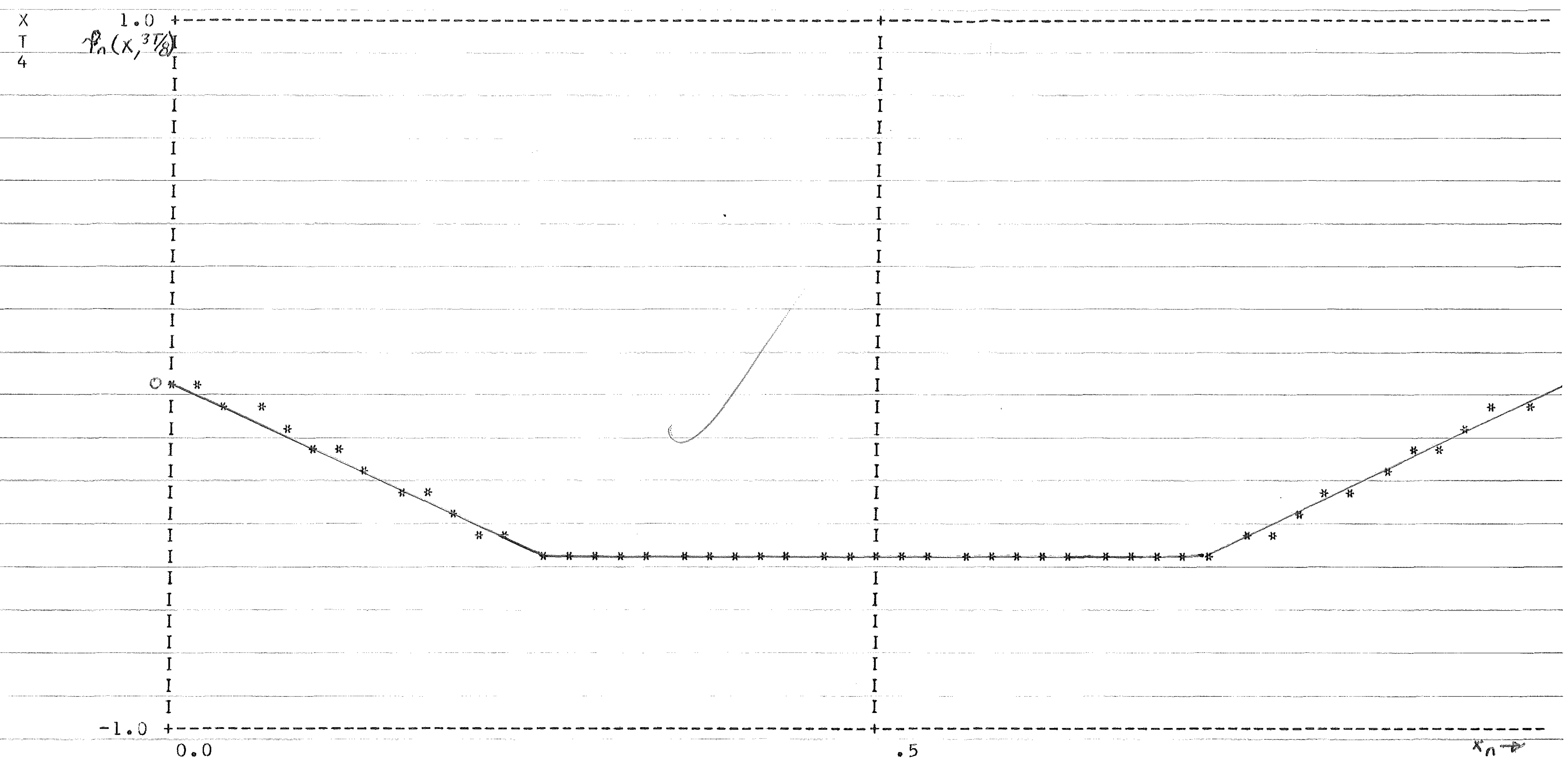


X
T
2
$\varphi_n(x, T/8)$
1.0
-1.0
0.0
.5
x_n

************ PRINTED VALUES OF Y ARE 1.0E 15 TIMES ACTUAL VALUES ************




X
T
5
$\varphi_n(x, T/2)$
1.0
0
-1.0
0.0
.5
x_n

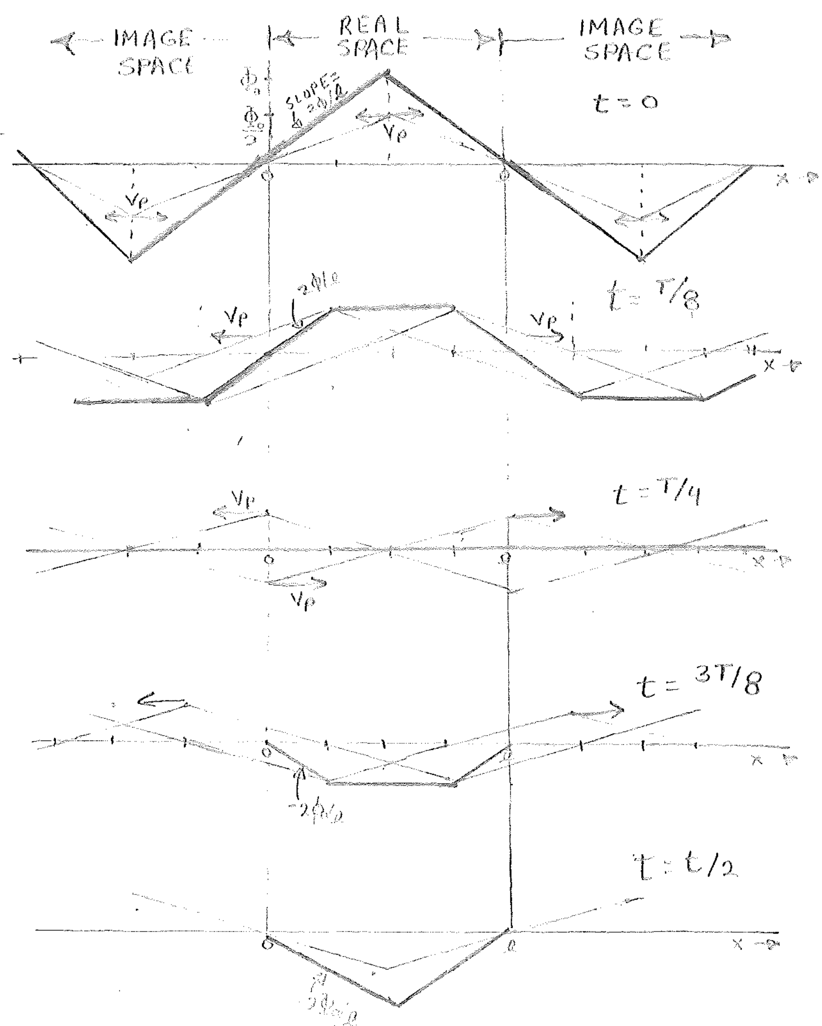

X
T
4
1.0
-1.0
0.0
.5

c) THE RESULTING MOTION OF THE DISPLACEMENT $\varphi(x,t)$ IN TIME AND SPACE MAY INTERPRETED AS AN INFINITE NUMBER OF STANDING WAVES, THE AMPITUDES OF WHICH ARE DETERMINED BY THE FOURIER SERIES EXPANSION OF THE INITIAL DISPLACLMENT $\varphi_0(x)$. FROM THE EQUATION DESCRIBING THE BARS MOTION:

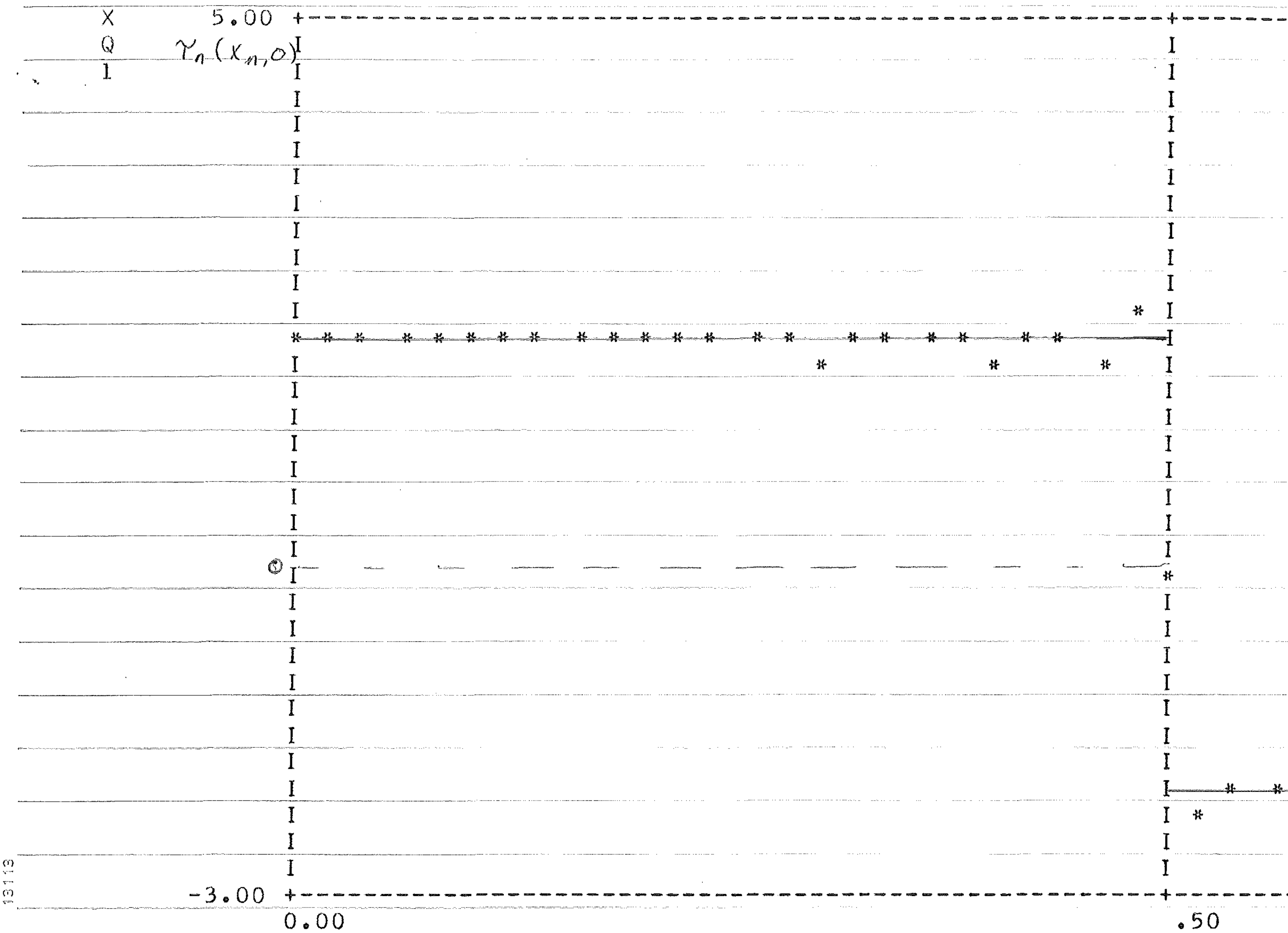
$$\varphi(x,t) = \sum_{m=1}^{\infty} \frac{8\Phi_0 \sin n_0\pi/2}{(n_0\pi)^2} \cos \frac{n_0\pi V_p t}{\ell} \sin \frac{n_0\pi x}{\ell},$$

IT CAN BE SEEN THAN THE FREQUENCY OF VIBRATION OF THE n_0TH VIBRATING COMPONENT IS $n_0\pi V_p/\ell$, WHERE $V_p (=\sqrt{J/C'})$ IS DETERMINED BY PHYSICAL PROPERTIES OF THE BAR. THE WAVELENGTH OF THE n_0TH COMPONENT (FROM $\sin$ TERM) IS $2\ell/n_0$. THE PRODUCT OF THE FREQUENCY (IN HZ $= n_0 V_p/2\ell$) AND THE WAVELENGTH OF THE n_0TH COMPONENT IS THE PHASE VELOCITY, V_p. HENCE, BY INCREASING THE WAVELENGTH OF THE BAR BY A FACTOR OF Q WILL REDUCE THE FREQUENCY BY A FACTOR OF Q (ie $f\lambda = V_p$)

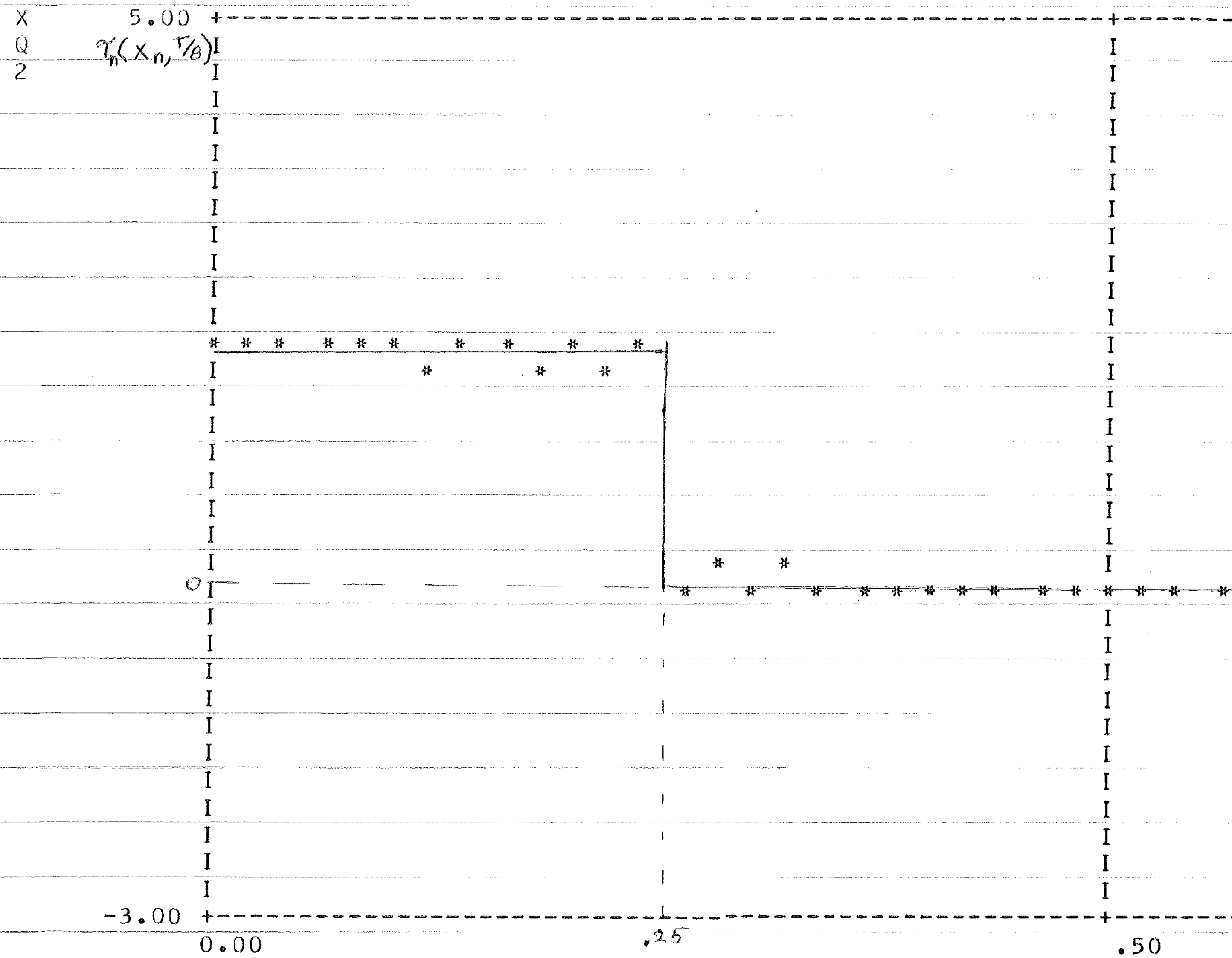
ANOTHER INTERPRETATION OF THE RESULTANT MOTION OF THE BAR IS THRU REPRESENTATION BY TRIANGULAR WAVEFORMS PROPAGATING IN REAL AND IMAGE SPACE AT VELOCITY V_p.



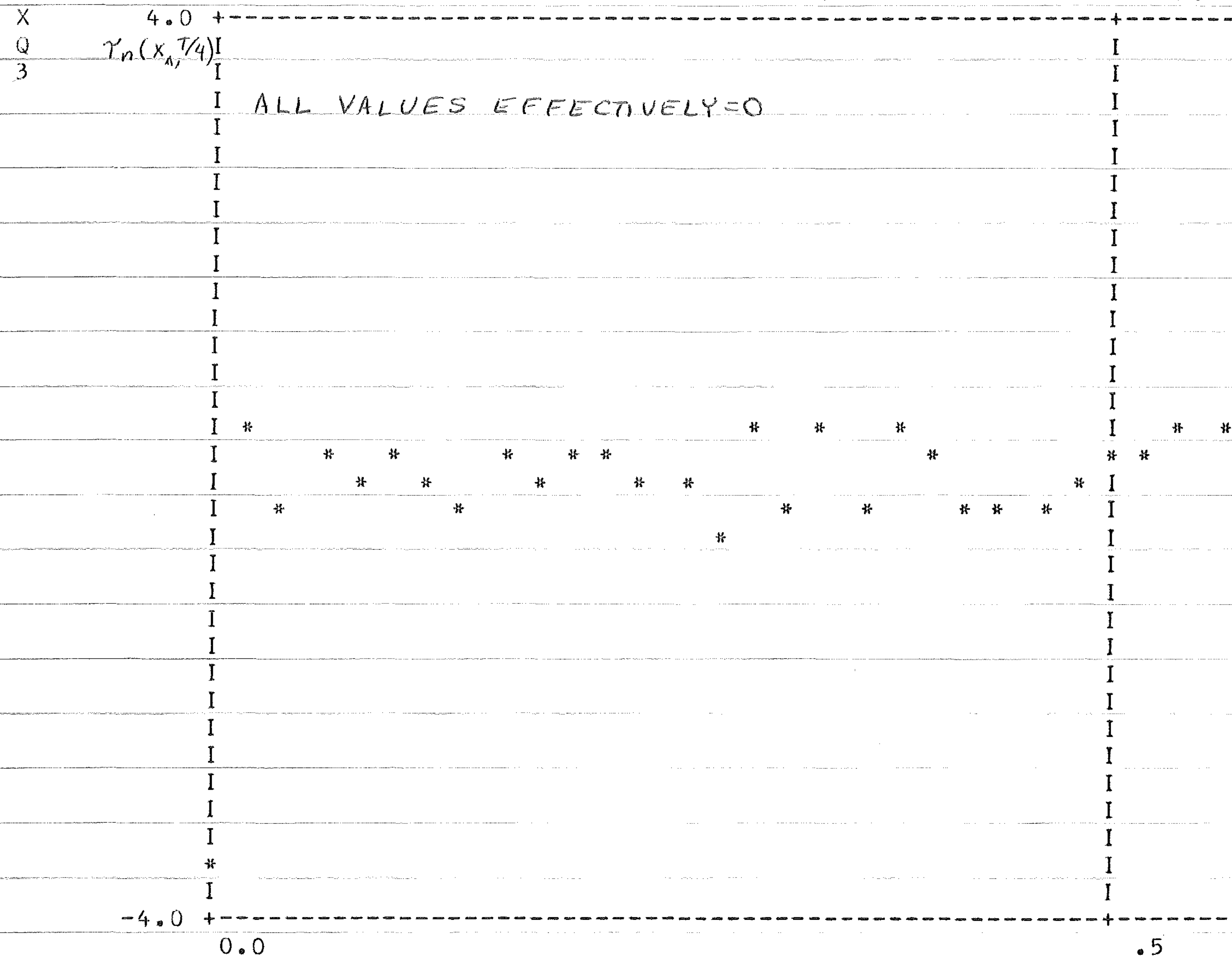
Good illustration. Recall that you assumed a periodic initial function so that the image space consists of the half-periods of the original function.

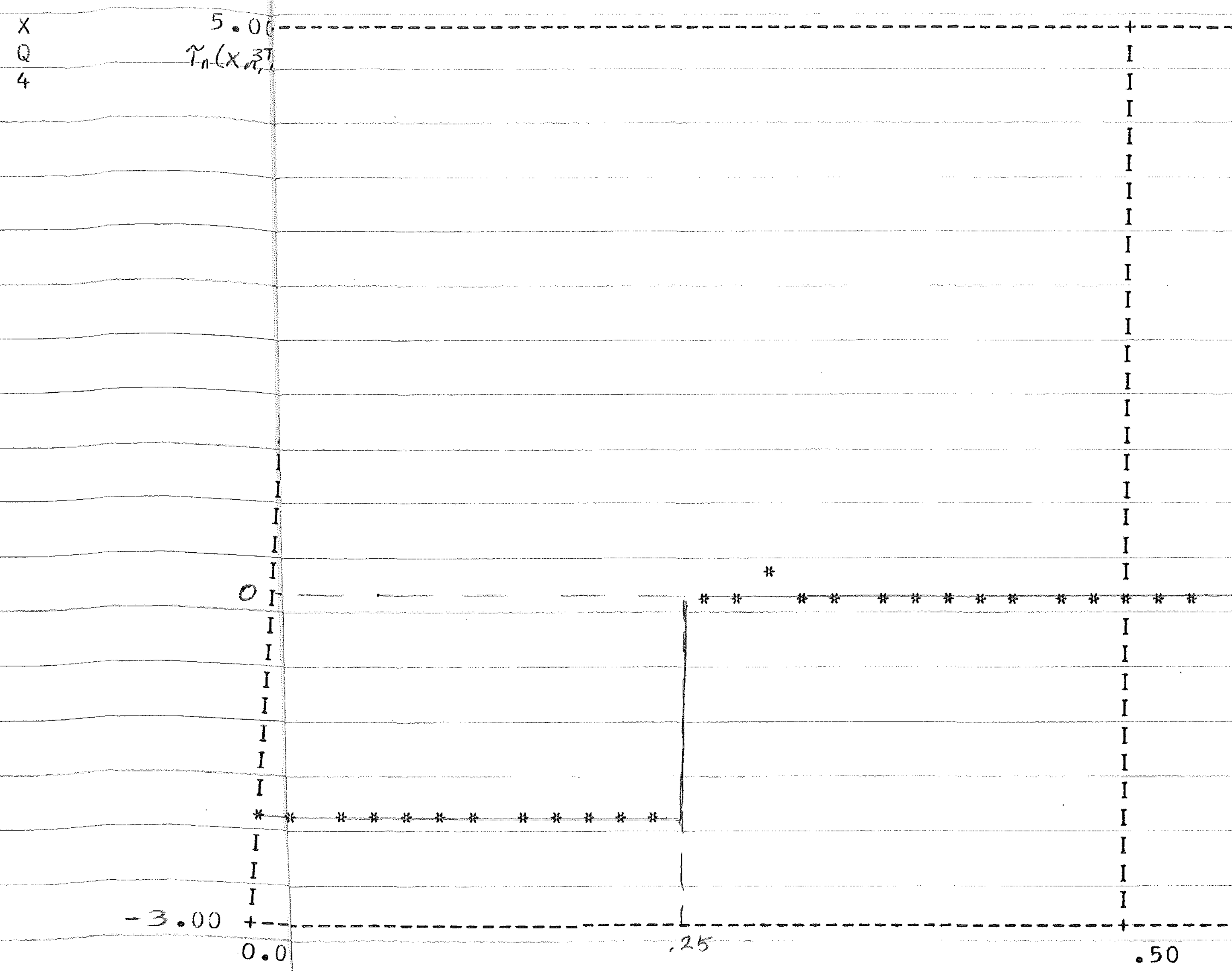


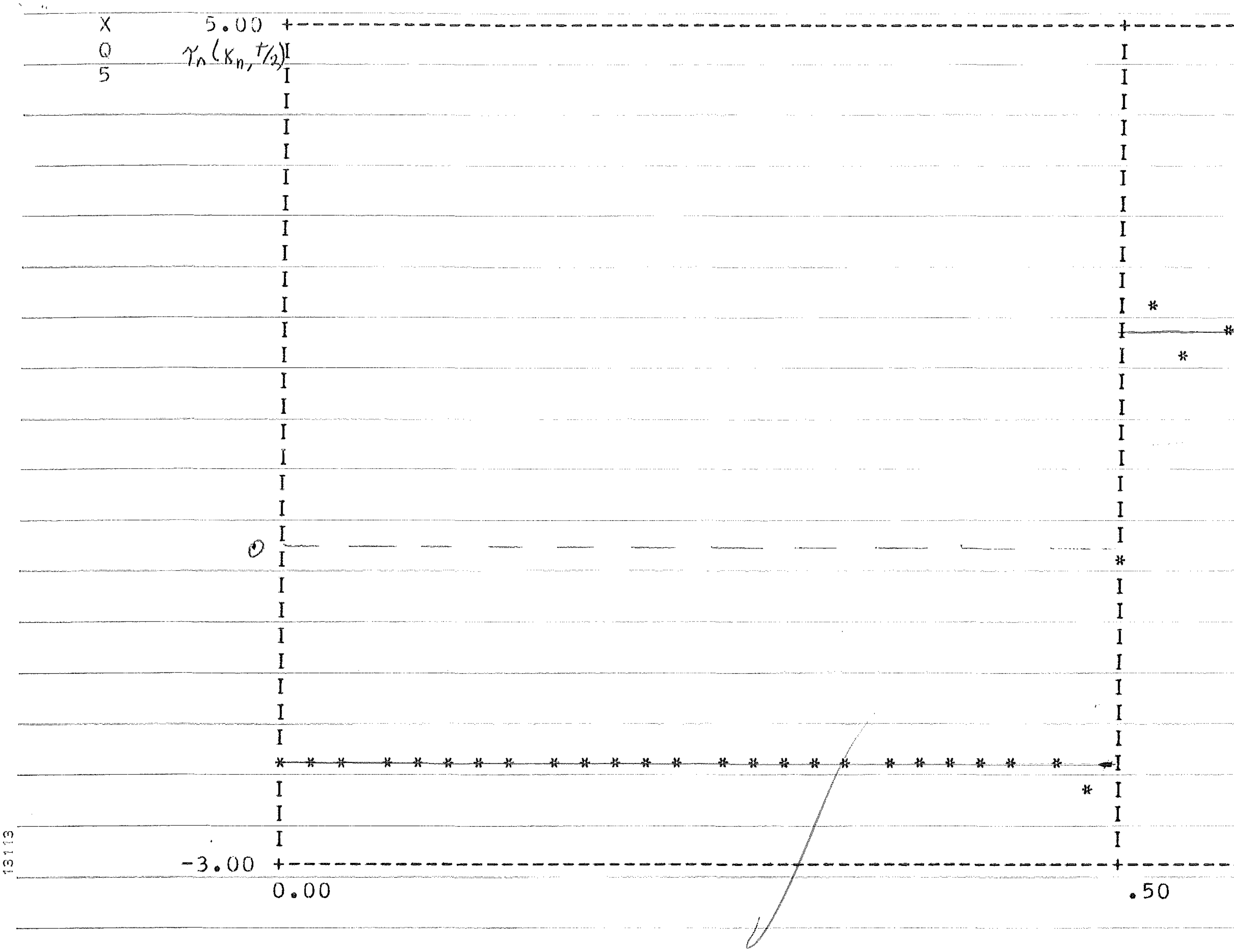
X
Q
1
5.00
$\Upsilon_n(X_n, 0)$
-3.00
0.00
.50


X
Q
2
5.00
ηn(xn, T/8)
0
-3.00
0.00
.25
.50

************ PRINTED VALUES OF Y ARE 1.0E 13 TIMES A




X
Q
4
5.00
0
-3.00
0.0
.25
.50

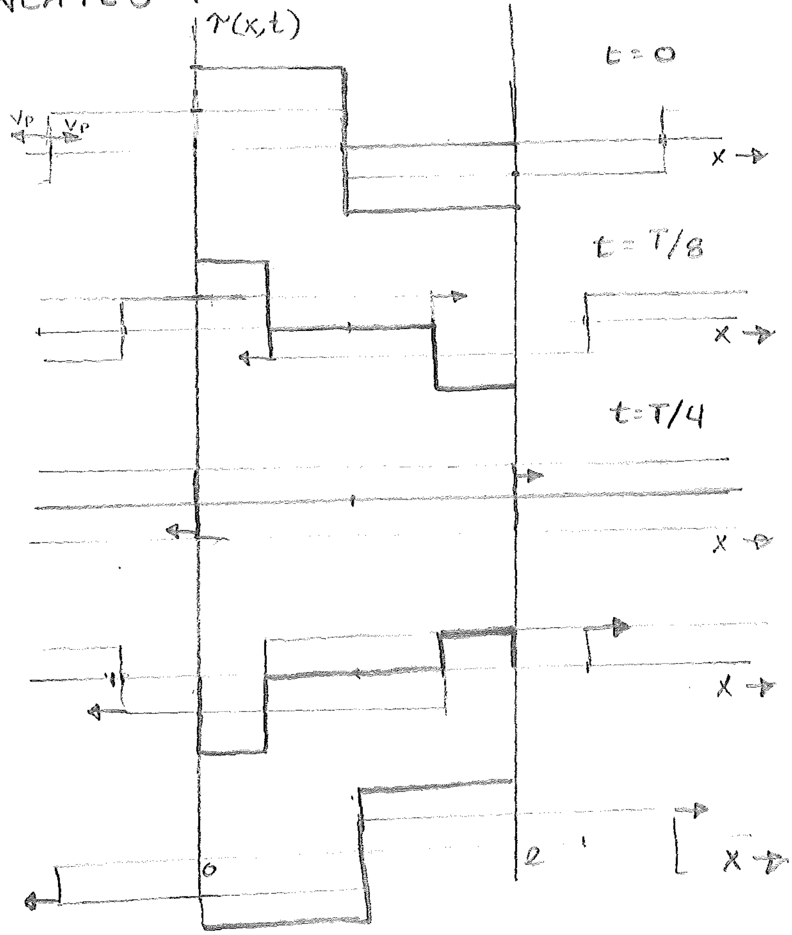


X
Q
5
5.00
-3.00
0.00
.50

d) THE TORQUE, BEING PROPORTIONAL TO THE DERIVATIVE OF DISPLACEMENT, CAN BE OBTAINED BY DIFFERENTIATING ALL FOURIER TERMS DESCRIBING THE DISPLACEMENT. THE RESULT, HOWEVER, IS NOT A STANDING WAVE:

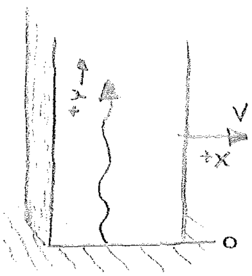
$$\tau(x,t) = \sum_{n_0=1}^{\infty} \frac{8\Phi_0 C \sin \frac{n_0\pi}{2}}{n_0\pi} \cos \frac{n_0\pi V_p t}{\ell} \cos n_0\pi x/\ell$$

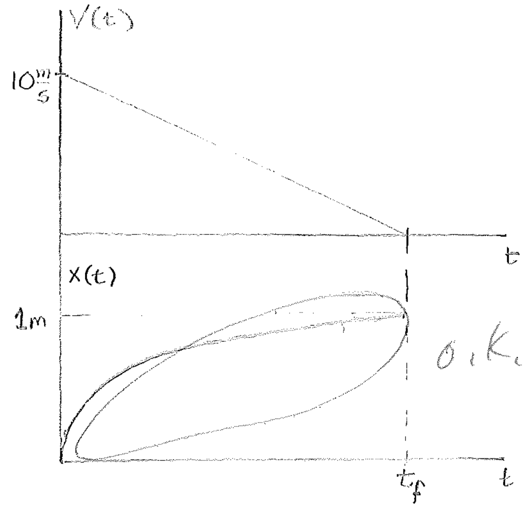
BUT, AS WITH DISPLACEMENT MAY BE THOUGHT OF IN TERMS OF TWO EQUIVALENT WAVES TRAVELING IN OPPOSITE DIRECTIONS, AGAIN WITH VELOCITY V_p. IT THE PREVIOUS CASE, THE WAVES WERE TRIANGULAR. TORQUE, BEING PROPORTIONAL TO THE DERIVITIVE OF DISPLACEMENT (WITH RESPECT TO X) WOULD YIELD2 SQUARE WAVES, A SQUARE WAVEFORM, BEING THE DERIVATIVE OF A TRIANGULAR WAVEFORM. WHILE THE DISPLACEMENT WAVEFORM YIELDED NODES @ X=0 AND X=ℓ (FROM $\sin \frac{n_0\pi x}{\ell}$ TERM, WHICH IS 0 @ X=0 AND X=0 FOR $n_0 = 1,3,5,7,\ldots$), DIFFERINTIATING THE $\sin \frac{n_0\pi x}{\ell}$ TERM GAVE A $\cos \frac{n_0\pi x}{\ell}$ TERM, WHICH WILL YIELD A NORMALIZED ±1 @ X=0 AND X=ℓ, WITH THE POSSIBILITY OF A 0 @ X=0 OR X=ℓ IF A PREVIOUS TERM (SPECIFICALLY $\cos n_0\pi V_p t/\ell$) TRUNCATES TO 0.



2) $\frac{\delta^2 q}{\delta^2 Y} = CR\frac{\delta q}{\delta t}$

$\frac{\delta^2 q}{\delta Y} = -C\frac{\delta T}{\delta t}$

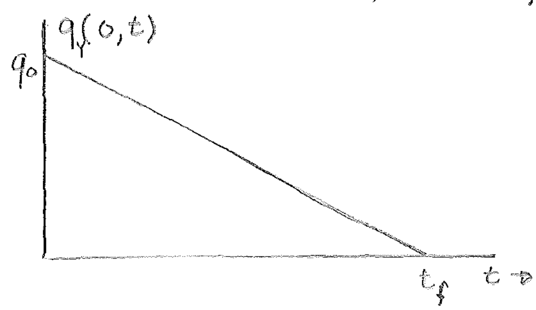




o.k. parabolic

LET $q(0,0^+) = q_0$

THE BRICK'S VELOCITY DECREASES LINEARLY, HENCE THE RATE OF ENERGY DISSIPATION CHANGES LINEARLY, AND WILL OBVIOUSLY BE 0 WHEN THE BRICK COMES TO REST, AT $t = t_f$:



$$q_Y(0,t) = q_0\left(\frac{t_f - t}{t_f}\right)$$

$$\mathcal{L}\{q_Y(0,t)\} = Q(0,s) = \frac{q_0}{s} - \frac{q_0}{t_f}\frac{(1-e^{-st_f})}{s^2}$$

NOW

$$\frac{d^2Q(Y,s)}{dY^2} = CR[sQ(Y,s) - q(0,s)] \quad (q(0,s) \to 0)$$

$$\Rightarrow Q(Y,s) = Q_0(s)e^{-\sqrt{CRs}\,Y}$$

WHERE $Q_0(s)$ IS INDEPENDENT OF Y.
TERM CONTAINING POSITIVE EXPONENT $(Q_{01}(s)e^{+\sqrt{sCR}\,Y})$ WAS NOT INCORPERATED INTO EXPRESSION FOR $Q(Y,s)$ BECAUSE HEAT WAVE IS ASSUMED TO TRAVEL ONLY IN POSITIVE Y DIRECTION

ALSO:

$$\frac{dQ(Y,s)}{dY} = -C[sT(Y,s) - T(Y,0)] \quad (T(Y,0) \to 0) \Rightarrow T(Y,s) = \frac{-1}{sC}\frac{dQ(Y,s)}{dY}$$

$$\Rightarrow T(Y,s) = \frac{Q_0(s)}{sC}(CRs)^{\frac{1}{2}}e^{-(CRs)^{\frac{1}{2}}Y}$$

$$= Q_0(s)\left(\frac{R}{sC}\right)^{\frac{1}{2}}e^{-(sCR)^{\frac{1}{2}}Y}$$

@Y = 0 (AT BRICK, TABLE INTERFACE)

$$T(s,0) = \left(\frac{R}{sC}\right)^{\frac{1}{2}} Q_0(s)$$

$$= \left(\frac{R}{sC}\right)^{\frac{1}{2}} \left[\frac{q_0}{s} - \frac{q_0}{t_f}\frac{1}{s^2}(1 - e^{-st_f})\right]$$

$$= q_0\left(\frac{R}{C}\right)^{\frac{1}{2}} \left[\frac{1}{s^{3/2}} - \frac{1}{t_f s^{5/2}} + \frac{e^{-st_f}}{t_f s^{5/2}}\right]$$

$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{1}{\sqrt{\pi t}}\mu(t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^{3/2}}\right) = \int_0^t \mathcal{L}^{-1}\left(\frac{1}{\sqrt{s}}\right)dt = \int_0^t \frac{1}{\sqrt{\pi t}}\,dt = 2\left(\frac{t}{\pi}\right)^{\frac{1}{2}}\mu(t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^{5/2}}\right) = \int_0^t \mathcal{L}^{-1}\left(\frac{1}{s^{3/2}}\right)dt \int_0^t 2\left(\frac{t}{\pi}\right)^{\frac{1}{2}}dt = \frac{4}{3}\pi^{-\frac{1}{2}}t^{3/2}\mu(t)$$

$$\mathcal{L}^{-1}\left(e^{-st_0}F(s)\right) = f(t-t_0)\mu(t-t_0)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{e^{-st_f}}{s^{5/2}}\right) = \frac{4}{3\sqrt{\pi}}(t-t_f)^{\frac{1}{2}}\mu(t-t_f)$$

$$\therefore T(0,t) = q_0\left(\frac{R}{C}\right)^{\frac{1}{2}}\left[\left\{2\left(\frac{t}{\pi}\right)^{\frac{1}{2}} - \frac{4}{3t_f\sqrt{\pi}}t^{3/2}\right\}\mu(t) + \frac{4}{3t_f\sqrt{\pi}}(t-t_f)\mu(t-t_f)\right]$$

FINDING MAXIMUM TEMPERATURE AT INTERFACE, REALIZING THE TERMS CONTAINING $\mu(t-t_f)$ DO NOT EFFECT $T(0,t)$ UNTIL AFTER THE BRICK HAS STOPPED:

$$\frac{dT(0,t)}{dt} = q_0\left(\frac{R}{C}\right)^{\frac{1}{2}}\left[\left(\frac{1}{\sqrt{\pi t}} - \frac{2}{t_f\sqrt{\pi}}\sqrt{t}\right)\mu(t) + \left(2\sqrt{\frac{t}{\pi}} - \frac{4}{3t_f\sqrt{\pi}}t^{3/2}\right)\delta(t)\right];\ t < t_f$$

$$= q_0\left(\frac{R}{C}\right)^{\frac{1}{2}}\left[\frac{1}{\sqrt{\pi t}} - \frac{2}{t_f\sqrt{\pi}}\sqrt{t}\right];\ 0 < t < t_f$$

SETTING DERIVATIVE TO 0

$$\Rightarrow \frac{1}{\sqrt{\pi t}} = \frac{2}{t_f\sqrt{\pi}}\sqrt{t}$$

$\therefore t = t_f/2$ FOR MAXIMUM TEMPERATURE

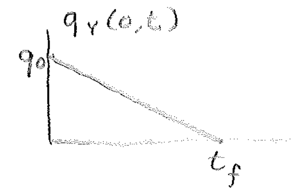
COMPUTING PARAMETERS:

$$q_0 = 5\,\text{kg} \times 10\,\frac{\text{m}}{\text{sec}}\,\frac{1}{(8\times 4)\,\text{IN}^2}\,\frac{\text{IN}^2}{16.387\times 10^{-4}\,\text{m}^2} = \quad ? \quad \frac{\text{WATTS}}{\text{m}^2}$$

$$t_f = \frac{\text{MAXIMUM } X}{V_{AVER}} = \frac{1}{5}\,\text{SEC} = .2\,\text{SEC}$$

Answer? (-3)

THIS SYSTEM IS ANALAGOUS TO AN INFINITLY LONG RC TRANSMISSION LINE, WITH $T \sim$ VOLTAGE, $Q \sim$ CURRENT, AND $Z_c(s) = (R/sC)^{\frac{1}{2}}$, ($R$ = THERMAL CONDUCTIVITY, C = HT CAPACITY)



3) $\frac{\delta^2 U}{\delta t^2} = c^2 \frac{\delta^2 U}{\delta z^2} + Ac^2 \frac{\delta S_x}{\delta z}$

$\frac{\delta S_x}{\delta t} = -\omega_0 S_y$

$\frac{\delta S_y}{\delta t} = \omega_0 S_x + \omega_0 A \frac{\delta U}{\delta z}$

a) TAKING LAPLACE TRANSFORMS

$s^2 U = c^2 \frac{d^2 U}{dz^2} + Ac^2 \frac{dS_x}{dz}$

$s S_x = -\omega_0 S_y$

$s S_y = \omega_0 S_x + \omega_0 A \frac{dU}{dz}$

$s = j\omega$

$\Rightarrow -\omega^2 U = c^2 \frac{d^2 U}{dz^2} + Ac^2 \frac{dS_x}{dz}$

$j\omega S_x = -\omega_0 S_y \Rightarrow S_x = j\omega_0 S_y / \omega$

$j\omega S_y = \omega_0 S_x + \omega_0 A \, dU/dz \Rightarrow S_y = \frac{1}{j\omega}(\omega_0 S_x + \omega_0 A \frac{dU}{dz})$

$\Rightarrow S_x = \frac{-\omega_0}{j\omega} S_y = \frac{+\omega_0^2}{\omega^2}(S_x + A\frac{dU}{dz}) \Rightarrow S_x(1 - (\frac{\omega_0}{\omega})^2) = (\frac{\omega_0}{\omega})^2 A \frac{dU}{dz}$

$\Rightarrow S_x = \frac{(\frac{\omega_0}{\omega})^2 A}{(1 - (\frac{\omega_0}{\omega})^2)} \frac{dU}{dz} = \frac{\omega_0^2 A}{(\omega^2 - \omega_0^2)} \frac{dU}{dz}$

AND $\frac{dS_x}{dz} = \frac{\omega_0^2 A}{(\omega^2 - \omega_0^2)} \frac{d^2 U}{dz^2}$

SUBSTITUTING INTO EXPRESSION FOR $U(\omega)$

$-\omega^2 U = c^2 \frac{d^2 U}{dz^2} + Ac^2 \left(\frac{\omega_0 A}{(\omega^2 - \omega_0^2)} \frac{d^2 U}{dz^2} \right)$

$= c^2 \left[\frac{A^2 \omega_0}{(\omega^2 - \omega_0^2)} + 1 \right] \frac{d^2 U}{dz^2}$

$= c^2 \left[\frac{A^2 \omega_0 + \omega^2 - \omega_0^2}{(\omega^2 - \omega_0^2)} \right] \frac{d^2 U}{dz^2}$

$= c^2 \left[\frac{\omega_0^2 (A^2 - 1) + \omega^2}{(\omega^2 - \omega_0^2)} \right] \frac{d^2 U}{dz^2}$

LET $\omega_1^2 = \omega_0^2 (1 - A^2)$

$\Rightarrow -\omega^2 U = c^2 \left[\frac{\omega^2 - \omega_1^2}{\omega^2 - \omega_0^2} \right] \frac{d^2 U}{dz^2}$

$\therefore \frac{d^2 U}{dz^2} = -\left(\frac{\omega}{c}\right)^2 \left[\frac{\omega^2 - \omega_0^2}{\omega^2 - \omega_1^2} \right] U$

b) $\beta(\omega) = \left(\frac{\omega}{c}\right)\left[\frac{\omega^2-\omega_0^2}{\omega^2-\omega_1^2}\right]^{\frac{1}{2}} \Rightarrow c\beta(\omega) = \omega\left[\frac{\omega^2-\omega_0^2}{\omega^2-\omega_1^2}\right]^{\frac{1}{2}}$

$\beta(\omega) = -\beta(-\omega)$

$\omega_1^2 = \omega_0^2(1-A^2)$; $0<A<1 \Rightarrow 0<A^2<1 \Rightarrow 0<1-A^2<1$

$\Rightarrow \omega_1^2 < \omega_0^2$

FOR $0<\omega<\omega_1$; $c\beta(\omega)$ IS REAL

$c\beta(0)=0$

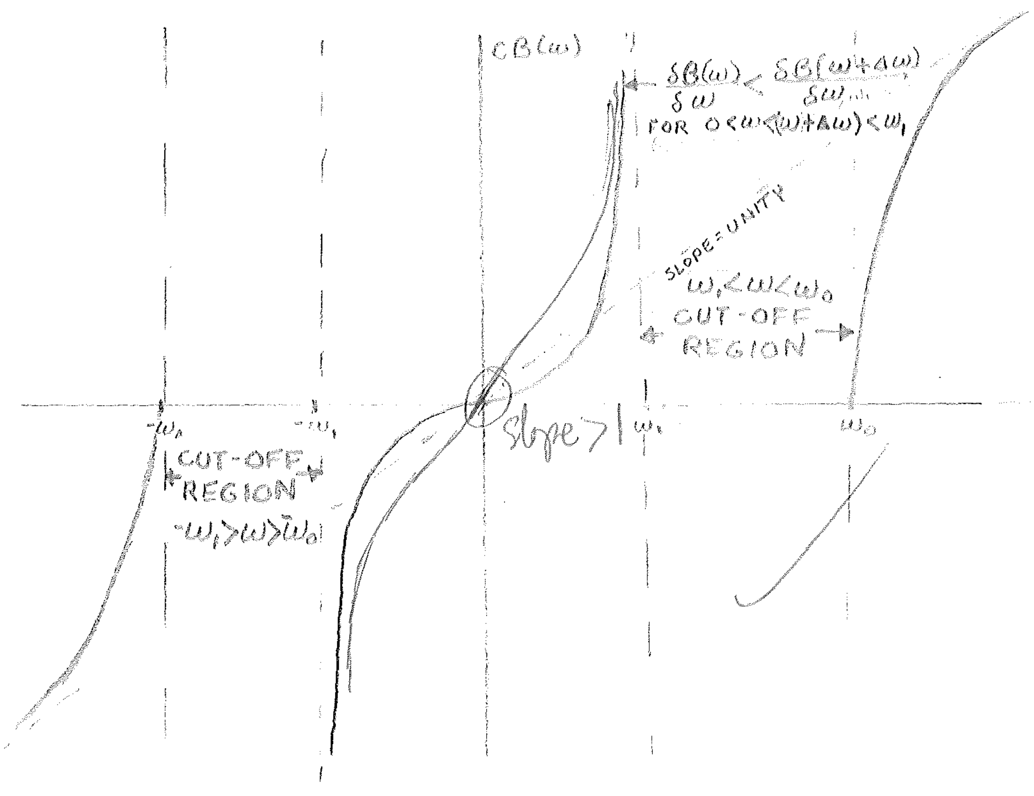
$c\beta(\omega_1)=\pm\infty$

FOR $\omega_1<\omega<\omega_0$; $c\beta(\omega)$ IS PURE IMAGINARY

FOR $\omega_0<\omega<+\infty$; $c\beta(\omega)$ IS REAL

$c\beta(\omega_0)=0$

$\lim_{\omega\to\infty} c\beta(\omega) = \lim_{\omega\to\infty}\omega$



$$cB(\omega) = \omega\left[\frac{\omega^2-\omega_0^2}{\omega^2-\omega_1^2}\right]^{\frac{1}{2}}$$

$$c\frac{\delta B(\omega)}{\delta\omega} = \frac{(\omega^2-\omega_1^2)^{\frac{1}{2}}\left[(\omega^2-\omega_0^2)^{\frac{1}{2}}+\omega^2(\omega^2-\omega_0^2)^{-\frac{1}{2}}\right]-\omega^2(\omega^2-\omega_0^2)^{\frac{1}{2}}(\omega^2-\omega_1^2)^{-\frac{1}{2}}}{(\omega^2-\omega_1^2)}$$

$$= \frac{(\omega^2-\omega_1^2)\left[(\omega^2-\omega_0^2)^{\frac{1}{2}}+\omega^2(\omega^2-\omega_0^2)^{-\frac{1}{2}}\right]-\omega^2(\omega^2-\omega_0^2)^{\frac{1}{2}}}{(\omega^2-\omega_1^2)^{3/2}}$$

$$= \frac{(\omega^2-\omega_1^2)\left[(\omega^2-\omega_0^2)+\omega^2\right]-\omega^2(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^{\frac{1}{2}}(\omega^2-\omega_1^2)^{3/2}}$$

$$= \frac{(\omega^2-\omega_1^2)(2\omega^2-\omega_0^2)-\omega^2(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^{\frac{1}{2}}(\omega^2-\omega_1^2)^{3/2}}$$

$$= \frac{2\omega^4-\omega_0^2\omega^2-2\omega_1^2\omega^2+\omega_1^2\omega_0^2-\omega^4+\omega^2\omega_0^2}{(\omega^2-\omega_0^2)^{\frac{1}{2}}(\omega^2-\omega_1^2)^{3/2}}$$

$$= \frac{\omega^4-2\omega_1^2\omega^2+\omega_1^2\omega_0^2}{(\omega^2-\omega_0^2)^{\frac{1}{2}}(\omega^2-\omega_1^2)^{3/2}}$$

$$c\frac{\delta B(\omega)}{\delta\omega} = c\frac{\delta B(-\omega)}{\delta\omega}$$

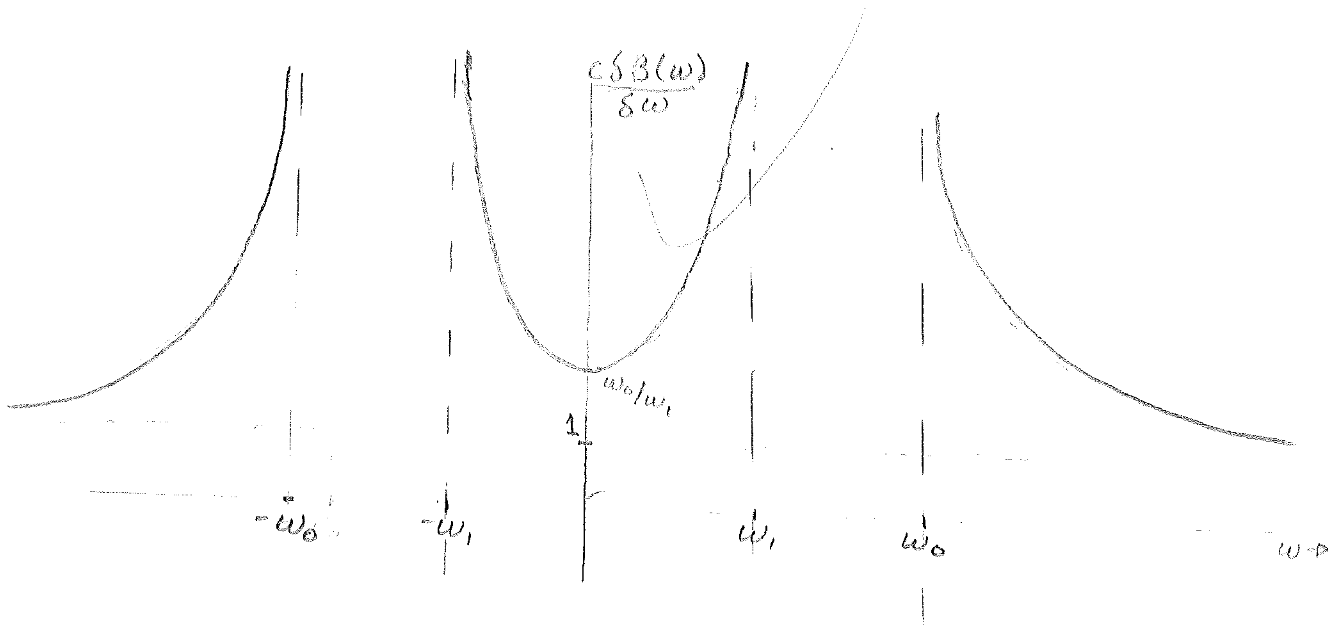
$$c\frac{\delta B(0)}{\delta\omega} = \frac{\omega_0}{\omega_1}$$

$$c\frac{\delta B(\omega_1)}{\delta\omega} = \infty$$

$$c\frac{\delta B(\omega_1)}{\delta\omega} = \infty$$

FOR LARGE VALUES OF ω, ω_0 AND ω_1 MAY BE CONSIDERED EQUAL ($k = \omega_1 = \omega_0$)
HENCE:

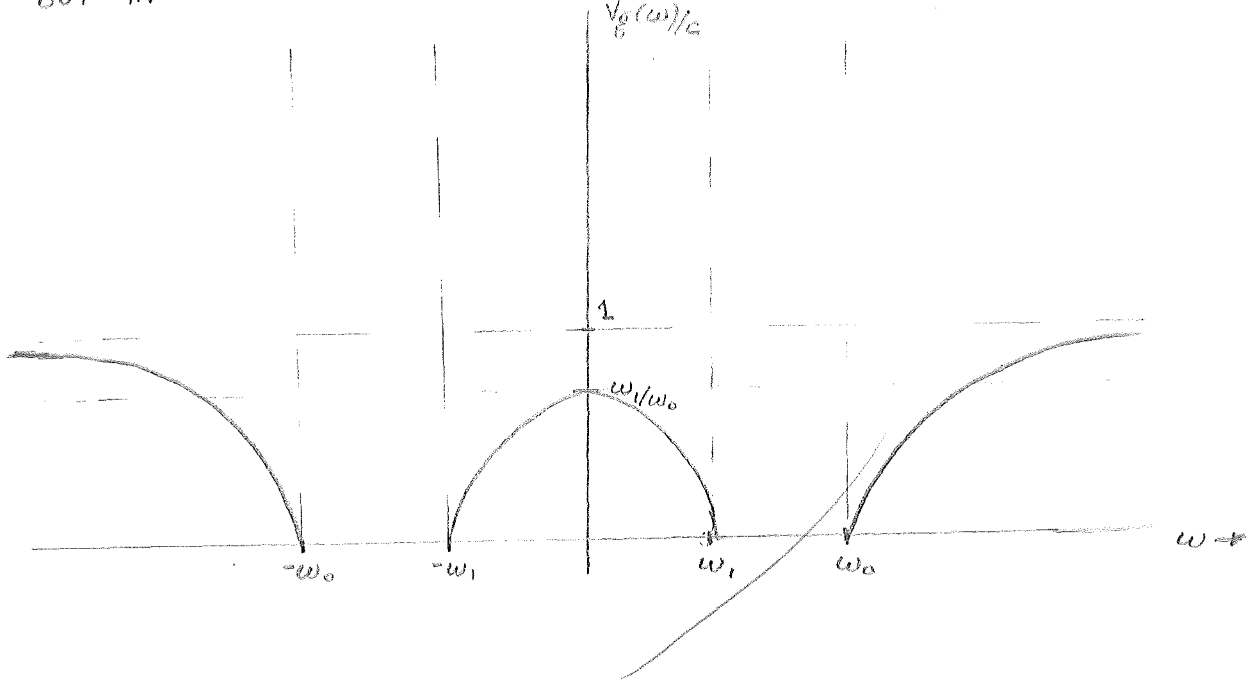
$$\lim_{\omega\to\infty} cB(\omega) = \frac{(\omega^2-k^2)^2}{(\omega^2-k^2)^2} = 1$$



$$\frac{1}{c} V_g(\omega) = \frac{1}{c}\left(\frac{\delta\beta}{\delta\omega}\right)^{-1}$$

$$= \frac{(\omega^2-\omega_0^2)^{\frac{1}{2}}(\omega^2-\omega_1^2)^{3/2}}{\omega^4 - 2\omega_1^2\omega^2 + \omega_1^2\omega_0^2}$$

$\frac{1}{c} V_g(\omega)$ WILL FOLLOW THE SAME ASYMPTOTES AS $c\,\frac{\delta\beta(\omega)}{\delta\omega}$, BUT IN AN INVERSE FASHION:



c) $j\beta(\omega) = (Z(\omega)Y(\omega))^{\frac{1}{2}}$; $Z_c(\omega) = (Z(\omega)/Y(\omega))^{\frac{1}{2}}$ (DERIVED IN PROBLEM 2-3)

$\ni Z(\omega)$ = SERIES IMPEDANCE OF INCREMENTAL TRANS. LINE EQUIVALENT CIRCUIT
$Y(\omega)$ = SHUNT ADMITTANCE " " " " " "
($Z(\omega)$ AND $Y(\omega)$ VALUES GIVEN PER UNIT LENGTH)

GIVEN: $Y(\omega) = SC = \frac{j\omega}{\mu}$

$\Rightarrow -\beta^2(\omega) = Z(\omega)\frac{j\omega}{\mu}$

FROM PART b:

$$\beta^2(\omega) = \left(\frac{\omega}{c}\right)^2\left[\frac{\omega^2-\omega_0^2}{\omega^2-\omega_1^2}\right]$$

$$\Rightarrow \frac{\omega^2}{c^2}\left[\frac{\omega^2-\omega_0^2}{\omega^2-\omega_1^2}\right] = Z(\omega)\frac{j\omega}{\mu}$$

$$\therefore Z(\omega) = j\frac{\omega\mu}{c^2}\left[\frac{\omega^2-\omega_0^2}{\omega^2-\omega_1^2}\right]$$

$$= j\omega\rho\left[\frac{\omega^2-\omega_0^2}{\omega^2-\omega_1^2}\right] \ni \rho = \mu/c^2$$

$$s = j\omega \Rightarrow Z(s) = s\rho\left[\frac{s^2+\omega_0^2}{s^2+\omega_1^2}\right]$$

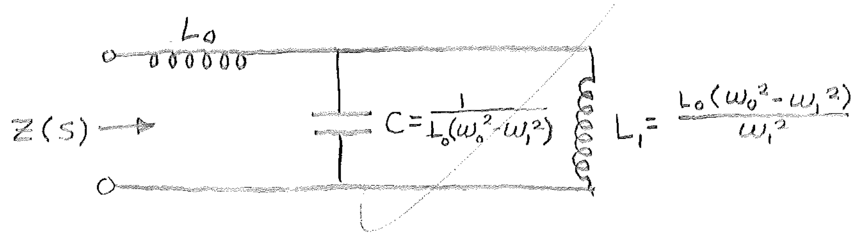
SYNTHESIS:

AT $\omega=\omega_0$, $Z(s)=0 \Rightarrow$ SERIES LC TANK TUNED TO ω_0

AT $\omega=\omega_1$, $Z(s)=0 \Rightarrow$ PARALLEL LC TANK TUNED TO ω_1

AS $\omega\to\infty$, $Z(s)\to s\rho \Rightarrow$ CIRCUIT APEARS AS AN INDUCTANCE AT HIGHER FREQUENCIES

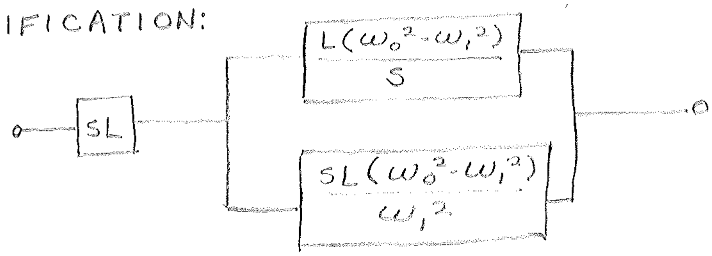
GIVEN THAT SYNTHESIZED CIRCUIT IS LC, AND WITH SOME TRIAL AND ERROR CRANKING, THE INCREMENTAL EQUIVALENT CIRCUIT FOR $Z(s)$ IS THE FOLLOWING: ($L_0 \sim \rho$)



L_1 AND C TUNED TO ω_1
L_0 AND C TUNED TO ω_0

AT HIGH FREQUENCIES, C WILL ACT AS A SHORT $\Rightarrow Z(s)\to L_0 s$

VERIFICATION:



$$Z(s) = sL + \frac{\frac{L^2(\omega_0^2-\omega_1^2)}{\omega_1^2}}{L(\omega_0^2-\omega_1^2)\left(\frac{1}{s}+\frac{s}{\omega_1^2}\right)}$$

$$= sL + \frac{sL(\omega_0^2-\omega_1^2)}{(\omega_1^2+s^2)}$$

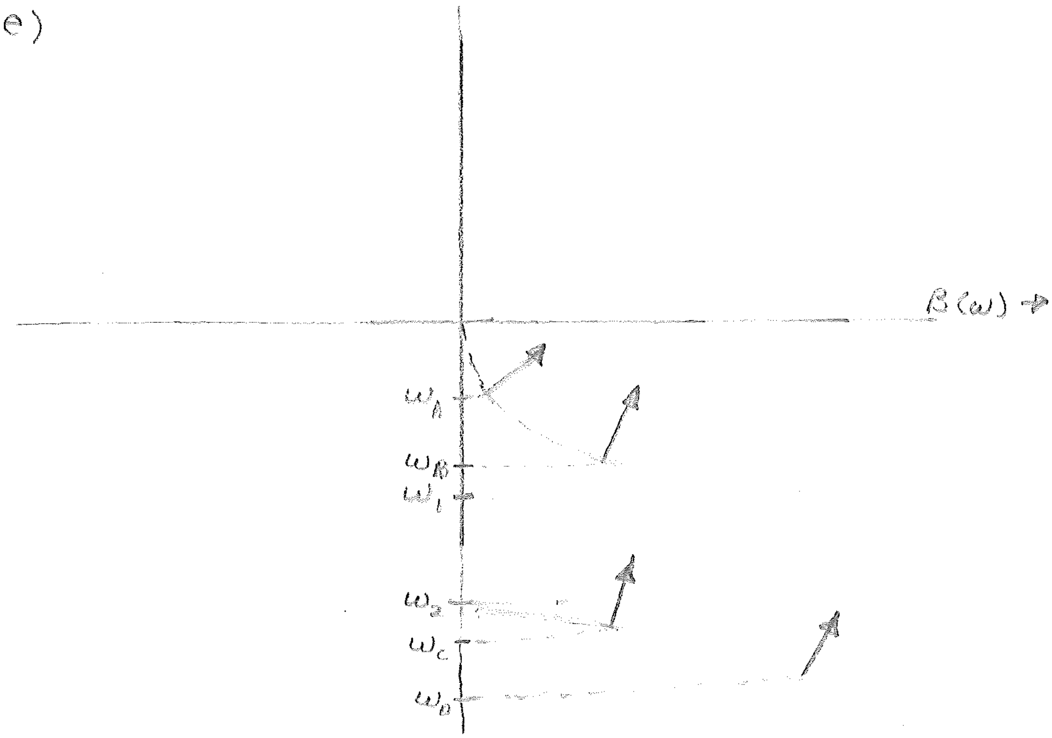
$$= sL\left[1+\frac{\omega_0^2-\omega_1^2}{\omega_1^2+s^2}\right]$$

$$= \frac{sL(s^2+\omega_0^2)}{s^2+\omega_1^2}$$

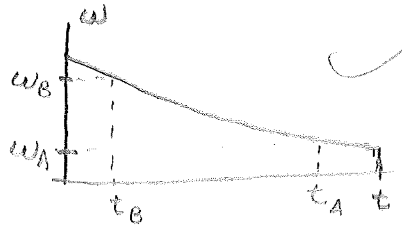
(d) ?

−10

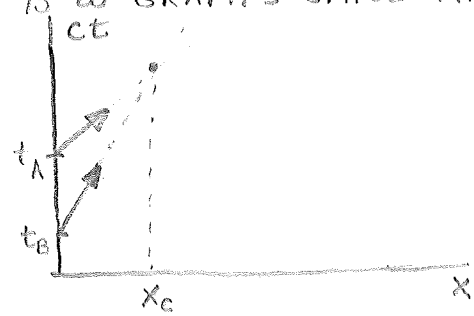
e)



USING THE INTERVAL, $0<\omega<\omega_1$, A MONOTONICALLY DECREASING "CHIRP" MUST BE USED FOR PULSE COMPRESSION.

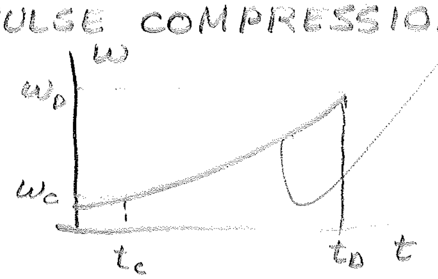


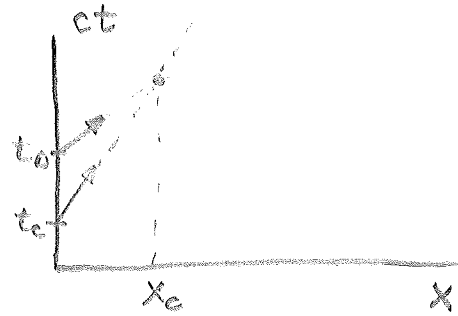
FROM β-ω GRAPH'S SPACE-TIME RAYS:



THIS INTERVAL $(0<\omega<\omega_1)$ COULD BE USED IF PULSE COMPRESSION WERE DESIRED OVER A LONG DISTANCE, FOR THE SPACE TIME RAYS MAY ACHIEVE A NORMALIZED SLOPE LESS THAN UNITY.

USING INTERVAL $\omega_D<\omega<\infty$, A MONOTONICALLY INCREASING "CHIRP" NEED BE USED FOR PULSE COMPRESSION:





THIS SEGMENT IS EQUIVALENT TO A SYSTEM WITH A SINGLE CUT OFF FREQ.

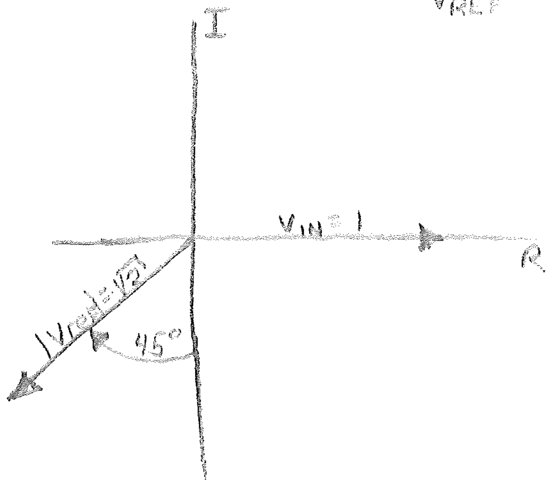
a) $\Gamma_r = \frac{Z_r - Z_0}{Z_r + Z_0} = \frac{100 - j100}{j100}$

$= \frac{\sqrt{2}\,100\, e^{-j\pi/4}}{100\, e^{j\pi/2}}$

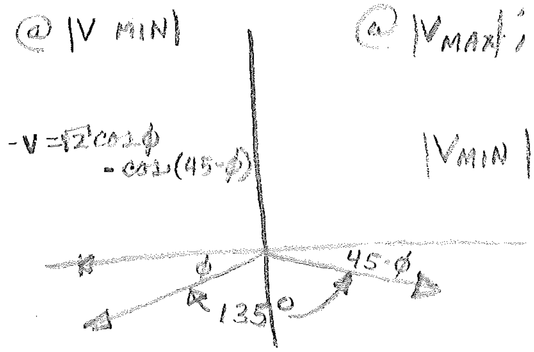
$\Rightarrow \Gamma_r = \sqrt{2}\, e^{-j3\pi/4} \left(= \frac{V_{REF}}{V_{IN}} \right)$

USING V_{IN} AS REFERENCE

$V_{REF} = V_{IN} \sqrt{2}\, e^{j3\pi/4}$



@ |V MIN|



@ $|V_{MAX}|$; $\sin\phi = \sqrt{2}\sin(\phi - \frac{\pi}{4})$

$\Rightarrow \phi = +45°$

$|V_{MIN}|$; $\Rightarrow \phi = -45°$

b) $\Rightarrow SWR = \left|\frac{V_{MAX}}{V_{MIN}}\right| = \sqrt{2} \left(= \frac{1 + |\Gamma_r|}{1 - |\Gamma_r|} \right)$

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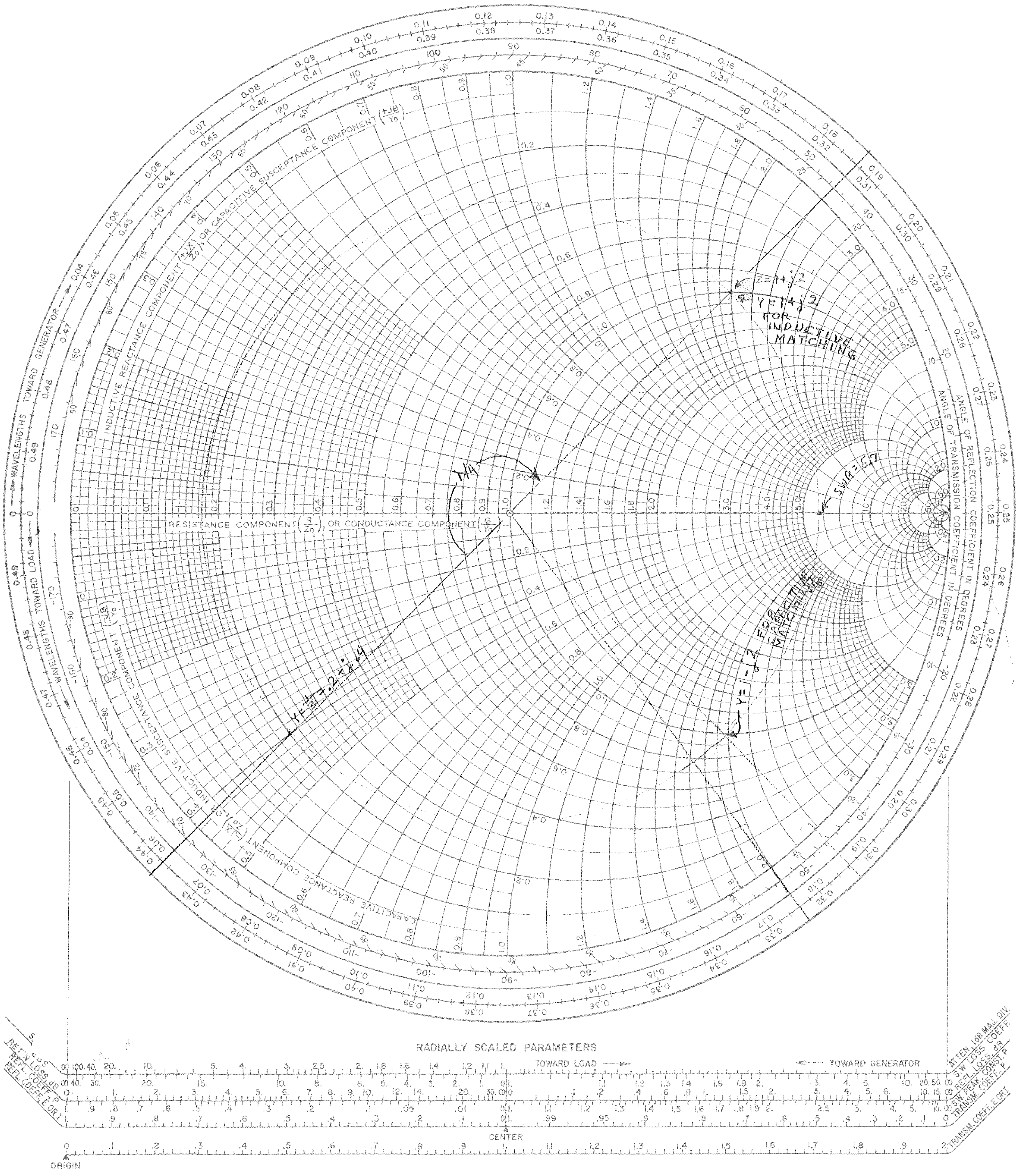
IMPEDANCE OR ADMITTANCE COORDINATES



A MEGA-CHART

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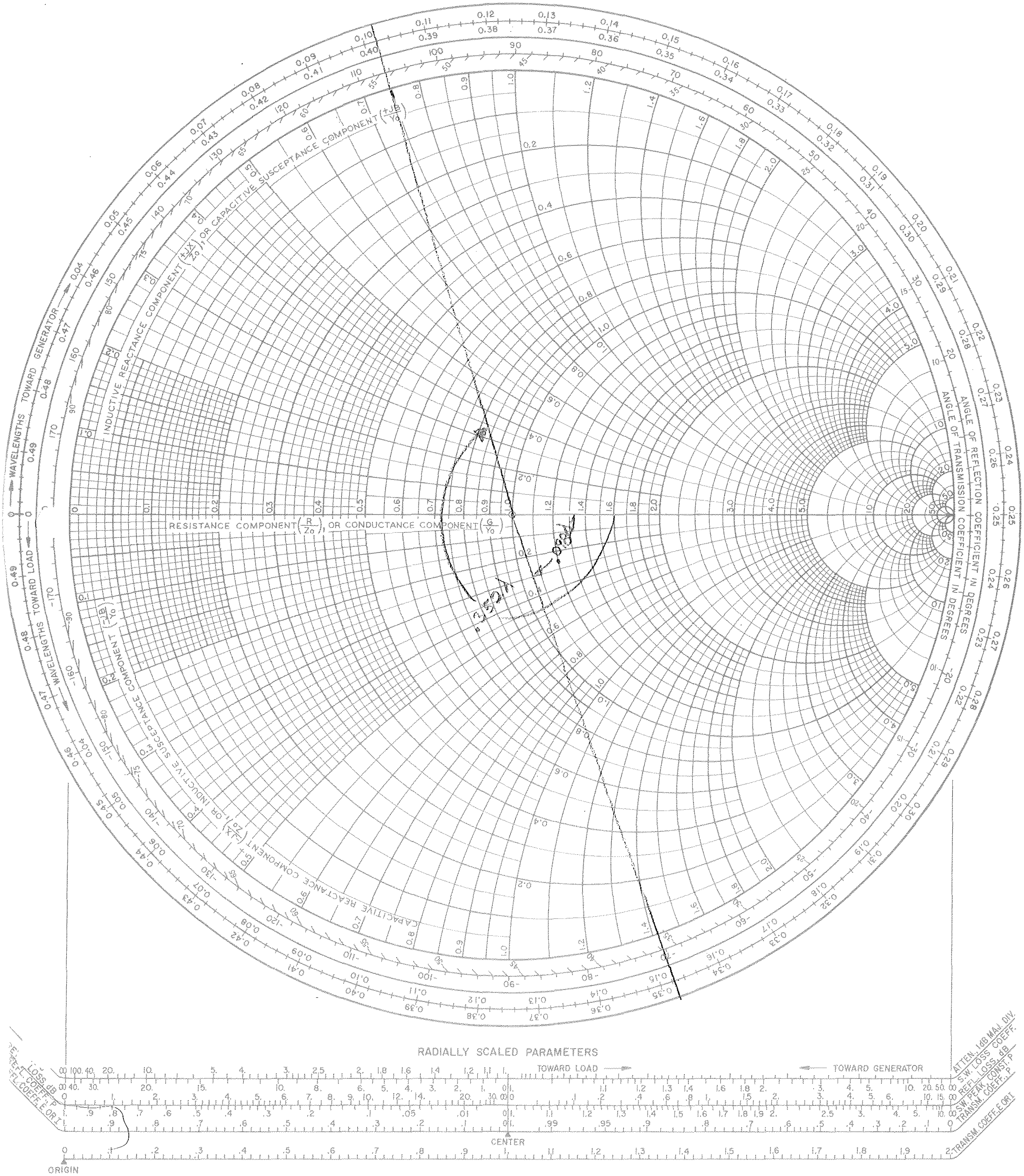


A MEGA-CHART

Chapter 1 [illegible]

The study of electromechanical energy [illegible] starts with a discussion of the electromagnetic field. With the background acquired from such a study we will be able to better understand the production of forces and torques and, in addition, we will be able to compute inductances and capacitances of machine windings and thereby make a first step in the analysis and design of electromechanical machines.

1-1 Fundamental Laws and Equations The fundamental laws of electromagnetic theory are: (1) Faraday's law relating induced voltage to change of magnetic flux, (2) Ampere's law relating magnetic field intensity to total current flow (conduction plus displacement), (3) Gauss' law for electric and magnetic fields relating the total (electric or magnetic) flux crossing the surface bounding a region of space to the total charge (electric or magnetic) within that region.

If we let $\bar{E}$ = electric field strength, $\bar{D}$ = electric flux density, $\bar{B}$ = magnetic induction (or flux density), $\bar{H}$ = magnetic field strength, I = electric current and Q = electric charge, the above laws become

(1-1) $$\oint \bar{E}\cdot d\bar{l} = -\frac{d}{dt}\iint \bar{B}\cdot d\bar{a}$$ (Faraday's law)

(1-2) $$\oint \bar{H}\cdot d\bar{l} = I + \frac{d}{dt}\iint \bar{D}\cdot d\bar{a}$$ (Ampere's law)

(1-3) $$\oiint \bar{D}\cdot d\bar{a} = Q$$ (Gauss' law for the electric field)

(1-4) $$\oiint \bar{B}\cdot d\bar{a} = 0$$ (Gauss' law for the magnetic field)

The single integral with a circle through it denotes the integral around a closed contour, whereas the double integral with the circle through it denotes the integral over the surface bounding a closed region of space. Without the circle, the double integral implies integration over an open surface (i.e., one which does not enclose a three dimensional region.) Often the integral of B is called magnetic flux, φ, and that of D, electric flux, ψ:

(1-5) $$\varphi = \iint \bar{B}\cdot d\bar{a}$$

(1-6) $$\psi = \iint \bar{D}\cdot d\bar{a}$$

It is assumed that the reader is familiar with the vector operators gradient, divergence and curl and with the fundamental mathematical statements known as the divergence theorem and Stokes' Theorem. We recall that the divergence theorem states that, for any vector D

... [illegible] triple integral ... [illegible] ... [illegible]
... [illegible] the surface bounded by the [illegible] ... [illegible]
Similarly, Stokes' theorem [illegible]

$$\oint \bar{V}\cdot d\bar{l} = \iint (\nabla \times \bar{V})\cdot d\bar{s} \qquad \text{(Stokes' theorem)}$$ (1-[illegible])

[illegible] vector V. The integral [illegible] ... [illegible]
closed region bounded by the closed [illegible] ... [illegible]

Using the theorems (1-7) and (1-8), [illegible] ... [illegible]
laws stated in (1-1) ... (1-3) into relations [illegible] ... [illegible]
space. Thus, utilizing (1-1) and (1-8), we [illegible]

$$\oint \bar{E}\cdot d\bar{l} = \iint (\nabla \times \bar{E})\cdot d\bar{s} = [\text{illegible}] \iint \bar{B} \cdot [\text{illegible}] = \iint [\text{illegible}] \frac{\partial \bar{B}}{\partial t} [\text{illegible}]$$ (1-9)

The time-derivative may be taken inside [illegible] ... [illegible]
stationary regions of integration.

Thus equating the surface integrands, we get

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$ (1-10)

This is the point form, or differential [illegible] equation [illegible] of
Faraday's law. Similarly, (1-2) and (1-8) [illegible] ... [illegible]

$$\oint \bar{H}\cdot d\bar{l} = \iint (\nabla \times \bar{H})\cdot d\bar{s} = [\text{illegible}] \frac{\partial \bar{D}}{\partial t} [\text{illegible}]$$ (1-11)

$$\iint \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right) \cdot d\bar{s}$$

Note that we have defined the [illegible] ... [illegible]
current density to be $\bar{J}$ [illegible] $\iint \bar{J}\cdot d\bar{s}$ [illegible] ... [illegible]
across a two dimensional [illegible] region [illegible] equal [illegible] ... [illegible]
of the current density [illegible] ... [illegible]

Again equating surface integrands [illegible] ... [illegible]
Maxwell equation stating Ampere's law [illegible] ... [illegible]

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$ (1-12)

(1-13) $\iiint (\nabla \cdot \bar{D})\, dv = \oiint \bar{D} \cdot d\bar{s} = Q = \iiint \rho\, dv$

where ρ is the volume charge density at each point of space. Upon equating the volume integrals in (1-13), we get

(1-14) $\nabla \cdot \bar{D} = \rho$

Applying exactly the same reasoning to (1-4) yields

(1-15) $\nabla \cdot \bar{B} = 0$.

The four differential equations (1-10), (1-12), (1-14), (1-15) are referred to as Maxwell's equations and are summarized in Table (1-1). Note that they are laws relating the electromagnetic variables, $\bar{E}, \bar{D}, \bar{B}, \bar{H}, \rho$ and $\bar{J}$ at a point rather than throughout a region of space.

Table 1-1. Maxwell's Equations and the Corresponding Integral Laws

Maxwell's equation		Integral Law	
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	(1-10)	(1-1) $\oint \bar{E} \cdot d\bar{l} = -\frac{d\Phi}{dt}$	(Faraday's Law).
$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$	(1-12)	(1-2) $\oint \bar{H} \cdot d\bar{l} = I + \frac{d\Psi}{dt}$	(Ampere's Law).
$\nabla \cdot \bar{D} = \rho$	(1-14	(1-3) $\oiint \bar{D} \cdot d\bar{s} = Q$	(Gauss' Law for electric field)
$\nabla \cdot \bar{B} = 0$	(1-15)	(1-4) $\oiint \bar{B} \cdot d\bar{s} = 0$	(Gauss' Law for magnetic field, implying absence of magnetic charges.

There are two fundamental vector-operator identities which are useful in the development of electromagnetic theory.

(1-16) $\nabla \cdot (\nabla \times \bar{V}) = 0$

and

(1-17) $\nabla \times (\nabla F) = 0$

where $\bar{V}$ is an arbitrary vector field and F is an arbitrary scalar field.

[illegible]

(1-24) $$\vec{\mathcal{E}} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$ [illegible]

The negative sign is used in order that V [illegible] potential energy per unit charge. It is again quite [illegible] demonstrate that (1-24) satisfies (1-23) [illegible]. Hence, we summarize the definitions of the potentials

(1-25) $$\vec{B} = \nabla \times \vec{A}$$

(1-26) $$\vec{\mathcal{E}} = -\frac{\partial \vec{A}}{\partial t} - \nabla V .$$

The utility of $\vec{A}$ and V rests in their relation to the charge and current densities. Upon taking the curl of B and substituting the result into (1-12), we get, upon using the constitutive relation $B = \mu \vec{H}$, where μ is the magnetic permeability of the medium

(1-27) $$\nabla \times \vec{B} = \mu \nabla \times \vec{H} = \mu \vec{J} + \mu \frac{\partial \vec{D}}{\partial t} = \nabla \times \nabla \times \vec{A} .$$

Upon using a second constitutive relation, $D = \varepsilon \mathcal{E}$, where ε is the dielectric constant, and then substituting (1-26) into (1-27) there results

(1-28) $$\nabla \times \nabla \times \vec{A} = \mu \vec{J} + \mu\varepsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} = \mu \vec{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\frac{\partial \vec{A}}{\partial t} - \nabla V\right)$$

Thus we have

(1-29) $$\nabla \times \nabla \times \vec{A} + \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} + \mu\varepsilon \nabla \left(\frac{\partial V}{\partial t}\right) = \mu \vec{J} .$$

The expression $\nabla \times \nabla \times \vec{A}$ may be expanded by the vector identity

(1-30) $$\nabla \times \nabla \times \vec{A} = -\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A}) .$$

This identity is taken to be the definition of the [illegible] of the vector A (it, of course, [illegible] arbitrary [illegible] statements of vector identity [illegible])

When we substitute (1-30) back into [illegible]

(1-31) [illegible]

[illegible]

$$(1\text{-}32) \qquad \nabla \cdot \vec{A} = -\mu\varepsilon \frac{\partial V}{\partial t},$$

we arrive at our desired result, an inhomogeneous wave equation for the vector potential A:

$$(1\text{-}33) \qquad \nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J},$$

This equation shows quite clearly that J serves as a source for the vector potential, A.

We may derive a similar equation for the potential V, viz., upon substitution of (1-26) into Gauss' law, (1-1d), there results

$$(1\text{-}34) \qquad \nabla \cdot \vec{E} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \frac{1}{\varepsilon}\nabla \cdot \vec{D} = \rho/\varepsilon$$

We have introduced the Laplacian of a scalar: $\nabla^2 V = \nabla \cdot (\nabla V) = \nabla^2 V$

Hence, if we substitute the Lorentz condition (1-32) in (1-34), we obtain

$$(1\text{-}35) \qquad \nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon,$$

which is of precisely the same form as (1-33) except that (1-35) involves scalars, only. From this result we conclude that the electric charge density is the source of the scalar potential V.

If we are concerned with static, i.e., time-independent, problems, then (1-33) and (1-35) reduce to the following:

$$(1\text{-}36) \qquad \nabla^2 \vec{A} = -\mu \vec{J}$$ Vector Poisson's equation

$$(1\text{-}37) \qquad \nabla^2 V = -\rho/\varepsilon$$ Scalar Poisson's equation

These equations are called the vector and scalar Poisson's equations. If the right-hand terms (the source terms) vanish, we arrive at Laplace's equation for a vector or scalar field:

$$(1\text{-}38) \qquad \nabla^2 \vec{A} = 0$$ Vector Laplace's equation

$$(1\text{-}39) \qquad \nabla^2 V = 0$$ Scalar Laplace's equation

[illegible] be most [illegible] in this book [illegible] degrees are magnetic [illegible]

1-2 Boundary Conditions When [illegible] dissimilar media, i.e., media in which [illegible] ϵ, μ or σ varies abruptly, it is necessary [illegible] conditions on the field components at the [illegible] different media. The conditions, which [illegible] fields courses (see also problem [illegible]), are

(1-40) $$\bar{a}_n \times (\bar{\mathcal{E}}_1 - \bar{\mathcal{E}}_2) = 0 \quad \text{or} \quad \mathcal{E}_{t_1} = \mathcal{E}_{t_2}$$

(1-41) $$\mathcal{D}_{n_1} - \mathcal{D}_{n_2} = \rho_s$$

(1-42) $$\mathcal{B}_{n_1} = \mathcal{B}_{n_2}$$

(1-43) $$\bar{a}_n \times (\bar{\mathcal{H}}_1 - \bar{\mathcal{H}}_2) = \bar{J}_s \quad \text{or} \quad \mathcal{H}_{t_1} - \mathcal{H}_{t_2} = J_s$$

The subscript "t" stands for the tangential component at the boundary, whereas the subscript "n" stands for the normal component. The positive unit normal vector is taken to point [illegible] ρ_s is the (free) surface charge density [illegible] surface separating the media. ρ_s has the dimensions of coulombs per meter squared. $\bar{J}_s$ is the surface current density flowing on the interface. It has the dimensions of amperes per meter, where the meter is measured along the boundary surface [illegible] the two media.

In terms of the vector and scalar potentials [illegible] boundary conditions become

(1-44) $$V_1 = V_2$$

(1-45) $$-\epsilon_1 \frac{\partial V_1}{\partial n} = -\epsilon_2 \frac{\partial V_2}{\partial n} + \rho_s$$

(1-46) $$\bar{A}_{t_1} = \bar{A}_{t_2}$$

(1-47) $$\bar{a}_n \times \left(\frac{1}{\mu_1} \nabla \times \bar{A}_1 - \frac{1}{\mu_2} \nabla \times \bar{A}_2\right) = \bar{J}_s$$

[illegible] electrical [illegible] field [illegible]

[illegible] vector to the [illegible]

1-3. Examples of Magnetic Boundary Value Problems. [illegible] apply the preceding concepts to some specific cases. [illegible] of more than academic interest for, as we shall see, they are [illegible] models of electromagnetic machines. In each of the examples [illegible] conditions prevail, so that we shall use either (1-36) or (1-39) and the appropriate boundary conditions.

Example (1-3-1). Cylinder in an External Magnetic Field

Calculate the fields $\bar{B}$, $\bar{H}$ inside and outside the cylinder carrying a net current (uniformly over the cross section) [illegible] in the z-direction (out of the page) (Fig. 1-1). [illegible]

Figure 1-1. Calculation of Internal and External Fields

Solution: We use cylindrical coordinates r, θ, z and [illegible] that the problem is two-dimensional, with all field vectors independent of z. In the region r > a, the vector potential [illegible] (1-30), because J vanishes in there, while for r < a, A [illegible] (1-36) with $\mu = \mu_1$, and $J = \frac{I}{\pi a^2} \bar{a}_z$.

Because all the differential equations of [illegible] (assuming that the permeability μ_1 does not vary with [illegible]) we may use superposition to resolve A into two parts: A_0, [illegible] the applied field B_0, and the secondary [illegible] due in part to the current I and, in part, to the induced magnetization of the cylinder.

It is a simple matter to verify that

$$\bar{A}_0 = B_0 y \bar{a}_z = B_0 r \sin\theta \, \bar{a}_z \tag{1-40}$$

is a vector potential from which $\bar{B}_0$ may be derived [illegible] with $\bar{B}_0 = \nabla \times \bar{A}_0$.

We anticipate that A_1, the potential due to I, [illegible] in the a_z direction, the same as J.

(1-49) $\vec{A}_1 = A(x,y)\,e_z = A(r,\theta)\,e_z$

and

(1-50) $$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} = -\mu_1 J = -\mu_1 \frac{I}{\pi a^2}$$

where we have expressed ∇^2 in cylindrical coordinates (see the Appendix).

The inhomogeneous equation, (1-50), possesses a complementary function which satisfies the homogeneous equation and a particular integral which satisfies (1-50).

In order to compute the particular integral, we multiply (1-50) by r^2 and observe that the resulting right side, $-\mu_1 \frac{I}{\pi}\left(\frac{r}{a}\right)^2$, is independent of θ. Therefore, we look for a particular integral, A_p, which is also independent of θ. A_p, therefore must satisfy

(1-51) $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_p}{dr}\right) = -\mu_1 \frac{I}{\pi a^2}.$$

It is a simple matter to determine that the solution of (1-51) is

(1-52) $$A_p(r) = -\frac{\mu_1 I}{4\pi}\left(\frac{r}{a}\right)^2$$

Our next task is to compute the complementary function, A_c, satisfying

(1-53) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_c}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_c}{\partial \theta^2} = 0.$$

We postulate a product solution of the form

(1-54) $$A_c(r,\theta) = R(r)\,\Theta(\theta),$$

where R depends only on r and Θ only on θ.

Substitution of (1-54) into (1-53) and division by $R\Theta$ yields after rearrangement

(1-55) $$\frac{r}{R(r)}\cdot\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta}\cdot\frac{d^2\Theta}{d\theta^2}$$

The left-hand side [illegible]
the right-hand side is independent of r. Hence, each side must be equal to a constant which we shall write as n^2.

$$(1\text{-}56) \qquad \frac{r}{R(r)} \cdot \frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

$$(1\text{-}57) \qquad -\frac{1}{\Theta} \cdot \frac{d^2\Theta}{d\theta^2} = n^2 .$$

We have, therefore, separated the <u>partial</u> differential equation (1-53) into two <u>ordinary</u> differential equations, (1-56), (1-57), which may be written

$$(1\text{-}58) \qquad \frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2 R}{r^2} = 0$$

$$(1\text{-}59) \qquad \frac{d^2\Theta}{d\theta^2} + n^2\Theta = 0 .$$

The solution of (1-59) which exhibits the periodicity 2π (the periodicity of the original physical configuration) is

$$(1\text{-}60) \qquad \Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta \qquad n = 1, 2, 3, \ldots$$

where a_n, b_n are arbitrary constants.

It is not at all difficult to prove that r^n, r^{-n} are, for any integer n, solutions of (1-58). Hence,

$$(1\text{-}61) \qquad R_n(r) = c_n r^n + d_n r^{-n} .$$

The most general solution, then, involves the sum of solutions of the form (1-54)

$$(1\text{-}62) \qquad A_c(r,\theta) = \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n + \sum_{n=0}^{\infty} \left(c_n \cos n\theta + d_n \sin n\theta\right) r^{-n}$$

Thus, (1-62) is in the form of a "generalized Fourier series."

We should recognize, however, that if (1-62) is to be part of the vector potential <u>within</u> the cylinder $r < a$, then the last summation must vanish or else r^{-n} will "explode" at $r = 0$. Thus, $c_n = d_n = 0$, and, therefore, the vector potential field inside the wire is the sum of (1-52) and (1-62)

$$(1\text{-}63) \qquad A^-(r,\theta) = -\frac{\mu_0 I}{4\pi}\left(\frac{r}{a}\right)^2 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n \qquad r < a$$

[illegible]

the same as A_c. We [illegible] now we must set a_n [illegible] b_n [illegible] 0 [illegible] which is in the region [illegible] to the [illegible] will be $\sum_{n=0}^{\infty}(c_n \cos n\theta + d_n \sin n\theta)$ [illegible] another part will be d_0 [illegible]

Finally, we must add the vector potential [illegible] cylinder due to the current [illegible] within the cylinder [illegible] Ampere's law, (1-2), neglecting $\partial D/\partial t$, we determine that the [illegible] is $H_\theta = I/2\pi r$. Therefore, $B_\theta = \mu_0 I/2\pi r$ [illegible] μ_0 because that is the permeability of the [illegible] as another simple exercise the proof that [illegible] potential which yields the above B_θ has only a z-component [illegible] This function does become infinite as r [illegible] only because it is produced by an infinitely long [illegible]

Hence, superposing the above three [illegible]

$$A^{+}(r,\theta) = B_0 r \sin\theta + \frac{\mu_0 I}{2\pi} \ln\left(\frac{1}{r}\right) + [illegible] \qquad (1\text{-}64)$$

At $r = a$, the boundary condition [illegible] that the z-direction, the direction of A^{+} [illegible] the cylinder surface)

$$A^{+}(a,\theta) = A^{-}(a,\theta) \qquad (1\text{-}65)$$

and (1-47) becomes (in the absence of a [illegible]

$$\frac{1}{\mu_1}\frac{\partial A^{-}}{\partial r} = \frac{1}{\mu_0}\frac{\partial A^{+}}{\partial r} \quad [illegible] \qquad (1\text{-}66)$$

Applying (1-65) to (1-63) and (1-64), we [illegible]

$$-\frac{\mu_0 I}{2\pi} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) [illegible] \qquad (1\text{-}67)$$

and application of (1-66) to (1-63) and (1-64) [illegible]

$$\frac{1}{\mu_1}\sum_{n=1}^{\infty} n(a_n \cos n\theta + b_n \sin n\theta) [illegible] \qquad (1\text{-}68)$$

[illegible]

by equating the coefficients of cos θ [illegible] doing the same for sin θ, cos 2θ, etc. [illegible] stant terms. For example, equating the [illegible] independent of θ) in (1-67) we get

$$(1\text{-}69) \qquad C_0 = \frac{I}{4\pi}\left([illegible] \ln a [illegible] \mu_1 \right).$$

Equating the coefficients of cos θ in (1-67) and (1-68) separately yields

$$(1\text{-}70) \quad (a) \qquad a\, a_1 = [illegible]$$

$$(b) \qquad \frac{1}{\mu_1} a_1 = \frac{1}{\mu_0 a^2} [illegible]$$

The solution of (1-70) (a), (b) is [illegible]. [illegible] the coefficients of sin θ in each of (1-67) and (1-68) we get

$$(1\text{-}71) \quad (a) \qquad a\, b_1 = a\, b_0 + \frac{d_1}{a}$$

$$(b) \qquad \frac{1}{\mu_1} b_1 = \frac{b_0}{\mu_0} - \frac{d_1}{\mu_0 a^2},$$

which has as its solution

$$(1\text{-}72) \qquad b_1 = \frac{2\mu_1}{\mu_1+\mu_0} B_0 \quad , \qquad d_1 = \frac{\mu_1 - \mu_0}{\mu_1+\mu_0} [illegible]$$

Continuing this process leads to the conclusion that all higher coefficients vanish. Hence, upon substituting (1-69) and (1-72) back into (1-63), (1-64) we have our solution for A

$$(1\text{-}73) \quad (a) \qquad A^-(r,\theta) = -\frac{\mu_1 I}{4\pi}\left(\frac{r}{a}\right)^2 + \frac{2\mu_1 [illegible]}{\mu_1 + [illegible]} [illegible]$$

$$(b) \qquad A^+(r,\theta) = -\frac{\mu_1 I}{4\pi} + \frac{\mu_0 I}{2\pi} \ln\left(\frac{a}{r}\right) + \left[[illegible] \right] B_0 [illegible]$$

[illegible]

From the expression for $\nabla \times \vec{A}$ [illegible] we [illegible] expressions for B

(1-74) (a) $\vec{B}^{+}(r,\theta)$ [illegible]

$= B_0 \cos\theta \left(1 + \left(\frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}\right) [illegible] \right) \hat{a}_r + [illegible]$

$+ \left(\frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}\right)\left(\frac{a}{r}\right)^2 B_0 \sin\theta \Big) \hat{a}_\theta$

$= B_0 \hat{a}_x + \frac{\mu_0 I}{2\pi r} \hat{a}_\theta + \left(\frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}\right)\left(\frac{a}{r}\right)^2 B_0 (\cos 2\theta \, \hat{a}_r + \sin 2\theta \, \hat{a}_\theta)$ [illegible]

(b) $\vec{B}^{-}(r,\theta) = \nabla \times \vec{A}^{-} = \frac{2\mu_1 B_0}{\mu_1 + \mu_0} \cos\theta \, \hat{a}_r + \left([illegible] - \frac{2\mu_1 B_0}{\mu_1 + \mu_0} \sin\theta \right) \hat{a}_\theta$

$= \frac{2\mu_1 B_0}{\mu_1 + \mu_0} \hat{a}_x + \frac{\mu_1 I r}{2\pi a^2} \hat{a}_\theta$

The first two terms in (1-74) (a) are [illegible] terms of the applied field, B_0, and imposed current [illegible]. The third term represents the "scattered" field due to the difference in permeability of the two media. The coefficient [illegible] may be called the scattering or reflection coefficient [illegible] and indicates the amount of the applied field that is scattered back into the region $r > a$ by the cylinder.

The second term in (1-74) (b) follows [illegible] application of Ampere's law, while the first is called the "refracted" or "transmitted" field into the cylinder. Note that [illegible] uniform and points in the $\hat{a}_x$ direction within the cylinder. The coefficient [illegible] may be called the transmission coefficient [illegible] the transmitted field within the cylinder [illegible] the applied field. This fact indicates why high permeability [illegible] magnetic devices—it offers an enhanced B field.

The scattered field (I = 0) of (1-74) [illegible] Figure 1-[illegible] for the values $\mu_1 = \infty$, a = [illegible] cm.

Figure 1-2. Field Scattered by a Cylinder ($\mu_1 = \infty$, $a = 5$ cm.)

Before concluding this example let us reexamine its solution in terms of <u>equivalent magnetization</u> or <u>amperian currents</u>. Let us set $I = 0$ and consider the applied and induced field functions only. The magnetic response of materials is defined in terms of a <u>volume magnetization</u> vector $\bar{M}$ which satisfies

$$\bar{M} = \frac{\bar{B}}{\mu_0} - \bar{H} \qquad (1\text{-}75)$$

In addition, for linear materials $\bar{M} = X_m \bar{H}$, where X_m is the <u>magnetic susceptibility</u>. Thus (1-75) may be rewritten

$$\bar{B} = \mu_0(\bar{H} + \bar{M}) = \mu_0 \bar{H}(1 + X_m) = \mu_1 \bar{H} \qquad (1\text{-}76)$$

where μ_1, is the usual permeability of the material. Hence, upon substitution of (1-76) into (1-75) we get.

$$\bar{M} = \bar{B}\left(\frac{\mu_1 - \mu_0}{\mu_1 \mu_0}\right) \qquad (1\text{-}77)$$

Now let us consider Ampere's model of a magnetized medium, Fig. (1-3)

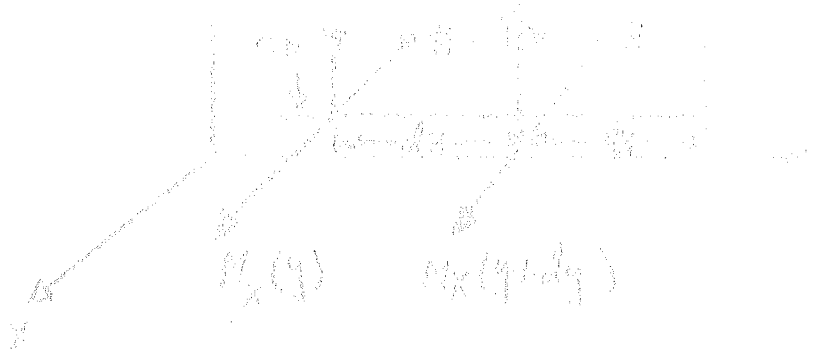


Fig. [illegible] Equivalent Amperian currents in a magnetized medium

Because $\bar{M}$ is the volume magnetization, or magnetic moment per unit volume, it follows that the magnetic moment of a single loop is

$$m = \bar{M}\,dx\,dy\,dz \qquad (1\text{-}78)$$

By its definition, the magnetic moment of a loop of current is $I\,Q\,a_n$, where Q is the area enclosed by the loop, and a_n is normal to the area and points in accordance with the right hand rule.

Therefore the current in loop #1 of Figure [illegible] is

$$I_1 = \frac{M_x(y)\,dx\,dy\,dz}{dy\,dz}, \qquad (1\text{-}79)$$

and by means of a Taylor expansion we may write the current in a nearby loop #2 to be

$$I_2 = \frac{M_x(y+dy)\,dx\,dy\,dz}{dy\,dz} = \frac{\left[M_x(y)+\frac{\partial M_x}{\partial y}dy\right]dx\,dy\,dz}{dy\,dz}. \qquad (1\text{-}80)$$

The net current in the z-direction flowing along the common boundary is

$$I_1 - I_2 = J_{m_z}\,dx\,dy = -\frac{\partial M_x}{\partial y}\,dx\,dy \qquad (1\text{-}81)$$

Hence, $J_{m_z} = -\partial M_x/\partial y$. If we consider loops in the x-z plane with M pointing in the y-direction we would have found $J_{m_z} = \partial M_y/\partial x$. Therefore, in general

$$J_{m_z} = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \qquad (1\text{-}82)\ (a)$$

(1-82) (b) $$J_{mx} = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

(c) $$J_{my} = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} .$$

Those three equations may be subsumed within the single vector equation

(1-83) $$\overline{J_m} = \nabla \times \overline{M} ,$$

where $\overline{J}_m$ is the equivalent magnetization current density. It is a hypothetical current density which, as far as magnetic fields are concerned, effectively replaces magnetic material whose permeability is μ_1 by a current system J_m, flowing in free space (permeability μ_0).

Equation (1-83) reminds us very much of (1-12) with the displacement current set equal to zero. Then just as the boundary condition (1-43), follows from (1-12) so does the magnetization surface current definition

(1-84) $$\overline{J}_{m_s} = \overline{a_n} \times \left(\overline{M_1} - \overline{M_2} \right)$$

follow from (1-83). In (1-84) a_n' is the unit vector pointing into region 1.

In order to apply (1-83) and (1-84) we must compute $\overline{M}$, both inside and outside the cylinder. Outside, of course, $\overline{M}$ vanishes because there is only free space, while within the cylinder we may apply (1-74) (b) and (1-77) to find

(1-85) $$\overline{M} = 2\left(\frac{B_0}{\mu_0}\right)\left(\frac{\mu_1 - \mu_0}{\mu_1 + \mu_0}\right)\overline{a_x}$$

Because $\overline{M}$ as given in (1-85) is spatially uniform within the cylinder, it follows that its curl vanishes, meaning that $J_m = 0$ within the cylinder (see (1-83)). At the surface, of course, there is a discontinuity of amount

(1-86) $$\overline{M_1} - \overline{M_2} = -2\left(\frac{B_0}{\mu_0}\right)\left(\frac{\mu_1 - \mu_0}{\mu_1 + \mu_0}\right)\overline{a_x}$$

Hence, by (1-84) there is an equivalent surface current

(1-87) $$\overline{J}_{m_s} = \left(\overline{a_n} \times \overline{a_x}\right)\left(-\frac{2B_0}{\mu_0}\right)\left(\frac{\mu_1 - \mu_0}{\mu_1 + \mu_0}\right)$$ [illegible]

[illegible] as [illegible]
[illegible], we cannot [illegible] cylinder having a field induced in it by [illegible] there is a surface current given by (1-67) [illegible] cylinder (i.e., the interior of the cylinder is free [illegible]). These alternatives are useful in discussing field [illegible]

Incidentally, as a by-product of this analysis we see that the surface current distribution (1-67) produces a uniform field within the cylinder (given by the first term of (1-74)(b)) and an external field given by (1-74)(a) with [illegible] = 0.

Example (1-3-2): Distributed Current Sheets

Calculate the magnetic field established in the air gap between two infinitely permeable cylindrical regions. There is a distributed current sheet at $r = R_a$ and a concentrated surface current at $r = R_b$. This is an idealized model used to calculate the armature and field flux in a rotating, commutator (usually d.c.) machine. The system is shown in Fig. 1-4.

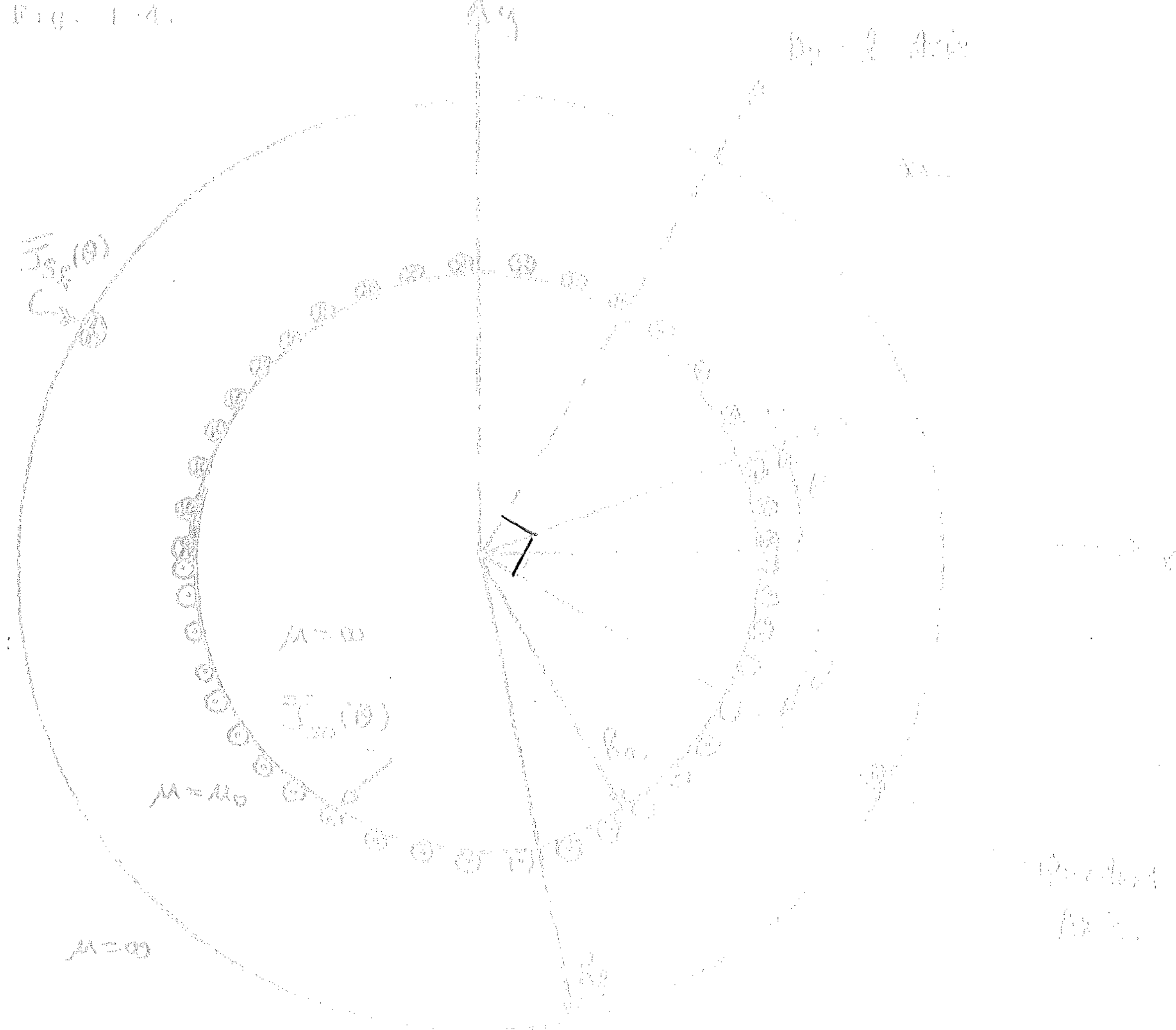


Figure 1-4. Model for calculating [illegible]
due to current sheets

[illegible] as shown in Fig. 1-5.

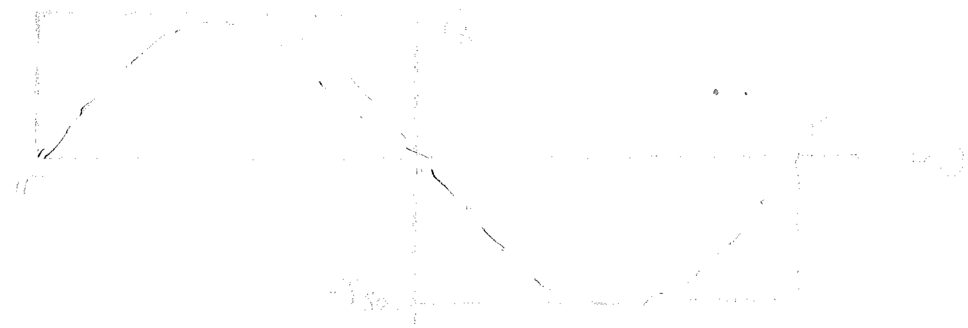


Fig. 1-5 Armature current sheet [illegible]

The concentrated field current [illegible]. The effect of the concentrated windings [illegible] is modeled as a system of impulse functions [illegible]

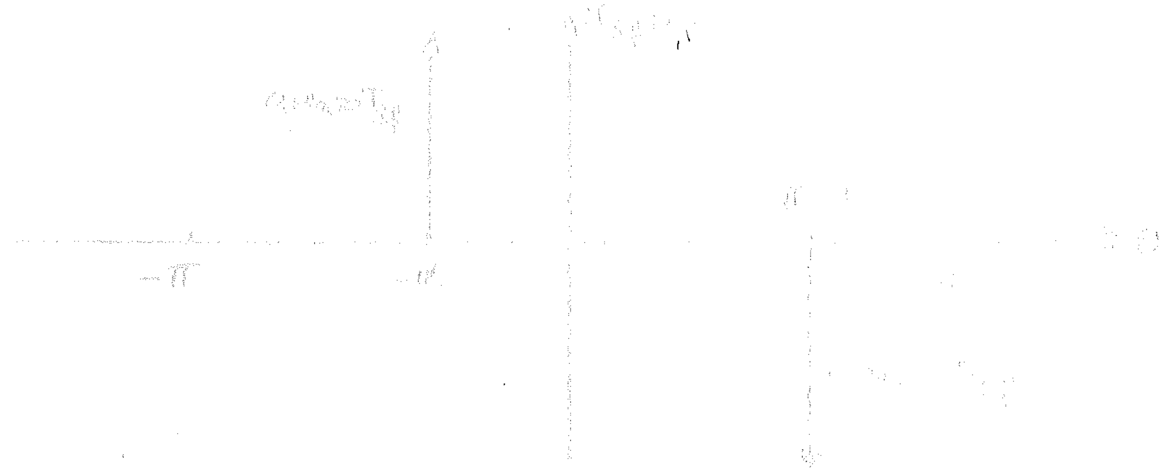


Fig. 1-6 $J_{sf}(\theta)$ [illegible]

Solution. Since we have cylindrical symmetry (all fields independent of z) and all currents flow in the z direction, we need only the z-component $A_z(r,\theta)$ for the vector potential [illegible] because the air-gap is void of currents A_z satisfies [illegible] equation there

$$\nabla^2 A_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} = 0 \qquad (1\text{-}80)$$

The solution of (1-80) has already been found [illegible] generalized Fourier series of (1-62)

$$A_z(r,\theta) = \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right) [\text{illegible}]$$

(1-81) [illegible]

[illegible] direction [illegible] contribute to B when we [illegible] the curl of A [illegible] involves differential operators).

In order to evaluate the constants a_n, b_n, c_n, d_n, we use the boundary conditions (1-47) evaluated at $r = R_f$ and $r = R_g$. At the former boundary we set $J_s = J_{sf}$, while at the latter $J_s = J_{sg}$.

Now in (1-47), $\bar{a}_n$ is in the radial direction, so that at $r = R_f$, $\bar{a}_n$ points into the infinitely permeable material (see Fig. (1-3)). Therefore in (1-47) $\mu_1 = \infty$ and $\mu_2 = \mu_0$, i.e. region 2 is the air gap only at $r = R_f$. Hence, (1-47) becomes

$$(1\text{-}90) \qquad \bar{a}_r \times \left(-\frac{1}{\mu_0} \nabla \times \bar{A}\right)\Big|_{r=R_f} = J_{sf}(\theta)$$

What we are effectively saying here is that the infinitely permeable region "short circuits" H within it, reducing it to zero.

But $\nabla \times \bar{A} = \frac{1}{r}\frac{\partial A_z}{\partial \theta}\bar{a}_\theta - \frac{\partial A_z}{\partial r}\bar{a}_\theta$ so that (1-90) reads for the z-component (which is tangential to the interface at $r = R_f$)

$$(1\text{-}91) \qquad \frac{1}{\mu_0} \cdot \frac{\partial A_z}{\partial r}\Big|_{r=R_f} = J_{sf}(\theta).$$

Upon substitution of (1-89) into (1-91), we obtain

$$(1\text{-}92) \qquad \sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\left\{n\left(a_n R_f^{(n-1)} - c_n R_f^{-(n+1)}\right)\cos n\theta + n\left(b_n R_f^{(n-1)} - d_n R_f^{-(n+1)}\right)\sin n\theta\right\}$$

$$= \mu_0 J_{sf}(\theta) = \int_{-\pi}^{\pi} \mu_0 J_{sf}\left(\delta(\theta+\alpha) - \delta(\theta+\alpha-\pi)\right) \cos m\theta$$

The coefficients of cos m θ and sin m θ may be found as in any Fourier series expansion.* We shall demonstrate the method for cos m θ. Multiplying each side of (1-92) by cos m θ and then integrating from $-\pi$ to π, we get

$$(1\text{-}93) \qquad \pi m\left(a_m R_f^{(m-1)} - c_m R_f^{-(m+1)}\right) = \mu_0 J_{sf}\int_{-\pi}^{\pi}\cos m\theta\left[\delta(\theta+\alpha) - \delta(\theta+\alpha-\pi)\right]d\theta$$

$$= \mu_0 J_{sf}\cos m\alpha\left(1 - \cos m\pi\right) = 0, \text{ if } m \text{ is even}$$

$$= 2\mu_0 J_{sf}\cos m\alpha, \text{ if } m \text{ is odd}$$

* See, for example, E.M. Sabbagh, Circuit Analysis, The Ronald Press, New York, 1961, Ch. 25.

(1-94) $\pi m \left(b_m R_f^{(m-1)} - d_m R_f^{-(m+1)} \right)$

$= -2\mu_0 J_{sa} \sin m\alpha$, if m is odd

Equations (1-93) and (1-94) are two equations in the four unknowns a_m, b_m, c_m, d_m. Of course, we need two more equations and we get them from the conditions at $r = R_a$. We use the same reasoning that led to (1-90), with the exception that at $r = R_a$, $\mu_1 = \mu_0$ and $\mu_2 = \infty$. Hence, (1-47) becomes

$$(1\text{-}95) \qquad \bar{a}_r \times \left(\frac{1}{\mu_0} \nabla \times \bar{A} \right) \Big|_{r=R_a} = -\frac{1}{\mu_0} \frac{\partial A_z}{\partial r} \Big|_{r=R_a} \bar{a}_z = \bar{J}_{sa}(\theta).$$

In place of (1-92) we have

$$(1\text{-}96) \qquad \sum_{n=1}^{\infty} \left\{ n \left(a_n R_a^{(n-1)} - c_n R_a^{-(n+1)} \right) \cos n\theta + n \left(b_n R_a^{(n-1)} - d_n R_a^{-(n+1)} \right) \sin n\theta \right\} = -\mu_0 J_{sa}(\theta).$$

$\int_{-\pi}^{\pi} \mu_0 J_{sa} \sin\theta \sin m\theta$

$m = 1 = n$

$b_1 - d_1 R_a^{-2} = \mu_0 J_{sa}$

Because $J_{sa}(\theta)$ is an odd function, as shown in Fig. (1-4), the coefficient of cos mθ must vanish for each m

$$(1\text{-}97) \qquad a_m R_a^{(m-1)} - c_m R_a^{-(m+1)} = 0, \quad \text{for each } m.$$

In addition, the equation for b_m, d_m is easily found to be

$$(1\text{-}98) \qquad b_m R_a^{(m-1)} - d_m R_a^{-(m+1)} = \frac{4\mu_0 J_{sa}}{m^2 \pi}, \quad \text{if } m \text{ is odd};$$

$$= 0, \quad \text{if } m \text{ is even}.$$

We now have the necessary number of equations, (1-93), (1-94), (1-97) and (1-98). Clearly all coefficients vanish for m even, so we need only concern ourselves with the odd order harmonics. The system to be solved is

$$(a) \quad a_m R_f^{(m-1)} - c_m R_f^{-(m+1)} = \frac{2\mu_0}{m\pi} J_{sf} \cos m\alpha$$

$$(1\text{-}99) \quad (b) \quad a_m R_a^{(m-1)} - c_m R_a^{-(m+1)} = 0$$

$$(c) \quad b_m R_f^{(m-1)} - d_m R_f^{-(m+1)} = -\frac{2\mu_0}{m\pi} J_{sa} \sin m\alpha$$

$$(d) \quad b_m R_a^{(m-1)} - d_m R_a^{-(m+1)} = \frac{4\mu_0 J_{sa}}{m^2 \pi} \quad \mu_0 J_{sa}$$

[illegible] two equations [illegible]

(1-100) (a)

$$a_m = \frac{\begin{vmatrix} \frac{2\mu_0}{m\pi} J_{sf} \cos m\alpha & -R_f^{-(m+1)} \\ 0 & -R_a^{-(m+1)} \end{vmatrix}}{\begin{vmatrix} R_f^{(m-1)} & -R_f^{-(m+1)} \\ R_a^{(m-1)} & -R_a^{-(m+1)} \end{vmatrix}}$$

$$= \frac{\frac{2\mu_0}{m\pi} J_{sf} \cos m\alpha \, R_f R_a^{-m}}{\left(\frac{R_f}{R_a}\right)^m - \left(\frac{R_a}{R_f}\right)^m}$$

(b)

$$c_m = R_a^{2m} \cdot a_m = \frac{\frac{2\mu_0}{m\pi} J_{sf} \cos m\alpha \, R_f R_a^{m}}{\left(\frac{R_f}{R_a}\right)^m - \left(\frac{R_a}{R_f}\right)^m},$$

as are the last two

(1-101) (a)

$$b_m = \frac{\frac{2\mu_0}{m\pi} R_a^{-m} R_f J_{sf} \sin m\alpha + \frac{4\mu_0}{m^2\pi} J_{sa} R_a R_f^{m}}{\left(\frac{R_a}{R_f}\right)^m - \left(\frac{R_f}{R_a}\right)^m}$$

(b)

$$d_m = \frac{\frac{2\mu_0}{m\pi} J_{sf} R_f R_a^{m} \sin m\alpha + \frac{4\mu_0}{m^2\pi} J_{sa} R_a R_f^{m}}{\left(\frac{R_a}{R_f}\right)^m - \left(\frac{R_f}{R_a}\right)^m}$$

$$\bar{B} = \bar{a}_r \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \bar{a}_\theta \frac{\partial A_z}{\partial r}$$

$$= \bar{a}_r \left\{ \sum_{n=odd} \left[-n\left(a_n r^{(n-1)} + c_n r^{-(n+1)}\right) \sin n\theta + n\left(b_n r^{(n-1)} + d_n r^{-(n+1)}\right) \cos n\theta \right] \right\}$$

(1-102)

$$- \bar{a}_\theta \left\{ \sum_{n=odd} \left[n\left(a_n r^{(n-1)} - c_n r^{-(n+1)}\right) \cos n\theta + n\left(b_n r^{(n-1)} - d_n r^{-(n+1)}\right) \sin n\theta \right] \right\}$$

$$R_a \geq r$$

After substituting (1-100) and (1-101) into (1-102) and then combining terms, we get

$$\bar{B} = \frac{2\mu_0}{\pi r} R_f J_{sf} \sum_{n=odd} \left\{ - \frac{\left(\frac{r}{R_a}\right)^n + \left(\frac{R_a}{r}\right)^n}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} \sin n(\phi+\theta)\,\bar{a}_r - \frac{\left(\frac{r}{R_a}\right)^n - \left(\frac{R_a}{r}\right)^n}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} \cos n(\phi+\theta)\,\bar{a}_\theta \right\}$$

(1-103)

$$+ \frac{2\mu_0}{\pi r} R_a J_{sa} \sum_{n=odd} \left\{ -\frac{1}{n} \frac{\left(\frac{r}{R_f}\right)^n + \left(\frac{R_f}{r}\right)^n}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} \cos n\theta\,\bar{a}_r + \frac{1}{n} \frac{\left(\frac{r}{R_f}\right)^n - \left(\frac{R_f}{r}\right)^n}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} \sin n\theta\,\bar{a}_\theta \right\}$$

We have written (1-103) in a form which clearly demonstrates the superposition of the partial fields due to J_{sf} and J_{sa}. The results of (1-103) are plotted in Figure (1-7) with R_a/R_f = 1.01, which indicates a small air gap of 0.01 R_f.

@ r = Ra (SURFACE)

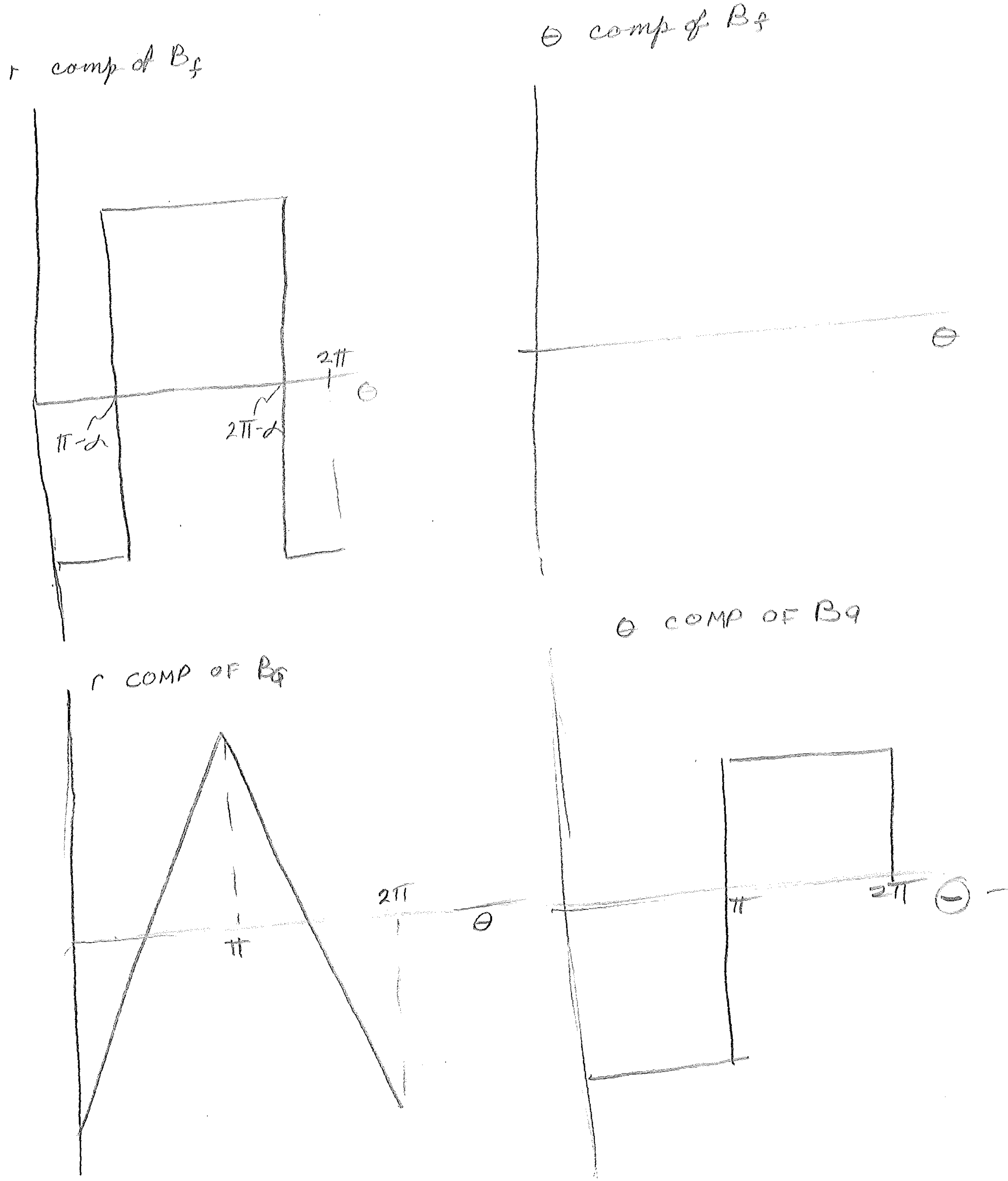


Fig. 1-7. Field plots of equation (1.103)

In Figure 1-8 we show an idealized model of a [illegible] "synchronous" machine. It is similar to the [illegible] 1-3-2 except that the current sheets are sinusoidally distributed along the periphery of the air-gap at $r = R_i$ and $r = R_o$. In addition at $r = R_a$ there are three identical current [illegible] 120° "electrical degrees" apart from each other.

The current sheet located at $r = R_i$ is called the [illegible] current sheet because it is on the rotor, the element free to rotate. The current sheet located at $r = R_o$ is called the stator current sheet

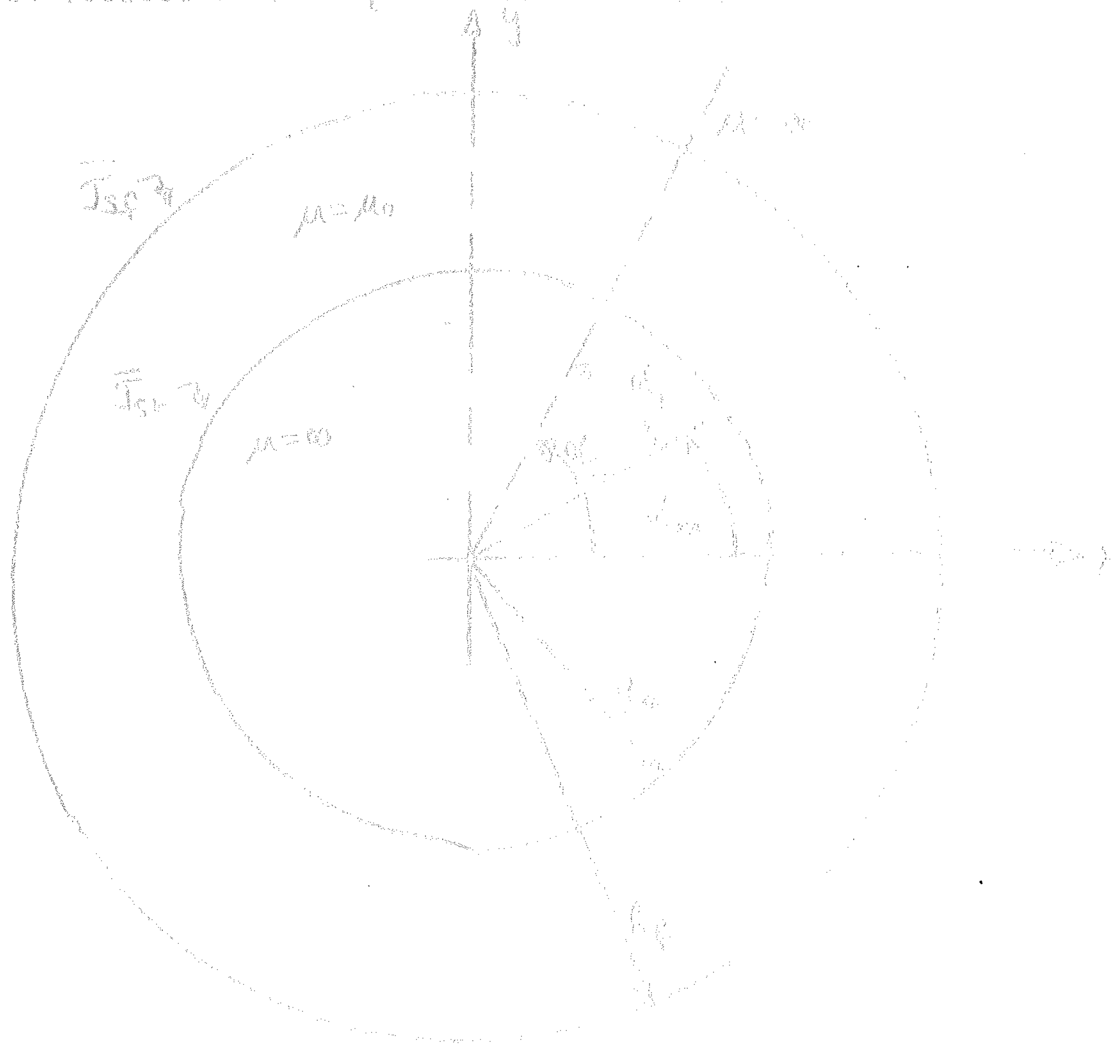


Figure 1-8. Illustrating equivalent current sh[illegible] [illegible] gap machine. A d.c. wind[illegible] [illegible] and three phase windings are [illegible] the [illegible] is the angle of the rotor relat[illegible]

(1-104) (a) $\bar{J}_{sf}(\alpha_2) = J_f \sin P\alpha_2 \ \bar{a}_z$

(b) $\bar{J}_{sr}(\alpha_2) = [J_a \sin P\alpha_1 + J_b \sin(P\alpha_1 - 120°) + J_c \sin(P\alpha_1 - 240°)]\bar{a}_z$

$$= [J_a \sin P(\alpha_2 - \theta_m) + J_b \sin(P\alpha_2 - 120° - P\theta_m) + J_c \sin(P\alpha_2 - 240° - P\theta_m)]\bar{a}_z$$

P is an integer equal to the <u>number of pairs of poles on the machine</u> or, what is the same thing, the <u>electrical periodicity of the current sheets</u>. α_1 is the angle measured with respect to a fixed point on the rotor, while θ_m is the angle that point makes with the stator reference. J_f, J_a, J_b and J_c are independent of angle.

An interesting, and very important, feature of $J_{sr}(\alpha_2)$ is that if the current densities J_a, J_b and J_c are balanced, three phase,

(1-105)
(a) $J_a = N' I_m \sin \omega t$
(b) $J_b = N' I_m \sin(\omega t - 120°)$
(c) $J_c = N' I_m \sin(\omega t - 240°)$,

where N' is the density of turns-per-meter in the rotor windings and I_m is the maximum value of current, then upon substitution of (1-105) into (1-104) (b), we get

(1-106)
$$J_{sr}(\alpha_1) = N' I_m \{\sin P\alpha_1 \sin \omega t + \sin(P\alpha_1 - 120°)\sin(\omega t - 120°) + \sin(P\alpha_1 - 240°)\sin(\omega t - 240°)\}$$

$$= \frac{N' I_m}{2}\{\cos(\omega t - P\alpha_1) - \cos(\omega t + P\alpha_1) + \cos(\omega t - P\alpha_1) - \cos(\omega t + P\alpha_1 - 240°) + \cos(\omega t - P\alpha_1) - \cos(\omega t + P\alpha_1 - 120°)\}$$

$$= \frac{3}{2} N' I_m \cos \omega\left(t - \frac{P}{\omega}\alpha_1\right)$$

[illegible] we [illegible] to balanced, sinusoidally distributed winding, [illegible] traveling wave of current whose phase velocity, [illegible], is

(1-107) $$v_p = \frac{\omega}{p} \quad \text{rad/sec.}$$

The phase velocity is also called the synchronous velocity of the wave in machine theory.

We are interested in calculating $\vec{B}$ within the air-gap.

<u>Solution</u>: Within the air gap $A_z(r,\alpha_s)$ is given by (1-89)

(1-108) $$A_z(r,\alpha_s) = \sum_{n=1}^{\infty} \left(a_{1n} r^n \sin n\alpha_s + a_{2n} r^n \cos n\alpha_s + a_{3n} r^{-n} \sin n\alpha_s + a_{4n} r^{-n} \cos n\alpha_s \right).$$

At $r = R_f$, the boundary condition on [illegible] is

(1-109) $$\frac{1}{\mu_0} \cdot \frac{\partial A_z}{\partial r}\Big|_{r=R_f} = H_{tang}\Big|_{r=R_f} = [illegible],$$

while that at $r = R_a$ is

(1-110) $$-\frac{1}{\mu_0} \cdot \frac{\partial A_z}{\partial r}\Big|_{r=R_a} = H_{tang}\Big|_{r=R_a} = [illegible]$$

Upon substitution of (1-108) into (1-109) and then equating corresponding sine and cosine terms, we get

(1-111)

(a) $$p\left(a_{1p} R_f^{(p-1)} - a_{3p} R_f^{-(p+1)}\right) = \mu_0 I_f$$

(b) $$a_{1n} R_f^{(n-1)} - a_{3n} R_f^{-(n+1)} = 0, \quad \text{for all } n \neq p$$

(c) $$a_{2n} R_f^{(n-1)} - a_{4n} R_f^{-(n+1)} = 0, \quad \text{for all } n$$

$$\text{(a)}\quad p\left[a_{1p} R_a^{(p-1)} - a_{3p} R_a^{-(p+1)}\right] = \text{[illegible]} - \mu_0 J_c \cos(240° + p\theta_a)$$

$$\text{(b)}\quad p\left[a_{2p} R_a^{(p-1)} - a_{4p} R_a^{-(p+1)}\right] = \mu_0 J_a \sin p\theta_a + \mu_0 J_b \sin(120° + p\theta_a) + \mu_0 J_c \sin(240° + p\theta_a) \qquad (7\text{-}112)$$

$$\text{(c)}\quad n\left[a_{1n} R_a^{(n-1)} - a_{3n} R_a^{-(n+1)}\right] = 0, \quad \text{for all } n \neq p$$

$$\text{(d)}\quad n\left[a_{2n} R_a^{(n-1)} - a_{4n} R_a^{-(n+1)}\right] = 0, \quad \text{for all } n \neq p.$$

Equations (7-111) and (7-112) have the following solutions:

$$\text{(a)}\quad a_{1n} = a_{3n} = a_{2n} = a_{4n} = 0, \quad \text{for all } n \neq p.$$

$$\text{(b)}\quad a_{1p} = \frac{-A R_a^{-2p} + B R_f^{-2p}}{-R_a^{-2p} + R_f^{-2p}} \qquad (7\text{-}113)$$

$$\text{(c)}\quad a_{3p} = \frac{B - A}{-R_a^{-2p} + R_f^{-2p}}$$

$$\text{(d)}\quad a_{2p} = \frac{C R_f^{-2p}}{-R_a^{-2p} + R_f^{-2p}}$$

$$\text{(e)}\quad a_{4p} = \frac{C}{-R_a^{-2p} + R_f^{-2p}}$$

where

$$A = \mu_0 J_f / p R_f^{(p-1)}, \quad B = \frac{-\mu_0 J_a \cos p\theta_a - \mu_0 J_b \cos(120° + p\theta_a) - \mu_0 J_c \cos(240° + p\theta_a)}{p R_a^{(p-1)}}$$

$$C = \frac{\mu_0 J_a \sin p\theta_a + \mu_0 J_b \sin(120° + p\theta_a) + \mu_0 J_c \sin(240° + p\theta_a)}{p R_a^{(p-1)}}$$

When (7-113) is substituted into (7-105) and use is made of the relation $\bar{B} = \bar{a}_r \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \bar{a}_\theta \frac{\partial A_z}{\partial r}$

we obtain

(1-114)

$$(b)\quad B_\theta = \frac{[illegible]}{R_c^{2p} - R_o^{2p}} \Big[[illegible] \Big]$$

Later, we will need to know the values of B_r and B_θ at the rotor surface $r = R_o$ under the conditions of balanced three-phase excitation, (1-105). When we substitute (1-105) [illegible] for A, B and C, we get

(1-115)

$$(a)\quad B = -\frac{3}{2}\frac{\mu_0 N' I_m}{p R_o^{(p-1)}} \sin(\omega t + p\theta)$$

$$(b)\quad C = -\frac{3}{2}\frac{\mu_0 N' I_m}{p R_o^{(p-1)}} \cos(\omega t + p\theta)$$

$$(c)\quad A = \frac{\mu_0 I_f}{p R_o^{(p-1)}}$$

and then substituting these results into [illegible] we get

(1-116)

$$(a)\quad B_r(R_o, \theta_2) = \frac{1}{R_o (R_o^{2p} - R_c^{2p})} \Big\{ [illegible] + \frac{3}{2}\mu_0 N I_m (R_c^{2p} + R_o^{2p}) [illegible] \Big\}$$

$$(b)\quad B_\theta(R_o, \theta_2) = \mu_0 I_{S_x}(\theta_2) [illegible]$$

where $N = R_2 N'$.

[illegible] of the rotor [illegible] with phase velocity w/p rad/sec. The radial component [illegible] traveling portion with the same phase velocity, [illegible] space and time phase with the traveling component of B_θ. The remaining part of B_r is established by the d.c. field current and consists of a non-traveling sinusoidal spatial distribution.

1-4. **Energy and Power Relations.** If we take the dot product of (1-10) with $\overline{H}$ and (1-12) with $\overline{E}$ and then subtract the latter from the former result we obtain

$$\overline{H}\cdot\nabla\times\overline{E}-\overline{E}\cdot\nabla\times\overline{H}=-\overline{H}\cdot\frac{\partial\overline{B}}{\partial t}-\overline{E}\cdot\frac{\partial\overline{D}}{\partial t}-\overline{E}\cdot\overline{J}. \tag{1-117}$$

We make use of the vector identity

$$\nabla\cdot(\overline{E}\times\overline{H})=\overline{H}\cdot\nabla\times\overline{E}-\overline{E}\cdot\nabla\times\overline{H} \tag{1-118}$$

to rewrite (1-117) as

$$\nabla\cdot(\overline{E}\times\overline{H})=-\overline{H}\cdot\frac{\partial\overline{B}}{\partial t}-\overline{E}\cdot\frac{\partial\overline{D}}{\partial t}-\overline{E}\cdot\overline{J}. \tag{1-119}$$

The first two terms on the right are, respectively, the negative rate-of-change of magnetic stored energy per unit volume and negative rate-of-change of electric stored energy per unit volume. This result may be put in the more familiar form by assuming the linear constitutive relations $\overline{B}=\mu\overline{H}$, $\overline{D}=\varepsilon\overline{E}$.

$$\text{(a)}\quad \overline{H}\cdot\frac{\partial\overline{B}}{\partial t}=\mu\overline{H}\cdot\frac{\partial\overline{H}}{\partial t}=\frac{\mu}{2}\frac{\partial\overline{H}^2}{\partial t}=\frac{1}{2\mu}\frac{\partial\overline{B}^2}{\partial t} \tag{1-120}$$

$$\text{(b)}\quad \overline{E}\cdot\frac{\partial\overline{D}}{\partial t}=\varepsilon\overline{E}\cdot\frac{\partial\overline{E}}{\partial t}=\frac{\varepsilon}{2}\frac{\partial\overline{E}^2}{\partial t}=\frac{1}{2\varepsilon}\frac{\partial\overline{D}^2}{\partial t}$$

The third term on the right in (1-119) is the [illegible] heat loss rate (power) per unit volume. If we [illegible] $\overline{J}=\sigma\overline{E}$ then this term becomes

$$\overline{J}\cdot\overline{E}=\sigma\overline{E}^2=\frac{\overline{J}^2}{\sigma}. \tag{1-121}$$

Thus, we rewrite (1-119) as

$$\nabla\cdot(\overline{E}\times\overline{H})=-\frac{\partial}{\partial t}\left(\frac{\overline{B}^2}{2\mu}+\frac{\overline{D}^2}{2\varepsilon}\right) \text{[illegible]} \tag{1-122}$$

and p_J [illegible]
is the Joule power dissipated per unit volume.

Upon integration of (1-122) throughout a volume V, we get

$$\iiint \nabla\cdot(\bar{\mathcal{E}}\times\bar{\mathcal{H}})\,dV = -\frac{d}{dt}\iiint (W_m + W_e)\,dV - \iiint P_J\,dV \qquad (1\text{-}123)$$

$$= -\frac{d}{dt}(w_m + w_e) - p_J$$

where lower case letters imply net quantities associated with the region V.

The term on the left in (1-123) may be transformed by using the divergence theorem (1-7) with the result

$$\oint\!\!\oint (\bar{\mathcal{E}}\times\bar{\mathcal{H}})\cdot d\bar{a} = -\frac{d}{dt}(w_m + w_e) - p_J \qquad (1\text{-}124)$$

(energy balance without sources)

The vector $\bar{\mathcal{E}}\times\bar{\mathcal{H}} = \bar{\mathcal{S}}$ has units of power per unit area and is called the Poynting vector. Its integral in (1-124) represents the net outflow of electromagnetic energy across the surface bounding the region V, and the interpretation of (1-124) is that there will be a net outflow of energy leaving the region V and power dissipated as heat, p_J, throughout V only if the stored energy, $w_m + w_e$, decreases with time. Thus, (1-124) is an energy balance equation. If, in addition, there are sources of energy, p_s, within V then the energy (really, power) balance equation becomes

$$\oint\!\!\oint \bar{\mathcal{S}}\cdot d\bar{a} + p_J + \frac{d}{dt}(w_m + w_e) = p_s \qquad (1\text{-}125)$$

(energy balance with sources)

Schematically, (1-125) is shown in Figure (1-[illegible]).

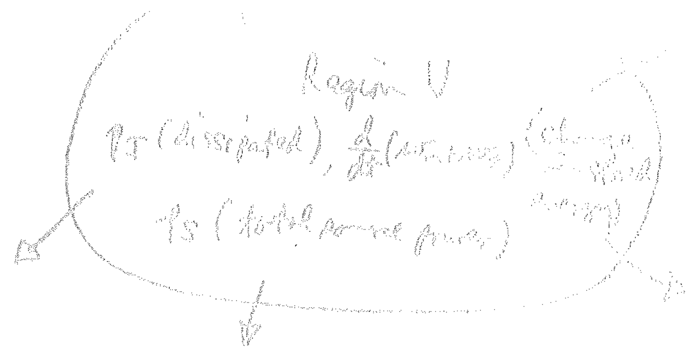


Fig. 1-9. Illustrating energy balance as given by (1-125). p_S supplies heat loss, p_J, change in stored energy, $\frac{d}{dt}(w_m + w_e)$, and net outflow of power $\oint \overline{S} \cdot d\overline{a}$.

The Poynting vector, $\overline{S} = \overline{\mathcal{E}} \times \overline{\mathcal{H}}$, may also be expressed in terms of the potentials V and $\overline{A}$ through the relations (1-25) and (1-26).

$$\nabla\cdot(\overline{\mathcal{E}}\times\overline{\mathcal{H}}) = \nabla\cdot\left(\left(-\nabla V - \frac{\partial \overline{A}}{\partial t}\right)\times\overline{\mathcal{H}}\right) = \overline{\mathcal{H}}\cdot\left(-\frac{\partial}{\partial t}(\nabla\times\overline{A})\right) + \left(\nabla V + \frac{\partial \overline{A}}{\partial t}\right)\cdot\nabla\times\overline{\mathcal{H}}$$

$$= -\overline{\mathcal{H}}\cdot\nabla\times\left(\frac{\partial \overline{A}}{\partial t}\right) + \frac{\partial \overline{A}}{\partial t}\cdot\nabla\times\overline{\mathcal{H}} + (\nabla V)\cdot(\nabla\times\overline{\mathcal{H}})$$

(1-126)

$$= \nabla\cdot\left(\overline{\mathcal{H}}\times\frac{\partial \overline{A}}{\partial t}\right) + \nabla V\cdot\left(\overline{J} + \frac{\partial \overline{D}}{\partial t}\right)$$

$$= \nabla\cdot\left(\overline{\mathcal{H}}\times\frac{\partial \overline{A}}{\partial t}\right) + \nabla\cdot\left[V\left(\overline{J} + \frac{\partial \overline{D}}{\partial t}\right)\right] - V\nabla\cdot\left(\overline{J} + \frac{\partial \overline{D}}{\partial t}\right)$$

$$= \nabla\cdot\left[\overline{\mathcal{H}}\times\frac{\partial \overline{A}}{\partial t} + V\left(\overline{J} + \frac{\partial \overline{D}}{\partial t}\right)\right]$$

In deriving this final form we have used Maxwell's equations, and the equation of charge continuity (1-19).

We realize that if the divergence of two vectors are equal the vectors themselves differ by the curl of an arbitrary third vector. The integral of the curl of such an arbitrary vector over a closed surface, however, will vanish, so that as far as conservation of energy is concerned, we may just as well equate $\overline{\mathcal{E}}\times\overline{\mathcal{H}}$ to $\overline{\mathcal{H}}\times\frac{\partial \overline{A}}{\partial t} + V(\overline{J} + \partial\overline{D}/\partial t)$ and call this latter vector the Poynting vector

(1-127)
$$\overline{\mathcal{S}} = \overline{\mathcal{H}}\times\frac{\partial \overline{A}}{\partial t} + V\left(\overline{J} + \frac{\partial \overline{D}}{\partial t}\right)$$

Now we are going to do something [illegible] and that is to define a stress tensor. As you [illegible] in field theory we like to relate volume [illegible] surface integrals over the corresponding boundary surfaces. This is precisely what we have done in the case of Gauss' law of electrostatics; it related the surface integral of electric flux density, D, to the volume integral of charge density.

Note that electric flux density is a vector point function, while volume charge density is a scalar point function. Thus Gauss' law relates a surface integral of a vector to the volume integral of a scalar. What we now wish to do is relate the volume integral of a body force density vector, f, to the surface integral of something. What should this something be? It can't be a vector because the surface integral of a vector dotted with the element of surface area yields a scalar. Right now let's just call this something, $\bar{\bar{\mathcal{T}}}$. The double bar over $\mathcal{T}$ will be interpreted later to mean a tensor. Thus we want $\bar{\bar{\mathcal{T}}}$ to satisfy

$$\oint\!\!\!\oint_S \bar{\bar{\mathcal{T}}} \cdot d\bar{a} = \iiint_V \bar{f}\, dv \qquad (1\text{-}128)$$

where S is the bounding surface of V. Of course, the right side of (1-128) is the net body force on the material in region V. This is due to the definition of f as a body force per unit volume. Hence, integrating force/vol times (volume) yields force. $\bar{\bar{\mathcal{T}}}$ must, therefore, have the dimensions of force/area, because when it is dotted with surface area and integrated, the result must be force. Thus, $\bar{\bar{\mathcal{T}}}$ may be called a force flux density. Recall from your physics that an entity bearing the dimensions of force/area is pressure. In the mechanics of continuous media, force/area is also called a stress. $\bar{\bar{\mathcal{T}}}$ therefore, may be called a stress tensor, although we still have not defined what we mean by a tensor.

Let's do that right now. For our purposes a tensor is a square matrix with 9 elements. The reason for calling it a tensor rather than a matrix has to do with transformation properties of the elements under an orthogonal rotation of the coordinate axes. We will not pursue this reasoning any longer.

Thus

$$\bar{\bar{\mathcal{T}}} = \begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} & \mathcal{T}_{13} \\ \mathcal{T}_{21} & \mathcal{T}_{22} & \mathcal{T}_{23} \\ \mathcal{T}_{31} & \mathcal{T}_{32} & \mathcal{T}_{33} \end{bmatrix} \qquad (1\text{-}129)$$

[illegible] system [illegible]
[illegible] with corresponding components. [illegible]
column vectors.

The question now becomes [illegible]. The dot product of a tensor with a vector [illegible] in (1-128) between $\bar{\bar{T}}$ and $d\bar{a}$ is nothing [illegible] product of a matrix with a column vector (i.e., [illegible] column only). For example, in (1-128) let

(1-130) $$d\bar{a} = \bar{a}_n\, da = da \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

where da is the magnitude of the surface element, $\bar{a}_n$ is the unit positive normal taken in the outward direction [illegible], and n_x, n_y, and n_z, are the x, y, and z components of $\bar{a}_n$. Of course, $d\bar{a}$ could have been written in a more standard notation as

(1-131) $$d\bar{a} = \bar{a}_n\, da = da\,(n_x \bar{a}_x + n_y \bar{a}_y + n_z \bar{a}_z)$$

According to what we have just [illegible] the dot product of a tensor with a vector, we [illegible]

$$\bar{\bar{T}} \cdot d\bar{a} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} [illegible] \\ [illegible] \\ [illegible] \end{bmatrix}$$

(1-132)

$$= \begin{bmatrix} da\,(T_{11} n_x + T_{12} [illegible]) \\ da\,(T_{21} n_x + T_{22} [illegible]) \\ da\,(T_{31} n_x + T_{32} [illegible]) \end{bmatrix}$$

ordinary vector

$$\overline{\overline{\sigma}} \cdot d\overline{a} = da\left(\sigma_{11} n_x + \sigma_{12} n_y + \sigma_{13} n_z\right)\overline{a}_x \qquad (1\text{-}133)$$

$$+ da\left(\sigma_{21} n_x + \sigma_{22} n_y + \sigma_{23} n_z\right)\overline{a}_y$$

$$+ da\left(\sigma_{31} n_x + \sigma_{32} n_y + \sigma_{33} n_z\right)\overline{a}_z \, .$$

Using (1-133), we rewrite (1-128) in component form

(1-134)

x component: $$\oint\!\!\oint_S \left(\sigma_{11} n_x + \sigma_{12} n_y + \sigma_{13} n_z\right) da = \iiint_V f_x \, dV$$

y component: $$\oint\!\!\oint_S \left(\sigma_{21} n_x + \sigma_{22} n_y + \sigma_{23} n_z\right) da = \iiint_V f_y \, dV$$

z component: $$\oint\!\!\oint_S \left(\sigma_{31} n_x + \sigma_{32} n_y + \sigma_{33} n_z\right) da = \iiint_V f_z \, dV \, .$$

Next, calling

$$\overline{\sigma}_1 = \sigma_{11}\overline{a}_x + \sigma_{12}\overline{a}_y + \sigma_{13}\overline{a}_z$$

$$\overline{\sigma}_2 = \sigma_{21}\overline{a}_x + \sigma_{22}\overline{a}_y + \sigma_{23}\overline{a}_z \qquad (1\text{-}135)$$

$$\overline{\sigma}_3 = \sigma_{31}\overline{a}_x + \sigma_{32}\overline{a}_y + \sigma_{33}\overline{a}_z$$

and recalling that $d\overline{a} = da(n_x\overline{a}_x + n_y\overline{a}_y + n_z\overline{a}_z)$, then it follows that (1-134) may be rewritten as

$$\oint\!\!\oint_S \overline{\sigma}_1 \cdot d\overline{a} = \iiint_V f_x \, dV \qquad (1\text{-}136)$$

$$\oint\!\!\oint_S \overline{\sigma}_2 \cdot d\overline{a} = \iiint_V f_y \, dV$$

$$\oint\!\!\oint_S \overline{\sigma}_3 \cdot d\overline{a} = \iiint_V f_z \, dV .$$

[illegible]
equation in (1-136):

$$\oiint_S \overline{T}_1 \cdot d\overline{a} = \iiint_V (\nabla \cdot \overline{T}_1)\, dV = \iiint_V f_x\, dV$$

$$\oiint_S \overline{T}_2 \cdot d\overline{a} = \iiint_V (\nabla \cdot \overline{T}_2)\, dV = \iiint_V f_y\, dV \qquad (1\text{-}137)$$

$$\oiint_S \overline{T}_3 \cdot d\overline{a} = \iiint_V (\nabla \cdot \overline{T}_3)\, dV = \iiint_V f_z\, dV.$$

from which follows that

$$f_x = \nabla \cdot \overline{T}_1 = \frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} + \frac{\partial T_{13}}{\partial z}$$

$$f_y = \nabla \cdot \overline{T}_2 = \frac{\partial T_{21}}{\partial x} + \frac{\partial T_{22}}{\partial y} + \frac{\partial T_{23}}{\partial z} \qquad (1\text{-}138)$$

$$f_z = \nabla \cdot \overline{T}_3 = \frac{\partial T_{31}}{\partial x} + \frac{\partial T_{32}}{\partial y} + \frac{\partial T_{33}}{\partial z}$$

Incidentally, the right side of (1-138) defines the divergence of a tensor, $\nabla \cdot \overline{\overline{T}}$. Note that it is a vector.

Now let us proceed to derive an equivalent stress tensor for the electromagnetic field. Such a tensor is called the Maxwell stress tensor. We shall consider first the electrostatic field, then the magnetic static field and, finally, the combination, the electrodynamic field.

A. Electric fields only. Let ρ be the electric "free" charge density within a region of space and $\overline{\mathcal{E}}$ the electric field at the point occupied by ρ. Then the electric force density (force/vol) acting at that point is

$$\overline{f}_E = \rho\, \overline{\mathcal{E}} = \overline{\mathcal{E}}\, (\nabla \cdot \overline{\mathcal{D}}) \qquad (1\text{-}139)$$

where the second equality follows from Gauss' law.

Thus,

$$(a)\quad f_{\mathcal{E}_x} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \mathcal{E}_x$$

$$(b)\quad f_{\mathcal{E}_y} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \mathcal{E}_y$$

(3-140)

$$(c)\quad f_{\mathcal{E}_z} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \mathcal{E}_z .$$

Consider (a)

$$\mathcal{E}_x \frac{\partial D_x}{\partial x} = \mathcal{E}_x \frac{\partial}{\partial x}(\varepsilon \mathcal{E}_x) = \frac{\partial}{\partial x}\left(\varepsilon \frac{\mathcal{E}_x^2}{2}\right)$$

$$\mathcal{E}_x \frac{\partial D_y}{\partial y} = \frac{\partial}{\partial y}(D_y \mathcal{E}_x) - D_y \frac{\partial \mathcal{E}_x}{\partial y} = \frac{\partial}{\partial y}(D_y \mathcal{E}_x) - D_y \frac{\partial \mathcal{E}_y}{\partial x}$$

(3-141)

$$\mathcal{E}_x \frac{\partial D_z}{\partial z} = \frac{\partial}{\partial z}(D_z \mathcal{E}_x) - D_z \frac{\partial \mathcal{E}_x}{\partial z} = \frac{\partial}{\partial z}(D_z \mathcal{E}_x) - D_z \frac{\partial \mathcal{E}_z}{\partial x}$$

$$= \frac{\partial}{\partial z}(D_z \mathcal{E}_x) - \frac{\partial}{\partial x}\left(\varepsilon \frac{\mathcal{E}_z^2}{2}\right)$$

In deriving the final forms of the last two equations we have used the electrostatic condition, $\nabla \times \mathcal{E} = 0$, which in component form yields

$$\frac{\partial \mathcal{E}_x}{\partial y} = \frac{\partial \mathcal{E}_y}{\partial x}, \quad \frac{\partial \mathcal{E}_x}{\partial z} = \frac{\partial \mathcal{E}_z}{\partial x}, \quad \frac{\partial \mathcal{E}_y}{\partial z} = \frac{\partial \mathcal{E}_z}{\partial y} .$$

Because each equation in (3-141) [illegible] form of [illegible], we have

(3-142)

$$(a)\quad f_{\mathcal{E}_x} = \frac{\partial}{\partial x}\left(\frac{\varepsilon \mathcal{E}_x^2}{2} - \frac{\varepsilon \mathcal{E}_y^2}{2} - \frac{\varepsilon \mathcal{E}_z^2}{2} \right) + \frac{\partial}{\partial y}(\varepsilon \mathcal{E}_y \mathcal{E}_x) + \frac{\partial}{\partial z}(\varepsilon \mathcal{E}_z \mathcal{E}_x)$$

Similar reasoning gives

$$(b)\quad f_{\mathcal{E}_y} = \frac{\partial}{\partial x}(\varepsilon \mathcal{E}_x \mathcal{E}_y) + \frac{\partial}{\partial y}\left(\frac{\varepsilon \mathcal{E}_y^2}{2} - \frac{\varepsilon \mathcal{E}_z^2}{2} - \frac{\varepsilon \mathcal{E}_x^2}{2} \right) + \frac{\partial}{\partial z}(\varepsilon \mathcal{E}_z \mathcal{E}_y)$$

(3-142)

$$(c)\quad f_{\mathcal{E}_z} = \frac{\partial}{\partial x}(\varepsilon \mathcal{E}_x \mathcal{E}_z) + \frac{\partial}{\partial y}(\varepsilon \mathcal{E}_y \mathcal{E}_z) + \frac{\partial}{\partial z}\left(\frac{\varepsilon \mathcal{E}_z^2}{2} - \frac{\varepsilon \mathcal{E}_x^2}{2} - \frac{\varepsilon \mathcal{E}_y^2}{2} \right)$$

Upon comparing [illegible]

$$(1\text{-}143)\quad T_{11E} = \frac{\varepsilon}{2}\left(E_x^2 - E_y^2 - E_z^2\right),\; T_{12E} = \varepsilon E_x E_y,\; T_{13E} = \varepsilon E_x E_z$$

$$(1\text{-}144)\quad T_{21E} = \varepsilon E_y E_x,\; T_{22E} = \frac{\varepsilon}{2}\left(E_y^2 - E_z^2 - E_x^2\right),\; T_{23E} = \varepsilon E_y E_z$$

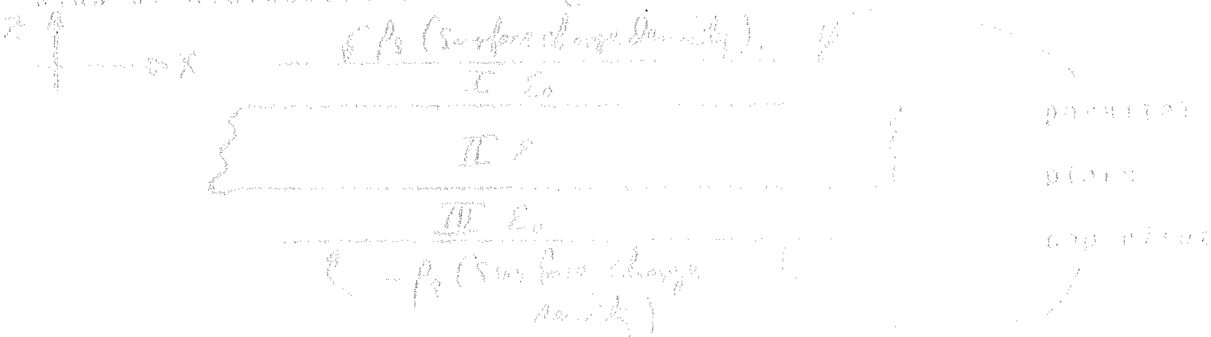
$$(1\text{-}145)\quad T_{31E} = \varepsilon E_x E_z,\; T_{32E} = \varepsilon E_y E_z,\; T_{33E} = \frac{\varepsilon}{2}\left(E_z^2 - E_x^2 - E_y^2\right)$$

The electric stress tensor becomes

$$(1\text{-}146)\quad \bar{\bar{T}}_E = \begin{bmatrix} \frac{\varepsilon}{2}(E_x^2 - E_y^2 - E_z^2) & \varepsilon E_x E_y & \varepsilon E_x E_z \\ \varepsilon E_x E_y & \frac{\varepsilon}{2}(E_y^2 - E_z^2 - E_x^2) & \varepsilon E_y E_z \\ \varepsilon E_x E_z & \varepsilon E_y E_z & \frac{\varepsilon}{2}(E_z^2 - E_x^2 - E_y^2) \end{bmatrix}$$

Note that it is symmetric.

Example (1-5-1). Calculate the pressure on the faces of the slab of dielectric constant ε. Discuss and interpret the results.



Solution: The electric flux density is given by $\vec{D} = \rho_s \vec{a}_z$ in all three regions. In regions I and III, therefore, $\vec{E} = \vec{D}/\varepsilon_0 = (\rho_s/\varepsilon_0)\vec{a}_z$ while in region II $\vec{E} = (\rho_s/\varepsilon)\vec{a}_z$.

The stress tens[illegible]

$$\bar{\bar{\sigma}}_{\epsilon_I} = \bar{\bar{\sigma}}_{\epsilon_{III}} = \begin{bmatrix} -\frac{\epsilon_0}{2}\mathcal{E}_{0z}^2 & 0 & 0 \\ 0 & -\frac{\epsilon_0}{2}\mathcal{E}_{0z}^2 & 0 \\ 0 & 0 & \frac{\epsilon_0}{2}\mathcal{E}_{0z}^2 \end{bmatrix}$$

$$\bar{\bar{\sigma}}_{\epsilon_{II}} = \begin{bmatrix} -\frac{\epsilon}{2}\mathcal{E}_z^2 & 0 & 0 \\ 0 & -\frac{\epsilon}{2}\mathcal{E}_z^2 & 0 \\ 0 & 0 & \frac{\epsilon}{2}\mathcal{E}_z^2 \end{bmatrix}$$

Since all surfaces in this problem have their normals in the ± z-direction, we have for the net pressure on the upper face of the slab

$$(\sigma_{\epsilon_I})_{33} - (\sigma_{\epsilon_{II}})_{33} = \frac{\epsilon_0}{2}\mathcal{E}_{0z}^2 - \frac{\epsilon}{2}\mathcal{E}_z^2 = \frac{\rho_s^2}{2}\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon}\right)$$

and is upward if $\epsilon_0 < \epsilon$. The net pressure on the lower face is

$$(\sigma_{\epsilon_{II}})_{33} - (\sigma_{\epsilon_{III}})_{33} = \frac{\epsilon}{2}\mathcal{E}_z^2 - \frac{\epsilon_0}{2}\mathcal{E}_{0z}^2 = \frac{\rho_s^2}{2}\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right)$$

and is downward if $\epsilon_0 < \epsilon$.

Thus, while the net force on the slab is zero, there are surface stresses (tensions) which tend to pull the slab apart so that it occupies the entire region between the plates (if $\epsilon_0 < \epsilon$). The converse is true if $\epsilon_0 > \epsilon$.

Note that in either case the resultant action is to increase the capacitance of the system (recall that $C = \frac{\epsilon A}{d}$ for a parallel plate capacitor of plate area A and separation d).

The energy stored in a capacitor is $1/2\ Q^2/C$. For fixed charge (as we are assuming here) the stored energy decreases if C increases. Physical systems tend to occupy states of lower potential energy. Hence, for fixed ρ_s the capacitance tends to increase. We shall see later that this result (based on field theory and stress tensors) is consistent with our circuit theory based on energy considerations.

B. Magnetic fields only. Let $\bar{J}$ be the conduction current density within a region of space (i.e., the "body") and $\bar{B}$ be the magnetic induction at the point occupied by $\bar{J}$. Then the magnetic force density (force/vol) is given by

(7-107) $$\bar{f}_M = \bar{J} \times \bar{B} = (\nabla \times \bar{H}) \times \bar{B}$$

$$(a)\quad f_{Mx} = \left(\frac{\partial \mathcal{H}_x}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial x}\right) B_z - \left(\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y}\right) B_y$$

(1-148)
$$(b)\quad f_{My} = \left(\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y}\right) B_x - \left(\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z}\right) B_z$$

$$(c)\quad f_{Mz} = \left(\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z}\right) B_y - \left(\frac{\partial \mathcal{H}_x}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial x}\right) B_x$$

Consider (a)

$$-B_z \frac{\partial \mathcal{H}_z}{\partial x} - B_y \frac{\partial \mathcal{H}_y}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{B_z^2 + B_y^2}{2\mu}\right)$$

$$B_z \frac{\partial \mathcal{H}_x}{\partial z} + B_y \frac{\partial \mathcal{H}_x}{\partial y} = \frac{\partial}{\partial z}\left(B_z \mathcal{H}_x\right) + \frac{\partial}{\partial y}\left(B_y \mathcal{H}_x\right) - \mathcal{H}_x\left(\frac{\partial B_z}{\partial z} + \frac{\partial B_y}{\partial y}\right)$$

(1-149)
$$= \frac{\partial}{\partial z}\left(B_z \mathcal{H}_x\right) + \frac{\partial}{\partial y}\left(B_y \mathcal{H}_x\right) + \mathcal{H}_x \frac{\partial B_x}{\partial x}$$

$$= \frac{\partial}{\partial z}\left(B_z \mathcal{H}_x\right) + \frac{\partial}{\partial y}\left(B_y \mathcal{H}_x\right) + \frac{\partial}{\partial x}\left(\frac{B_x^2}{2\mu}\right)$$

In going from the second to the third equation we used $\nabla \cdot \vec{B} = 0$ which implies that $\partial B_x/\partial x = -\partial B_y/\partial y - \partial B_z/\partial z$.

Thus

(1-150) $$(a)\quad f_{Mx} = \frac{\partial}{\partial x}\left(\frac{B_x^2}{2\mu} - \frac{B_y^2}{2\mu} - \frac{B_z^2}{2\mu}\right) + \frac{\partial}{\partial y}\left(B_y \mathcal{H}_x\right) + \frac{\partial}{\partial z}\left(B_z \mathcal{H}_x\right)$$

Similar reasoning gives

$$(b)\quad f_{My} = \frac{\partial}{\partial x}\left(B_x \mathcal{H}_y\right) + \frac{\partial}{\partial y}\left(\frac{B_y^2}{2\mu} - \frac{B_z^2}{2\mu} - \frac{B_x^2}{2\mu}\right) + \frac{\partial}{\partial z}\left(B_z \mathcal{H}_y\right)$$

(1-150)

$$(c)\quad f_{Mz} = \frac{\partial}{\partial x}\left(B_x \mathcal{H}_z\right) + \frac{\partial}{\partial y}\left(B_y \mathcal{H}_z\right) + \frac{\partial}{\partial z}\left(\frac{B_z^2}{2\mu} - \frac{B_x^2}{2\mu} - \frac{B_y^2}{2\mu}\right)$$

Upon comparing (1-150) with (1-138), we make the assignments:

(1-151) $\sigma_{11M} = \left(\frac{B_x^2}{2\mu} - \frac{B_y^2}{2\mu} - \frac{B_z^2}{2\mu}\right)$, $\sigma_{12M} = \mu \mathcal{H}_x \mathcal{H}_y$, $\sigma_{13M} = \mu \mathcal{H}_x \mathcal{H}_z$

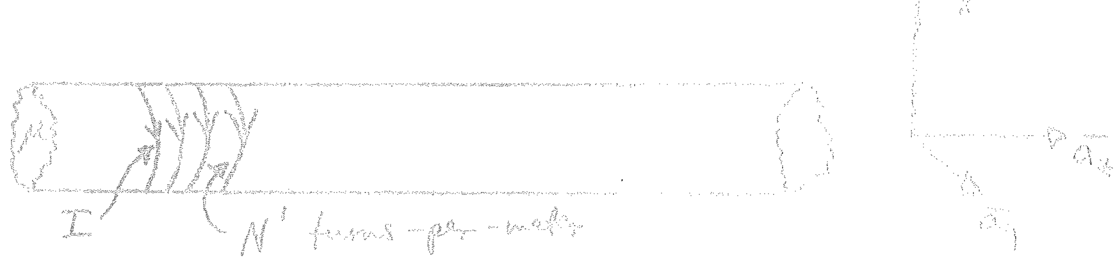
(1-152) $\sigma_{21M} = \mu \mathcal{H}_x \mathcal{H}_y$, $\sigma_{22M} = \left(\frac{B_y^2}{2\mu} - \frac{B_z^2}{2\mu} - \frac{B_x^2}{2\mu}\right)$, $\sigma_{23M} = \mu \mathcal{H}_y \mathcal{H}_z$

(1-153) $\sigma_{31M} = \mu \mathcal{H}_x \mathcal{H}_z$, $\sigma_{32M} = \mu \mathcal{H}_y \mathcal{H}_z$, $\sigma_{33M} = \left(\frac{B_z^2}{2\mu} - \frac{B_x^2}{2\mu} - \frac{B_y^2}{2\mu}\right)$.

(1-154)

$$\overline{\overline{\sigma}}_M = \begin{bmatrix} \frac{\mu}{2}(H_x^2 - H_y^2 - H_z^2) & \mu H_x H_y & \mu H_x H_z \\ \mu H_x H_y & \frac{\mu}{2}(H_y^2 - H_z^2 - H_x^2) & \mu H_y H_z \\ \mu H_x H_z & \mu H_y H_z & \frac{\mu}{2}(H_z^2 - H_x^2 - H_y^2) \end{bmatrix}$$

Example (1-5-2): Calculate the stresses acting on the solenoid. The solenoid is of "infinite" length, radius R and tightly wound with N' turns-per-unit length. Permeability within the solenoid is μ. Total current is I.



Solution: The surface current density is $\overline{J}_S = N' I \,\overline{a}_\varphi$. The magnetic field outside the solenoid is zero, while that inside is $\overline{H} = N' I \,\overline{a}_z$. (prove this statement).

Therefore, the stress tensor outside vanishes, while that inside becomes

$$\overline{\overline{\sigma}}_{inside} = \begin{bmatrix} -\frac{\mu}{2} H_z^2 & 0 & 0 \\ 0 & -\frac{\mu}{2} H_z^2 & 0 \\ 0 & 0 & \frac{\mu}{2} H_z^2 \end{bmatrix} = \begin{bmatrix} -\frac{\mu}{2}(N'I)^2 & 0 & 0 \\ 0 & -\frac{\mu}{2}(N'I)^2 & 0 \\ 0 & 0 & \frac{\mu}{2}(N'I)^2 \end{bmatrix}$$

Because the surface of the solenoid has its normal in the radial direction, we get for the net pressure acting on the surface

$$\overline{p} = (\overline{\overline{\sigma}}_{out} - \overline{\overline{\sigma}}_{in}) \cdot \overline{a}_R = (\overline{\overline{\sigma}}_{out} - \overline{\overline{\sigma}}_{in}) \cdot (\overline{a}_x \cos\varphi + \overline{a}_y \sin\varphi)$$

$$= \begin{bmatrix} \frac{\mu}{2}(N'I)^2 & 0 & 0 \\ 0 & \frac{\mu}{2}(N'I)^2 & 0 \\ 0 & 0 & -\frac{\mu}{2}(N'I)^2 \end{bmatrix} \cdot \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\mu}{2}(N'I)^2 \cos\varphi \\ \frac{\mu}{2}(N'I)^2 \sin\varphi \\ 0 \end{bmatrix}$$

$$= \frac{\mu}{2}(N'I)^2 (\overline{a}_x \cos\varphi + \overline{a}_y \sin\varphi) = \frac{\mu}{2}(N'I)^2 \,\overline{a}_R .$$

[illegible] region of the [illegible] [illegible] cross section *did* increase [illegible] would increase, thereby increasing the [illegible].

The tendency of a system [illegible] amount to energy conversion, [illegible].

If we integrate $\bar{p}$ over a cross section of the surface [illegible] going from 0 to 2π), we find that the net force on the solenoid vanishes. This is in agreement with our everyday experience that a solenoid does not get up and move about when [illegible] excited. It *is* stressed, but the net force tending to [illegible] vanishes. Thus, its center of gravity is not accelerated.

From these two examples, we see that the [illegible] lines suffer a tension along their axes and also repel each other in the transverse (lateral) directions. Hence, electric and magnetic flux lines may be likened to stretched rubber bands repelling each other, the more crowded the flux lines the greater their repulsion.

<u>Example [illegible]:</u> <u>Torque on the Armature of a Commutator Machine</u>

Calculate the torque on the armature of Figure [illegible] (Example [illegible]-3-2).

<u>Solution:</u> Because there is no air gap flux density in the z-direction (out of the page) in Figure [illegible], it follows that the magnetic stress tensor ([illegible]-154), at $r = R_a$, is given in cylindrical coordinates by

$$\bar{\bar{T}}_M(R_a,\theta) = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix}$$

where B_r and B_θ are evaluated at $r = R_a$.

The traction (or stress) acting at any point on the armature surface (whose outward normal is $\bar{a}_R$) is given by

$$\bar{p} = \bar{\bar{T}}_M(R_a,\theta)\cdot\bar{a}_R = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left(\frac{B_r^2 - B_\theta^2}{2\mu_0}\right)\bar{a}_r + \frac{B_r B_\theta}{\mu_0}\,[illegible]$$

Upon taking the [illegible]
radius vector from the origin to any point on the armature surface, we get the torque-per-unit area on the armature as established by the air gap flux density

$$\bar{\tau}' = R_a \bar{a}_R \times \bar{f} = \frac{R_a B_r B_\theta}{\mu_0} \bar{a}_R \times \bar{a}_\theta = \frac{R_a B_r B_\theta}{\mu_0} \bar{a}_z .$$

The torque-per-unit length of the armature is then given by

$$\bar{\tau} = \frac{R_a}{\mu_0} \bar{a}_z \int_0^{2\pi} B_r B_\theta R_a \, d\theta$$

In order to evaluate this integral we must know B_r and B_θ on the armature surface. From (1-95) we see that $B_\theta(R_a, \theta)$ is simply equal to $\mu_0 J_{sa}(\theta)$ and from (1-[illegible]) we have for $B_r(R_a, \theta)$

$$B_r(R_a, \theta) = \frac{2\mu_0}{\pi}\left(\frac{R_f}{R_a}\right) J_{sf} \sum_{n=odd} \left\{ \frac{2}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} \right\} \sin n(\alpha + \theta)$$

$$+ \frac{4\mu_0}{\pi} J_{sa} \sum_{n=odd} \frac{1}{n} \left\{ \frac{\left(\frac{R_a}{R_f}\right)^n + \left(\frac{R_f}{R_a}\right)^n}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} \right\} \cos n\theta .$$

Upon substituting these results, together with Figure (-5), into the integral for $\bar{\tau}$, we obtain

$$\bar{\tau} = R_a^2 \bar{a}_z J_{sa} \int_{-\pi}^{0} \sum_{n=odd} \left\{ \frac{-\frac{4\mu_0}{\pi}\left(\frac{R_f}{R_a}\right) J_{sf}}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} \sin n(\alpha + \theta) - \frac{\frac{4\mu_0}{\pi} J_{sa} \cos n\theta}{n\left[\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n\right]} \right\} d\theta$$

$$- R_a^2 \bar{a}_z J_{sa} \int_{0}^{\pi} \sum_{n=odd} \left\{ \frac{-\frac{4\mu_0}{\pi}\left(\frac{R_f}{R_a}\right) J_{sf} \sin n(\alpha + \theta)}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} - \frac{\frac{4\mu_0}{\pi} J_{sa} \cos n\theta}{n\left[\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n\right]} \right\} d\theta$$

Note that it was necessary to decompose [illegible] into two integrals because $J_{sa}(\theta)$ is discontinuous [illegible] is a straight-forward matter to evaluate the integrals appearing above. Using the fact that n is an odd integer, the result is

$$\bar{\tau} = \frac{16\mu_0}{\pi} R_a R_f J_{sa} J_{sf} \sum_{n=\text{odd}}^{\infty} \frac{\cos n\alpha}{n\left[\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n\right]} \hat{i}_z$$

The infinite sum is plotted versus α in Figure 1-10 for $\frac{R_a}{R_f} = 0.99$. In practice one usually arranges the system (through the use of brushes and a commutator) so that $\alpha = 0$, thereby producing maximum torque. Referring to Figure 1-4, this condition implies that the armature current sheet is aligned along the quadrature axis (i.e., $\alpha = 0$ in Figure 1-4). If $\alpha = \pi/2$, the net torque vanishes. There does exist a surface traction at each point tending to rotate the armature, however, but when $\alpha = \pi/2$ these tractions cancel out over the circumference of the armature.

Figure 1-10 [illegible]

[illegible] v_{sa} [illegible] armature current, i_a, and field current i_f, we conclude that $\tau = k_f i_f i_a$, where k_f is a constant called the torque constant. Our analysis gives us a means of calculating it in terms of α, μ_0, R_a, R_f and the density of turns in the armature and field coils.

Example (1-5-4): Torque in a Polyphase Machine

Calculate the torque on the rotor of the machine of Figure 1-8, Example 1-3-3.

Solution: The magnetic stress tensor at the surface of the rotor is again

$$\bar{\bar{T}}_M(R_a, x_2) = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & \frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix}$$

with B_r and B_θ being evaluated at $r = R_a$.

The surface traction acting on the rotor, whose outward normal is $\bar{a}_R$, is

$$\bar{p} = \bar{\bar{T}}_M(R_a, x_2) \cdot \bar{a}_R = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left(\frac{B_r^2 - B_\theta^2}{2\mu_0}\right)\bar{a}_R + \frac{B_r B_\theta}{\mu_0}\bar{a}_\theta .$$

The torque-per-unit area is given by

$$\bar{\tau}' = R_a(\bar{a}_R \times \bar{p}) = R_a \frac{B_r B_\theta}{\mu_0}\bar{a}_z$$

and the torque-per-unit length of rotor is

$$\bar{\tau} = \bar{a}_z \frac{R_a}{\mu_0}\int_0^{2\pi} B_r B_\theta R_a \, dx_2 ,$$

[illegible]
three phase excitation (see (1-105)):

When (1-116) is substituted into this integral there results easily

$$\bar{\tau} = \bar{a}_z \cdot \frac{3}{2} \frac{\mu_0 N I_a I_f}{\left(\frac{R_f}{R_a}\right)^p - \left(\frac{R_a}{R_f}\right)^p} \cos(\omega t + p\alpha_m)$$

In this expression, unlike that for $\bar{\tau}$ in the preceding example, α_m is not fixed in space but changes with time at a fixed rate (under steady-state conditions). The time-average torque vanishes unless $d\alpha_m/dt = \omega_m = -\omega/p$. Under this condition $\alpha_m = -\omega t/p + \theta/p$ where θ is an arbitrary reference phase angle measured in "electrical degrees." Upon substitution of this expression for α_m back into $\bar{\tau}$ we find that $\bar{\tau}$ is constant in time

$$\bar{\tau} = \frac{3}{2} \cdot \frac{\mu_0 N I_a I_f}{\left(\frac{R_f}{R_a}\right)^p - \left(\frac{R_a}{R_f}\right)^p} \cos\theta \cdot \bar{a}_z$$

Thus, we conclude that in such a balanced, polyphase machine, torque is produced, on the average, at only one speed, the synchronous speed given by $\omega_m = -\omega/p$. Such a machine is called a synchronous machine and will be studied in depth later.

C. Time-varying Electromagnetic Field.

We must now use the coupled (Maxwell) electromagnetic equations.

The net force density in the presence of electric and magnetic fields is

(1-155)

$$\bar{f} = \bar{f}_E + \bar{f}_m = \rho\bar{E} + \bar{J}\times\bar{B} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) E_x \bar{a}_x$$
$$+ \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) E_y \bar{a}_y + \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) E_z \bar{a}_z$$
$$+ \left(\nabla\times\bar{H} - \frac{\partial\bar{D}}{\partial t}\right)\times\bar{B}.$$

(1-156)

$$(\nabla\times\overline{\mathcal{H}})\times\overline{\mathcal{B}} = \Big[\frac{\partial}{\partial x}\Big(\frac{\mu}{2}(\mathcal{H}_x^2-\mathcal{H}_y^2-\mathcal{H}_z^2)\Big)+\frac{\partial}{\partial y}(\mu\mathcal{H}_x\mathcal{H}_y) + \frac{\partial}{\partial z}(\mu\mathcal{H}_x\mathcal{H}_z)\Big]\overline{a}_x + \Big[\frac{\partial}{\partial x}(\mu\mathcal{H}_x\mathcal{H}_y)+\frac{\partial}{\partial y}\Big(\frac{\mu}{2}(\mathcal{H}_y^2-\mathcal{H}_z^2-\mathcal{H}_x^2)\Big) + \frac{\partial}{\partial z}(\mu\mathcal{H}_y\mathcal{H}_z)\Big]\overline{a}_y + \Big[\frac{\partial}{\partial x}(\mu\mathcal{H}_x\mathcal{H}_z)+\frac{\partial}{\partial y}(\mu\mathcal{H}_y\mathcal{H}_z) + \frac{\partial}{\partial z}\Big(\frac{\mu}{2}(\mathcal{H}_z^2-\mathcal{H}_x^2-\mathcal{H}_y^2)\Big)\Big]\overline{a}_z$$

Consider f_{Ex}:

(1-157)

$$f_{Ex} = \frac{\partial \mathcal{D}_x}{\partial x}\mathcal{E}_x + \frac{\partial \mathcal{D}_y}{\partial y}\mathcal{E}_x + \frac{\partial \mathcal{D}_z}{\partial z}\mathcal{E}_x$$

Then

(1-158)

$$\frac{\partial \mathcal{D}_x}{\partial x}\mathcal{E}_x = \frac{\partial}{\partial x}\Big(\frac{\varepsilon}{2}\mathcal{E}_x^2\Big)$$

(1-159)

$$\frac{\partial \mathcal{D}_y}{\partial y}\mathcal{E}_x = \frac{\partial}{\partial y}(\mathcal{D}_y\mathcal{E}_x) - \mathcal{D}_y\frac{\partial \mathcal{E}_x}{\partial y} = \frac{\partial}{\partial y}(\varepsilon\mathcal{E}_x\mathcal{E}_y) - \mathcal{D}_y\frac{\partial \mathcal{E}_y}{\partial x} - \mathcal{D}_y\frac{\partial \mathcal{B}_z}{\partial t}$$

$$= \frac{\partial}{\partial y}(\mathcal{D}_y\mathcal{E}_x) - \frac{\partial}{\partial x}\Big(\frac{\varepsilon}{2}\mathcal{E}_y^2\Big) - \mathcal{D}_y\frac{\partial \mathcal{B}_z}{\partial t}$$

(1-160)

$$\frac{\partial \mathcal{D}_z}{\partial z}\mathcal{E}_x = \frac{\partial}{\partial z}(\mathcal{D}_z\mathcal{E}_x) - \mathcal{D}_z\frac{\partial \mathcal{E}_x}{\partial z} = \frac{\partial}{\partial z}(\mathcal{D}_z\mathcal{E}_x) - \mathcal{D}_z\frac{\partial \mathcal{E}_z}{\partial x} + \mathcal{D}_z\frac{\partial \mathcal{B}_y}{\partial t}$$

$$= \frac{\partial}{\partial z}(\varepsilon\mathcal{E}_z\mathcal{E}_x) - \frac{\partial}{\partial x}\Big(\frac{\varepsilon}{2}\mathcal{E}_z^2\Big) + \mathcal{D}_z\frac{\partial \mathcal{B}_y}{\partial t}.$$

Therefore

(1-161)

$$f_{Ex} = \frac{\partial}{\partial x}\Big[\frac{\varepsilon}{2}(\mathcal{E}_x^2-\mathcal{E}_y^2-\mathcal{E}_z^2)\Big] + \frac{\partial}{\partial y}(\varepsilon\mathcal{E}_x\mathcal{E}_y) + \frac{\partial}{\partial z}(\varepsilon\mathcal{E}_x\mathcal{E}_z) - \mathcal{D}_y\frac{\partial \mathcal{B}_z}{\partial t} + \mathcal{D}_z\frac{\partial \mathcal{B}_y}{\partial t}.$$

The difference [illegible] of the Poynting term $\mathcal{D}_y \partial \mathcal{B}_z/\partial t - \mathcal{D}_z \partial \mathcal{B}_y/\partial t$ [illegible] from the electrodynamic [illegible] equation $\nabla \times \bar{\mathcal{E}} = -\partial \bar{\mathcal{B}}/\partial t$ rather than $\nabla \times \bar{\mathcal{E}} = 0$ for electrostatics.

Similarly

$$(1\text{-}162) \quad f_{\mathcal{E}y} = \frac{\partial}{\partial x}(\varepsilon \mathcal{E}_x \mathcal{E}_y) + \frac{\partial}{\partial y}\left(\frac{\varepsilon}{2}\left(\mathcal{E}_y^2 - \mathcal{E}_z^2 - \mathcal{E}_x^2\right)\right) + \frac{\partial}{\partial z}(\varepsilon \mathcal{E}_y \mathcal{E}_z) - \mathcal{D}_z \frac{\partial \mathcal{B}_x}{\partial t} + \mathcal{D}_x \frac{\partial \mathcal{B}_z}{\partial t}$$

$$(1\text{-}163) \quad f_{\mathcal{E}z} = \frac{\partial}{\partial x}(\varepsilon \mathcal{E}_x \mathcal{E}_z) + \frac{\partial}{\partial y}(\varepsilon \mathcal{E}_y \mathcal{E}_z) + \frac{\partial}{\partial z}\left(\frac{\varepsilon}{2}\left(\mathcal{E}_z^2 - \mathcal{E}_x^2 - \mathcal{E}_y^2\right)\right) - \mathcal{D}_x \frac{\partial \mathcal{B}_y}{\partial t} + \mathcal{D}_y \frac{\partial \mathcal{B}_x}{\partial t} .$$

The Poynting terms in (1-161)-(1-163) are the components of $-\bar{\mathcal{D}} \times \partial \bar{\mathcal{B}}/\partial t$. Therefore calling $\bar{\bar{T}}_{\mathcal{H}} + \bar{\bar{T}}_{\mathcal{E}} = \bar{\bar{T}}$, and using (1-155), (1-156), (1-161)-(1-163), we obtain

$$(1\text{-}164) \quad \bar{f} = \nabla \cdot \bar{\bar{T}} - \frac{\partial}{\partial t}(\bar{\mathcal{D}} \times \bar{\mathcal{B}})$$

$$= \nabla \cdot \bar{\bar{T}} - \frac{1}{c^2}\frac{\partial}{\partial t}(\bar{\mathcal{E}} \times \bar{\mathcal{H}})$$

where $\mu\varepsilon = 1/c^2$.

$\bar{f}$, the material, or mechanical, force density of electromagnetic origin, may be written

$$\frac{\partial \bar{p}_{mech}}{\partial t}$$

where $\bar{p}_{mech}$ is the mechanical momentum density.

If we call $\frac{1}{c^2}(\bar{\mathcal{E}} \times \bar{\mathcal{H}})$, which is the Poynting vector divided by the velocity of light squared, the <u>electromagnetic momentum density</u>, $\bar{p}_{elect}$ (check its dimensions to convince yourself that it <u>is</u> a momentum density), then (1-164) becomes a <u>continuity equation for momentum density</u>:

$$(1\text{-}165) \quad \nabla \cdot \bar{\bar{T}} = \frac{\partial}{\partial t}(\bar{p}_{mech} + \bar{p}_{elect}) = \frac{\partial}{\partial t}\bar{p}_{total} .$$

[illegible]
electrodynamics

$$(1\text{-}166) \qquad \nabla\cdot\overline{\overline{T}} = -\frac{\partial \bar{P}}{\partial t}\,.$$

In this context, therefore, $\overline{\overline{T}}$ may be called a momentum flux density tensor, Figure (1-11).

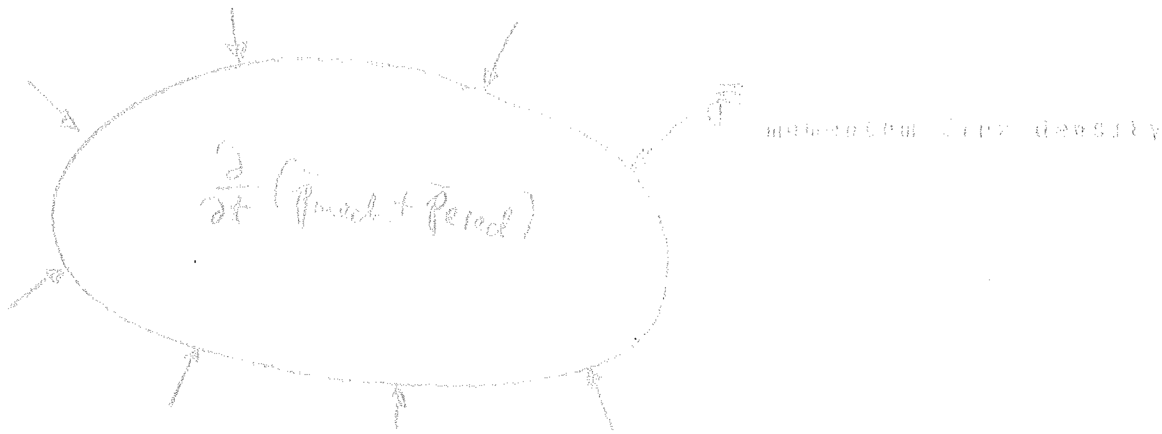
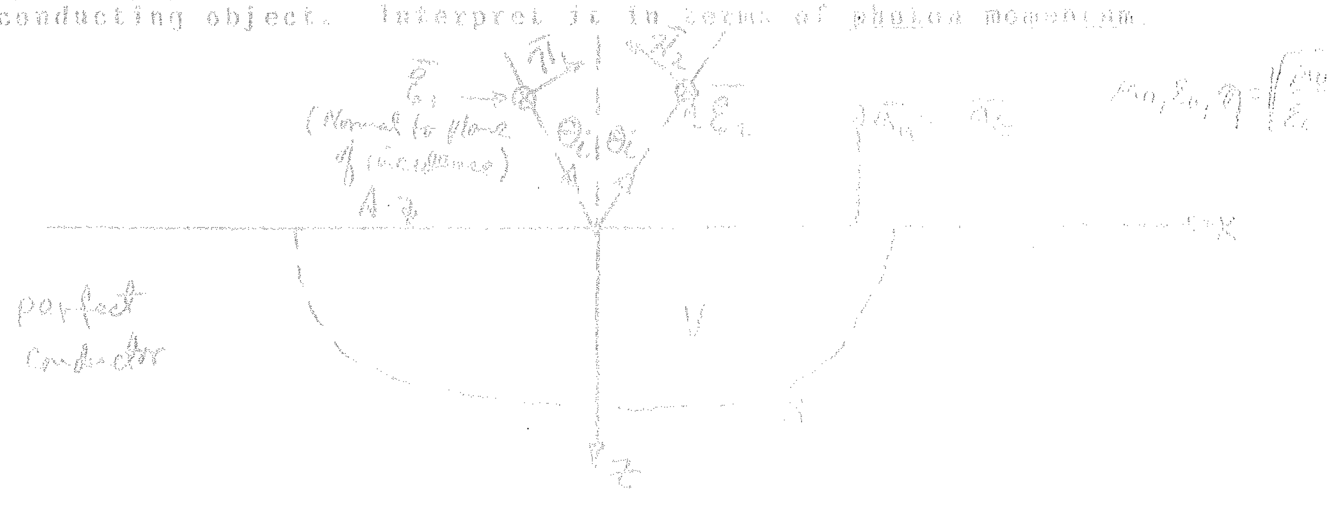


Figure 1-11. Illustrating the conservation of momentum (continuity equation for momentum density)

Example (1-5-5): Calculate the radiation pressure on a perfectly conducting object. Interpret it in terms of photon momentum.



Solution: Along that part of a closed surface, S, lying within the perfect conductor, $\bar{E} = \bar{H}$, and, hence $\overline{\overline{T}}$ vanish. In fact, the fields vanish everywhere within the conductor, so that $\bar{P}_{elect} \equiv 0$ within V. From (1-165), therefore,

$$\iiint_V (\nabla\cdot\overline{\overline{T}})\,dv = \bar{F}_{mech} + \frac{\partial}{\partial t}\iiint_V \bar{P}_{elect}\,dv = \bar{F}_{mech}\,.$$

But $\iiint_V (\nabla\cdot\overline{\overline{T}})\,dv = \oint\!\!\oint_{S+A} \overline{\overline{T}}\cdot d\bar{s} = \iint_A \overline{\overline{T}}(x,0)\cdot(\ldots)\,d\ldots$ [illegible].

Whinnery and Van Duzer [illegible]

$$E_y = j2E_i \sin(kz\cos\theta_i)\, e^{-jkx\sin\theta_i}, \qquad k = \omega\sqrt{\mu_0\epsilon_0}$$

$$H_x = \frac{2E_i}{\eta}\cos\theta_i \cos(kz\cos\theta_i)\, e^{-jkx\sin\theta_i}$$

$$H_z = j\frac{2E_i}{\eta}\sin\theta_i \sin(kz\cos\theta_i)\, e^{-jkx\sin\theta_i}$$

Because $\bar{E} = E_y\hat{a}_y = 0$ at the surface $z = 0$, it follows that the only stress-tensor contribution in the radiation pressure is $\bar{\bar{T}}_M$, which at $z = 0$, becomes

$$\bar{\bar{T}}_M(x,0) = \begin{bmatrix} \frac{\mu_0}{2}\mathcal{H}_x^2 & 0 & 0 \\ 0 & -\frac{\mu_0}{2}\mathcal{H}_x^2 & 0 \\ 0 & 0 & -\frac{\mu_0}{2}\mathcal{H}_x^2 \end{bmatrix}$$

where

$$\mathcal{H}_x(x,0,t) = \mathrm{Re}\left[H_x e^{j\omega t}\right] = \frac{2E_i}{\eta}\cos\theta_i \cos(\omega t - kx\sin\theta_i)$$

The radiation pressure is given by

$$\bar{P}_{rad} = \bar{\bar{T}}_M(x,0,t)\cdot(-\hat{a}_z) = \frac{\mu_0}{2}\mathcal{H}_x^2(x,0,t)\,\hat{a}_z$$

The time-average radiation pressure is, therefore,

$$\bar{P}_{rad,avg} = \frac{\mu_0}{2}\cdot\frac{4E_i^2}{\eta^2}\cdot\frac{\cos^2\theta_i}{2}\,\hat{a}_z$$

$$= \epsilon_0 E_i^2 \cos^2\theta_i\, \hat{a}_z$$

We interpret this result in terms of photon momentum in the following way. The incident radiation has a momentum density

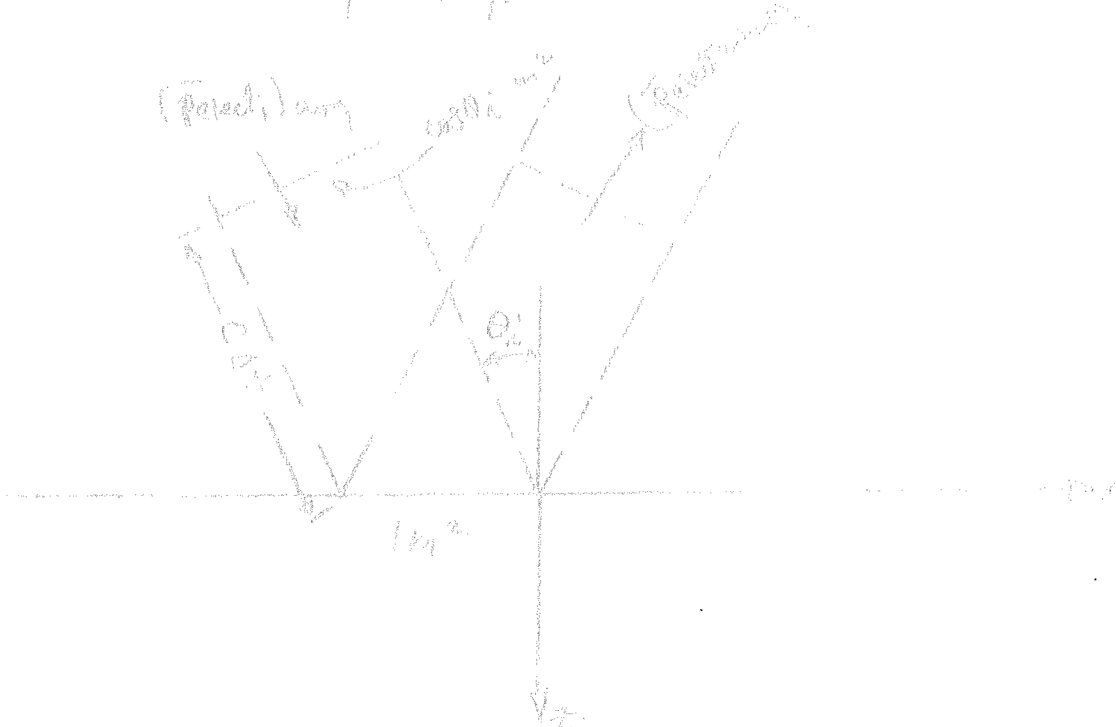
$$\bar{P}_{elect,i} = \frac{\bar{E}_i \times \bar{H}_i}{c^2}$$

$$(\bar{\wp}_{elect,i})_{avg} = \frac{1}{c^2}\cdot\frac{1}{2}\mathrm{Re}[\vec{E}_i \times \vec{H}_i^*]$$

$$= \frac{1}{2c^2}\cdot\frac{E_i^2}{\eta}(\sin\theta_i\,\hat{a}_x + \cos\theta_i\,\hat{a}_z)$$

The time-averaged momentum density of the reflected wave is

$$(\bar{\wp}_{elect,r})_{avg} = \frac{E_r^2}{2\eta c^2}(\sin\theta_r\,\hat{a}_x - \cos\theta_r\,\hat{a}_z)$$



Consider the "section" of incident radiation impinging on 1 m^2 of surface in time Δt. The volume of this "section" is $\Delta V = 1\cdot\cos\theta_i\cdot c\Delta t$, which means that the net (total) momentum (or radiation, or photon) average momentum carried by the "bundle of photons" is

$$(\bar{\wp}_{elect,i})_{avg}\cdot\Delta V = \frac{E_i^2}{2c\eta}(\sin\theta_i\,\hat{a}_x + \cos\theta_i\,\hat{a}_z)\cos\theta_i\,\Delta t$$

from the surface (i.e., reflected)

$$(\overline{P}_{elect_2})_{avg}\,\Delta V = \frac{\epsilon E_i^2}{2c}\left(\sin\theta_i\,\overline{a}_x - \cos\theta_i\,\overline{a}_z\right)\cos\theta_i\,\Delta t .$$

Since the "collision" of the bundle of photons occurs in time Δt, it follows that the rate-of-change of photon momentum is

$$\frac{\left[(\overline{P}_{elect_2})_{avg} - (\overline{P}_{elect_1})_{avg}\right]\Delta V}{\Delta t} = -\epsilon_0 E_i^2 \cos^2\theta_i\,\overline{a}_z$$

which is the negative of the result for $(\overline{P}_{rad})_{avg}$, as it should be from "action-reaction" principles.

Thus, we can equate the photon pressure on a material object with the time rate-of-change of photon momentum scattered by that object. To put it another way, the object puts pressure on the incident beam, thereby scattering it out of its original "path". The reaction of this pressure may propel the object, as in a "solar sail" or "photon rocket".

1-6. Faraday's Law for Moving Media

Recall that when we sought the point form of Faraday's law we started with (1-1)

$$\oint \overline{E}\cdot d\overline{l} = -\frac{d}{dt}\iint \overline{B}\cdot d\overline{a}$$

and then said that for a stationary region (or circuit) we could interchange $\frac{d}{dt}$ with $\iint$ to get equation (1-9),

$$\oint \overline{E}\cdot d\overline{l} = \iint -\frac{\partial \overline{B}}{\partial t}\cdot d\overline{a}$$

from which we concluded, by Stokes' theorem,

$$\oint \overline{E}\cdot d\overline{l} = \iint (\nabla\times\overline{E})\cdot d\overline{a},$$

that

$$(1\text{-}10) \qquad \nabla\times\overline{E} = -\frac{\partial \overline{B}}{\partial t} .$$

Now we wish to do the same thing for moving media.

Faraday's law, as expressed by (1-1), is valid for moving media as well as stationary media. What we seek, therefore, is a derivative, call it $\frac{D\overline{B}}{Dt}$, which satisfies

$$(1\text{-}167) \qquad \frac{d\Phi}{dt} = \frac{d}{dt}\iint_S \bar{B}\cdot d\bar{a} = \iint_S \frac{D\bar{B}}{Dt}\cdot d\bar{a}$$

As we have just stated, if S is stationary then $D\bar{B}/Dt = \partial\bar{B}/\partial t$. Thus, the question arises, if S is moving what additional terms must be added to $\partial\bar{B}/\partial t$ in order to get $\frac{D\bar{B}}{Dt}$?

Consider Fig. 1-12:

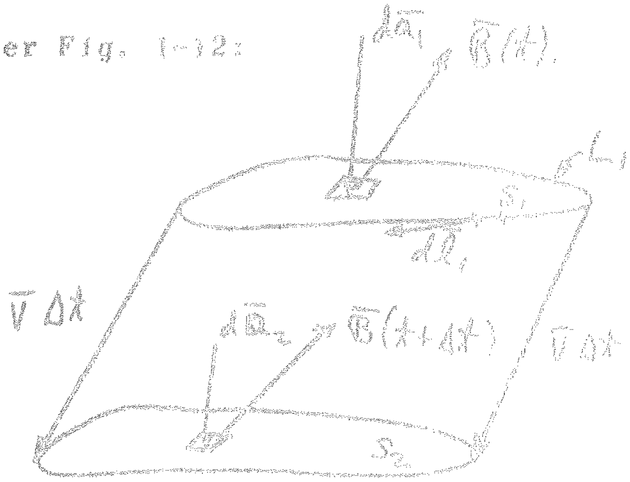


Fig. 1-12: Illustrating a moving circuit.

S_1 is the moving surface at time t, and S_2 is the same at time $t + \Delta t$.

By the definition of the derivative, we have

$$(1\text{-}168) \qquad \frac{d}{dt}\iint \bar{B}\cdot d\bar{a} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iint_{S_2}\bar{B}(t+\Delta t)\cdot d\bar{a} - \iint_{S_1}\bar{B}(t)\cdot d\bar{a}\right\}$$

We may think of S as tracing out a volume of space whose top face is S_1, bottom face is S_2, and whose sides have "generators" of length $\bar{V}\,\Delta t$. Note that $\bar{V}$ at some point of the circuit does not have to be identical to $\bar{V}$ at another point of the circuit.

Call the above defined volume V. We apply Gauss' divergence theorem to $\bar{B}$ at time t:

$$(1\text{-}169) \qquad \iiint_V (\nabla\cdot\bar{B})\,dV = \iint_{S_2}\bar{B}(t)\cdot d\bar{a} - \iint_{S_1}\bar{B}(t)\cdot d\bar{a}$$

$$- \oint_{L_1}\bar{B}(t)\cdot(\bar{V}\,\Delta t\times d\bar{l})\,,$$

is taken to be into the volume V. The last term represents an integral over the lateral surface. It is negative because $\overline{V} \times d\overline{l}$ points into the region V.

For a small increment of time, Δt, we have

$$\overline{B}(t+\Delta t) = \overline{B}(t) + \frac{\partial \overline{B}}{\partial t}(t)\,\Delta t \tag{1-170}$$

Thus (1-168) becomes

$$\frac{d}{dt}\iint \overline{B}\cdot d\overline{a} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{ \iint_{S_2} \left(\overline{B}(t) + \Delta t\, \frac{\partial \overline{B}}{\partial t}(t)\right)\cdot d\overline{a} - \iint_{S_1} \overline{B}(t)\cdot d\overline{a} \right\} \tag{1-171}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{ \iint_{S_2} \overline{B}(t)\cdot d\overline{a} - \iint_{S_1} \overline{B}(t)\cdot d\overline{a} + \Delta t \iint_{S_2} \frac{\partial \overline{B}}{\partial t}\cdot d\overline{a} \right\}.$$

The first term on the right may be rewritten by using (1-169)

$$\iint_{S_2} \overline{B}(t)\cdot d\overline{a} - \iint_{S_1} \overline{B}(t)\cdot d\overline{a} = \iiint_V \left(\nabla\cdot \overline{B}(t)\right) dV + \oint_{L} \overline{B}(t)\cdot\left(\overline{V}\,\Delta t \times d\overline{l}\right), \tag{1-172}$$

which means that

$$\frac{d}{dt}\iint \overline{B}\cdot d\overline{a} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big\{ \iiint_V \nabla\cdot \overline{B}(t)\, dV +$$

(1-173) $$+ \Delta t \oint_{L_1} \overline{B}(t)\cdot(\overline{V}\times d\overline{l}) + \Delta t \iint_{S_2} \frac{\partial \overline{B}}{\partial t}\cdot d\overline{a} \Big\}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iiint_V (\nabla\cdot \overline{B}(t))\, dV + \oint_{L_1} (\overline{B}\times\overline{V})\cdot d\overline{l}$$

$$+ \iint_{S_2} \frac{\partial \overline{B}}{\partial t}\cdot d\overline{a} ,$$

where we have used the identity $\overline{B}\cdot(\overline{V}\times d\overline{l}) = (\overline{B}\times\overline{V})\cdot d\overline{l}$, and we have divided out the Δt terms.
Finally, note that $dV = (\overline{V}\Delta t)\cdot d\overline{a}$, i.e., dV is a parallelepiped with base $d\overline{a}_1$, and height $V\Delta t$. Thus,

(1-174) $$\frac{d}{dt}\iint \overline{B}\cdot d\overline{a} = \lim_{\Delta t \to 0} \iint_{S_1} (\nabla\cdot\overline{B})\,\overline{V}\cdot d\overline{a} + \oint_{L_1} (\overline{B}\times\overline{V})\cdot d\overline{l}$$

$$+ \iint_{S_2} \frac{\partial \overline{B}}{\partial t}\cdot d\overline{a} .$$

Finally, as $\Delta t \to 0$, we have $S_2 \to S_1$, thus giving us

(1-175) $$\frac{d}{dt}\iint \overline{B}\cdot d\overline{a} = \iint_{S_1} \left(\frac{\partial \overline{B}}{\partial t} + (\nabla\cdot\overline{B})\overline{V}\right)\cdot d\overline{a} + \oint_{L_1} (\overline{B}\times\overline{V})\cdot d\overline{l} .$$

(7-176) $$\frac{d}{dt}\iint \overline{B}\cdot d\overline{a} = \iint_{S_1}\left(\frac{\partial \overline{B}}{\partial t} + \nabla\times(\overline{B}\times\overline{v}) + (\nabla\cdot\overline{B})\overline{v}\right)\cdot d\overline{a}$$

$$= \iint_{S_1} \frac{D\overline{B}}{Dt}\cdot d\overline{a}$$

Thus

(7-177) $$\frac{D\overline{B}}{Dt} = \frac{\partial \overline{B}}{\partial t} + \nabla\times(\overline{B}\times\overline{v}) + (\nabla\cdot\overline{B})\overline{v}$$

Hence,

(7-178) $$\oint_{C_1} \overline{E}'\cdot d\overline{l} = -\frac{d\psi}{dt} = -\iint_{S_1}\left(\frac{\partial \overline{B}}{\partial t} + \nabla\times(\overline{B}\times\overline{v}) + (\nabla\cdot\overline{B})\overline{v}\right)\cdot d\overline{a}$$

$$= \iint_{S_1}(\nabla\times\overline{E}')\cdot d\overline{a} ,$$

where $\overline{E}'$ is the field measured by the observer moving with the [illegible]. Because $\nabla\cdot\overline{B} = 0$, we have

(7-179) $$\nabla\times\overline{E}' = -\frac{\partial \overline{B}}{\partial t} - \nabla\times(\overline{B}\times\overline{v})$$

or

(7-180) $$\nabla\times(\overline{E}' - \overline{v}\times\overline{B}) = -\frac{\partial \overline{B}}{\partial t}$$

We know, however, that the [illegible] for a stationary observer who measures $\overline{E}$ [illegible] and $\overline{B}$ [illegible]

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t},$$

from which we conclude that

$$\bar{\mathcal{E}}' = \bar{\mathcal{E}} + \bar{V} \times \bar{\mathcal{B}} \tag{1-182}$$

If a stationary observer measures no $\bar{E}$ field, then a moving observer measures an effective field $\bar{V} \times \bar{B}$, i.e., a motional E'field

According to the theory of relativity, not only do the position and time coordinates of moving observers change, but also the electromagnetic field vectors. Thus if a reference system is moving with velocity $\bar{V}$ relative to another then the fields seen by two observers, each stationary in his own system, are given by

$$\begin{aligned} \mathcal{E}_{\parallel}' &= \mathcal{E}_{\parallel} \\ \mathcal{B}_{\parallel}' &= \mathcal{B}_{\parallel} \\ \bar{\mathcal{E}}_{\perp}' &= \gamma\left(\bar{\mathcal{E}}_{\perp} + \bar{V} \times \bar{\mathcal{B}}_{\perp}\right) \\ \bar{\mathcal{B}}_{\perp}' &= \gamma\left(\bar{\mathcal{B}}_{\perp} - \frac{\bar{V}}{c^2} \times \bar{\mathcal{E}}_{\perp}\right) \end{aligned} \tag{1-183}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{V}{c}$. The subscript "$\parallel$" means the component parallel to $\bar{V}$, and $\perp$ means component perpendicular to V. The unprimed quantities are measured in the rest frame and the primed in the moving frame. Comparing (1-183) with (1-182), we see that (1-182) is the low speed, i.e., $\beta^2 \to 0$ limit of (1-183).

<u>Example (1-6-1):</u> As an application of (1-182), let us investigate the phenomenon of <u>homopolar induction</u>, which occurs when a magnetized conductor rotates (Fig. 1-13)

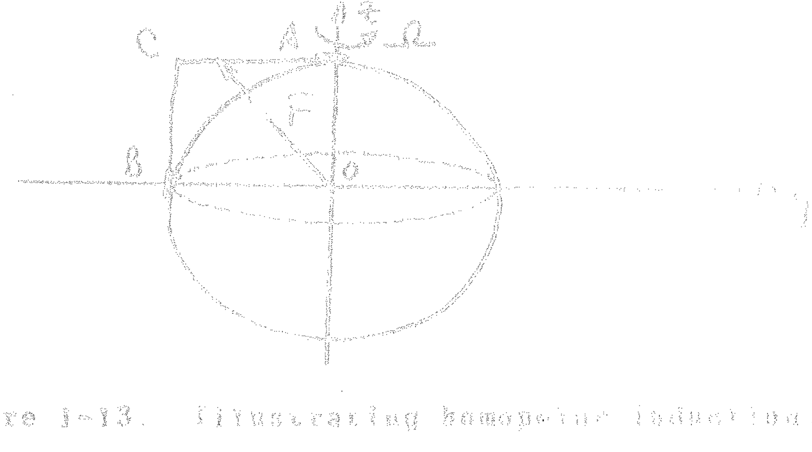


Figure 1-13. Illustrating homopolar induction.

velocity Ω. We have a wire ACB [illegible] ing contacts A and B. We may consider the magnet to be at rest and the wire rotating with angular velocity $-\Omega$. This gives rise to a wire moving in an external magnetic field. In the rest frame of the magnet, $E = 0$, because we have a static magnetic field B. Thus the wire sees a field $E' = V \times B$ (we ignore the reaction of the current in the wire on $\bar{B}$). If we take the path of integration to be OACB, we have:

$$EMF = \oint_{OACB} \bar{E}' \cdot d\bar{l} = \oint_{OACB} (\bar{V} \times \bar{B}) \cdot d\bar{l} =$$

(1-184)

$$= \int_{BOA} (\bar{V} \times \bar{B}) \cdot d\bar{l} + \int_{ACB} (\bar{V} \times \bar{B}) \cdot d\bar{l}$$

$$= \int_{ACB} (\bar{V} \times \bar{B}) \cdot d\bar{l} \, ,$$

because $\bar{V} = 0$ along BOA, i.e., nothing moves within the magnet.

Referring to Fig. 1-13, we have $\bar{V} = -\bar{\Omega} \times \bar{r}$ ($\bar{V}$ refers to the path of integration) so that

$$EMF = \int_{ACB} (\bar{V} \times \bar{B}) \cdot d\bar{l} = -\int_{ACB} [(\bar{\Omega} \times \bar{r}) \times \bar{B}] \cdot d\bar{l} \tag{1-185}$$

taken along the wire. Now $\bar{E}' = 0$ along the perfectly conducting wire, which means that the EMF appears as a voltage between the ends of the wire, i.e., points A and B.

Let us re-examine the implications of (1-182). Ohm's law for a moving observer is the same as for a stationary observer $\bar{J}' = \sigma \bar{E}'$, where primes denote quantities measured in a moving frame of reference. For motion with velocities much less than the speed of light, $J' \cong J$, where J is the current density measured at rest. Upon substituting (1-182) into Ohm's law we find

$$\bar{J} = \sigma (\bar{E} + \bar{V} \times \bar{B}) \tag{1-186}$$

(1-186) becomes

$$(1\text{-}187) \qquad \bar{J} = \sigma\left(-\nabla V - \frac{\partial \bar{A}}{\partial t} + \bar{v}\times\bar{B}\right)$$

upon using (1-26).

We shall now apply (1-187) to the circuit consisting of a movable part, b, and a stationary "load", a, as shown in Fig. 1-14

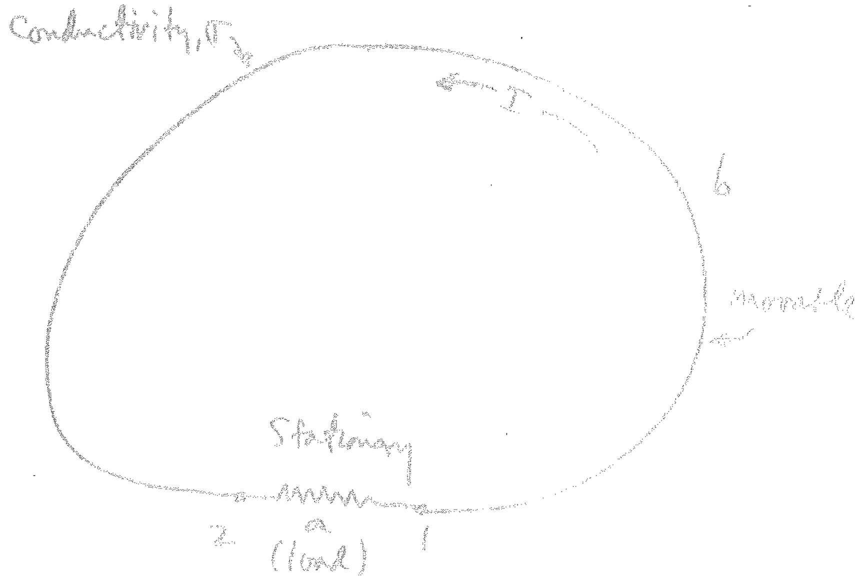


Figure 1-14. A circuit consisting of a moving portion and a stationary (load) portion

If we integrate $-\nabla V$ from point 1 to point 2 along path a we get the same result as is yielded by integration along path b, because the integral of the gradient of a scalar around a closed path vanishes. Hence

$$(1\text{-}188) \qquad V(1) - V(2) = \underset{(a)}{\int_1^2} (-\nabla V)\cdot d\bar{\ell} = \underset{(b)}{\int_1^2} -(\nabla V)\cdot d\bar{\ell} = \underset{(b)}{\int_1^2} \left(\frac{\bar{J}}{\sigma} + \frac{\partial \bar{A}}{\partial t} - \bar{v}\times\bar{B}\right)\cdot d\bar{\ell},$$

where the last equality follows from (1-187). $V(1) - V(2)$ is the electrostatic potential difference between points 1 and 2.

We assume the wire making up part b of the circuit is sufficiently thin that $\bar{J}$ is uniform over its cross-section. Then $\bar{J} = (I/A)\hat{a}_\ell$, where I is the total current flowing in the wire, A is the cross-sectional area of the wire, and $\hat{a}_\ell$ is the unit vector [illegible] flow.

It follows, then, that

(1-189) $$\int_{1\,(b)}^{2} \frac{\bar{J}}{\sigma} \cdot d\bar{l} = I \frac{l}{A\sigma} = I R_{int}$$

where l is the length of path b, and R_{int} is the internal resistance, i.e., the resistance of path b.

Equation (1-188) is now written

(1-190) $$V(1) - V(2) = I \cdot R_{int} + \int_{1\,(b)}^{2} \left(\frac{\partial \bar{A}}{\partial t} - \bar{V} \times \bar{B} \right) \cdot d\bar{l}$$

$$= I R_{int} + \oint \left(\frac{\partial \bar{A}}{\partial t} - \bar{V} \times \bar{B} \right) \cdot d\bar{l} - \int_{2\,(a)}^{1} \left(\frac{\partial \bar{A}}{\partial t} - \bar{V} \times \bar{B} \right) \cdot d\bar{l} .$$

Along path (a), $\bar{V} = 0$, so that (1-190) becomes

(1-192) $$\oint \left(\frac{\partial \bar{A}}{\partial t} - \bar{V} \times \bar{B} \right) \cdot d\bar{l} + I \cdot R_{int} = V(1) - V(2) + \int_{2\,(a)}^{1} \left(\frac{\partial \bar{A}}{\partial t} \right) \cdot d\bar{l}$$

$$= V(1) - V(2) - \int_{1\,(a)}^{2} \left(\frac{\partial \bar{A}}{\partial t} \right) \cdot d\bar{l} = \int_{1\,(a)}^{2} \left(-\nabla V - \frac{\partial \bar{A}}{\partial t} \right) \cdot d\bar{l} = \int_{1\,(a)}^{2} \bar{E} \cdot d\bar{l} = -V_L$$

where V_L is the load voltage drop from point 2 to point 1.

Note that V_L depends upon the path used to define the load circuit because the time-rate-of-change of the enclosed flux may vary.

The term $\oint (\bar{V} \times \bar{B}) \cdot d\bar{l}$ is called the generated or motional electromotive force (EMF), E_g, and $\oint (\partial \bar{A}/\partial t) \cdot d\bar{l}$ is the induced emf due to a time-rate-of-change of flux.
To prove this last statement we have:

(1-193) $$\oint \left(\frac{\partial \bar{A}}{\partial t} \right) \cdot d\bar{l} = \frac{d}{dt} \oint \bar{A} \cdot d\bar{l} = \frac{d}{dt} \iint (\nabla \times \bar{A}) \cdot d\bar{a} = \frac{d}{dt} \iint \bar{B} \cdot d\bar{a} = \frac{d\psi}{dt}$$

We may interchange the time-differ[illegible] in the first equality because A and $\frac{\partial A}{\partial t}$ are measured by a person at rest with the closed contour.

Finally, we derive our desired result by applying the above to (1-192)

$$E_g = \frac{d\varphi}{dt} + I R_{int} + V_L \tag{1-194}$$

In general $\frac{d\varphi}{dt}$ may result from self and mutual inductive effects, but if we assume, for the time being, only a self inductance in the closed circuit, then (1-194) has the circuit representation of Figure 1-15:

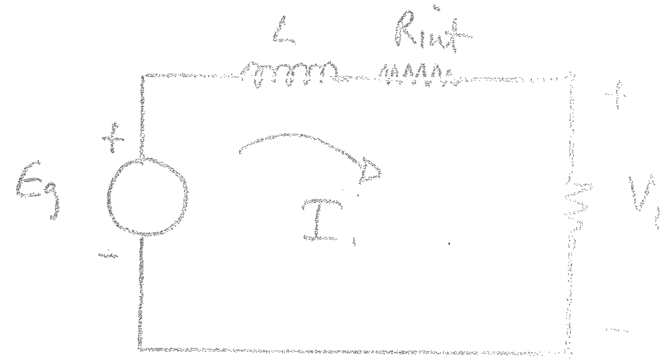


Figure 1-15. The equivalent circuit of (1-194) resulting from motion of a circuit in a magnetic field.

This equivalent circuit is the prototype of those associated with electromechanical energy conversion systems.

Example (1-6-2): Generated EMF in the Armature of a Commutator Machine.

Calculate the generated EMF in the armature of Figure 1-4 (example 1-3-2), if the armature rotates at angular velocity ω while commutators keep the magnetic fields stationary in space continuous distribution of very [illegible] the integration implied in the definition of E_g will go into the page of Figure 1-4 at some angle θ and return π radians (diametrically opposite) around the circumference. The path length (i.e., the length of the armature) is l. In order to complete the line integral, we must sum the contributions of the path integrals just described around the upper (or lower) half of the armature. Because the armature conductors are uniformly distributed with a density, $\frac{N_a}{\pi R_a}$ - conductors (or turns) per meter, there are, between θ and $\theta + d\theta$, $\left(\frac{N_a}{\pi R_a}\right) R_a d\theta = \frac{N_a}{\pi} d\theta$ conductors.

Hence, the summation alluded to above amounts to an integration from 0 to π of $\frac{N_a}{\pi}$ times the result for one conductor (which enters the page at θ and returns at $\theta + \pi$ radians).

By symmetry, we need only calculate $\overline{V} \times \overline{B}$ for the upper-half of the armature and multiply by two to take care of the lower-half. Because $\overline{V} = R_a \omega \overline{a}_\theta$, it follows from (1-103) that

$$\overline{V} \times \overline{B} = \overline{a}_z \cdot \frac{4\mu_0}{\pi} \left\{ R_f J_{sf} \sum_{n=odd}^{\prime} \frac{\sin n(\alpha+\theta)}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} + R_a J_{sa} \sum_{n=odd}^{\prime} \left(\frac{1}{n}\right) \frac{\left(\frac{R_a}{R_f}\right)^n + \left(\frac{R_f}{R_a}\right)^n}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n} \cos n\theta \right\}$$

Hence, by what was said above about the net path of integration we have

$$E_g = \oint (\overline{V} \times \overline{B}) \cdot d\overline{l} = \frac{8\mu_0}{\pi^2} \omega l N_a R_f J_{sf} \sum_{n=odd}^{\prime} \int_0^{\pi} \frac{\sin n(\alpha+\theta)\, d\theta}{\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n}$$

$$= \frac{16\mu_0}{\pi^2} \omega l N_a R_f J_{sf} \sum_{n=odd}^{\prime} \frac{\cos n\alpha}{n\left[\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n\right]}$$

Now if i_f [illegible]
through each armature conductor, [illegible]
are uniformly distributed with, respectively, $\frac{N_f}{\pi R_f}$ and $\frac{N_a}{\pi R_a}$ conductors (or turns) per meter, it follows that

$$J_{sf} = \frac{i_f N_f}{\pi R_f} \,, \quad J_{sa} = \frac{i_a N_a}{\pi R_a} \,.$$

Hence the expression for E_g becomes

$$E_g = \frac{16\mu_0}{\pi^3} \omega l N_a N_f i_f \sum_{n=odd} \frac{\cos n\alpha}{n\left[\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n\right]} = K_f i_f \omega$$

where

$$K_f = \frac{16\mu_0}{\pi^3} l N_a N_f \sum_{n=odd} \frac{\cos n\alpha}{n\left[\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n\right]}$$

is the same torque constant mentioned in Example 1-5-3.

Referring to Example 1-5-3, recall the expression for torque per-unit-length of armature. If we multiply that expression by l, the length of the armature, we get (using the same symbol, τ, for torque and torque-per-unit length)

$$\tau = \frac{16\mu_0}{\pi} l R_a R_f J_{sa} J_{sf} \sum_{n=odd} \left(\frac{1}{n}\right) \frac{\cos n\alpha}{\left[\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n\right]}$$

$$= \frac{16\mu_0}{\pi^3} l i_a i_f N_a N_f \sum_{n=odd} \left(\frac{1}{n}\right) \cdot \frac{\cos n\alpha}{\left[\left(\frac{R_f}{R_a}\right)^n - \left(\frac{R_a}{R_f}\right)^n\right]}$$

$$= K_f i_a i_f$$

$$= E_g \cdot \frac{i_a}{\omega} \,.$$

[illegible]

$$\tau\omega = \mathcal{E}_g i_a,$$

which, in words, states that mechanical power is completely transformed into electrical power and vice-versa. Therefore in a motor electrical power delivered into the rotor (after any Ohmic losses are subtracted), of amount $\mathcal{E}_g i_a$, is converted into mechanical power-out, of amount $\tau\omega$. Conversely in a generator mechanical power-input to the rotor (after subtraction of friction losses) of amount $\tau\omega$ is converted into electric power output, of amount $\mathcal{E}_g i_a$. The actual energy conversion process is lossless, or conservative. Losses are due to Ohmic losses or mechanical friction.

It is interesting to note that we derived this energy conversion rule by starting with a simple model and applying field theory to it. The fact that the results were consistent with conservation of energy attests to the internal consistency of the various field laws. We shall see this continually throughout the book.

1-7. THRUST AND LEVITATION FORCES IN A LINEAR INDUCTION MACHINE

As a final, and comprehensive, example of the application of the material in this chapter let us turn our attention to the calculation of thrust and levitation forces in a linear induction machine. Such machines have been proposed recently for high speed ground transportation purposes.* We shall consider the "single-sided" stator configuration which uses a normal force to produce levitation as well as a tangential force to produce thrust (Figure 1-16).

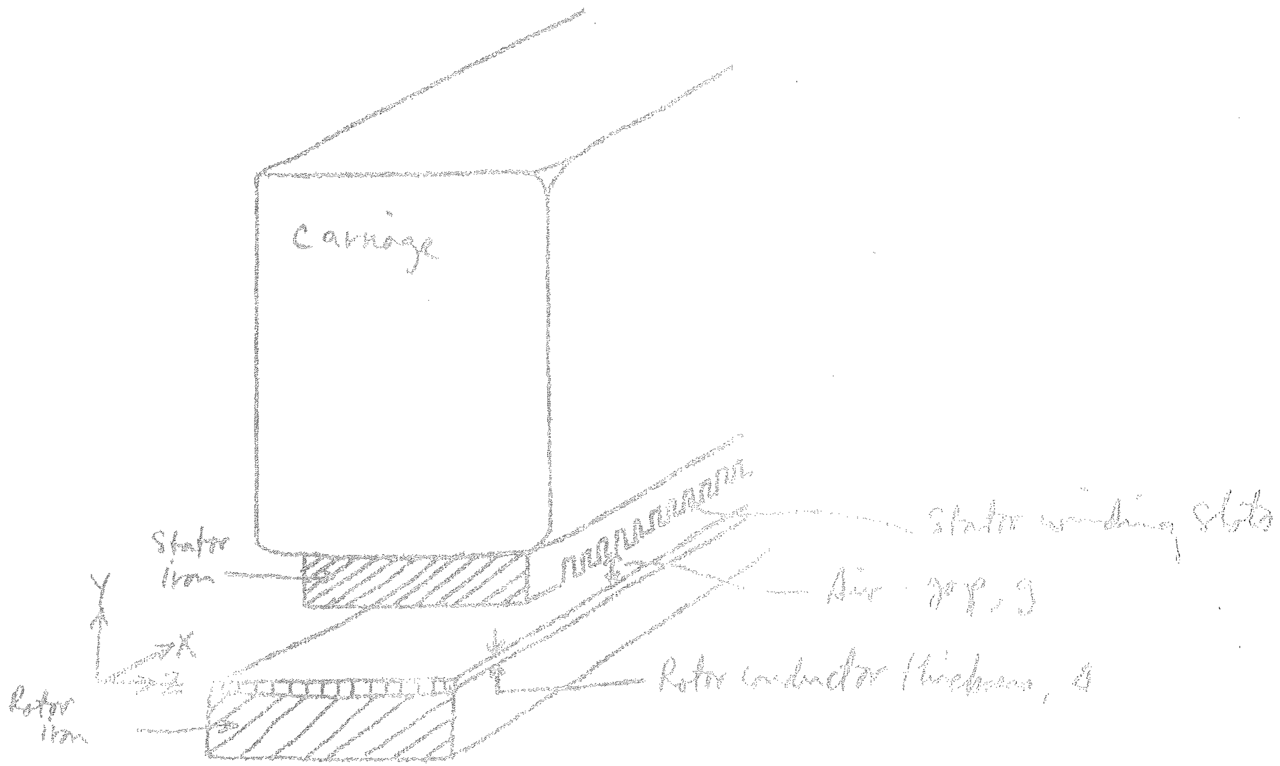


Figure 1-16. A linear induction machine suitable for a high-speed train. The single-sided stator configuration (single stator winding) produces a normal (levitation) as well as tangential (propulsive) force. Although the train is called the stator, it, in fact, actually moves (i.e., it pushes against the earth by virtue of magnetic stresses).

---

* M. Polowjadoff, "Linear Induction Machines" I. History and Theory of Operation, IEEE Spectrum, Feb. 1971, pp. 72-80.
_____________ II. Applications, IEEE Spectrum, March 1971, pp. 79-86. Boon-teck Ooi and David C. White, "Traction and Normal Forces in the Linear Induction Motor", IEEE Trans. on Power Apparatus and Systems", Vol. PAS-89, No. 4, April 1970.

We will make the problem two dimensional by ignoring all edge or end effects thereby giving rise to the idealized model of Figure 1-17

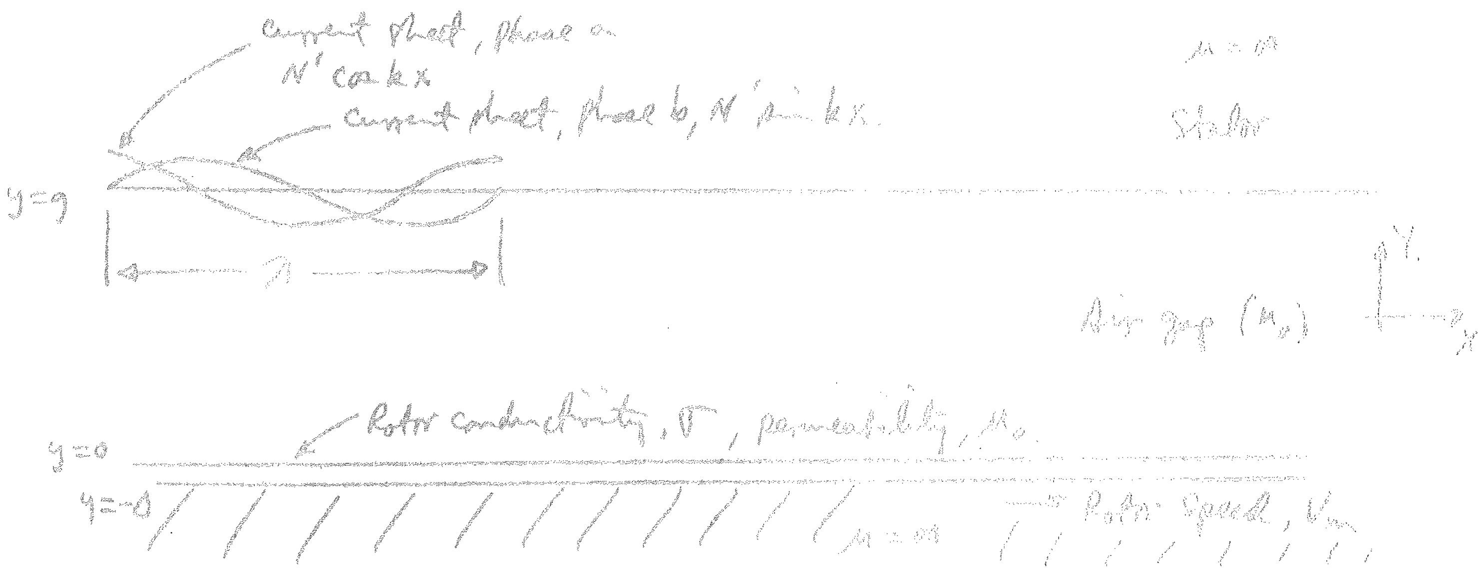


Figure 1-17. Idealized model of infinitely long (and broad) linear induction motor of the single sided stator configuration.

The stator current sheet consists of two current sheets displaced from each other by ninety electrical degrees (one-quarter of a wavelength) in space

(1-195) $\bar{J}_{ss}(x) = (J_a \cos kx + J_b \sin kx)\,\bar{a}_z$,

where $k = 2\pi/\lambda$, and λ is twice the "pole-pitch" of the windings. By "pole pitch" is meant the distance between consecutive north and south poles, or consecutive positive and negative current directions.

When J_a and J_b are two-phase currents ninety degrees out of time phase

(1-196)
$$J_a = N' I_m \cos \omega t$$
$$J_b = N' I_m \sin \omega t,$$

with N' being the density of turns-per-meter, then we have a <u>balanced two-phase system</u> producing the traveling wave of surface current on the stator

(1-197) $\bar{J}_{ss}(x) = N' I_m \cos(\omega t - kx)\,\bar{a}_z$,

[illegible]

Note that the ("synchronous") velocity of the travelling wave is given by

$$(1\text{-}198) \qquad v_{syn} = \frac{\omega}{k} = \frac{f}{\lambda}$$

In this sense we see that λ is the wavelength of the current wave and all other waves (e.g. magnetic fields, vector potentials, etc).

In Figure 1-17, we needn't worry about calculating the vector potential $\overline{A}$ anywhere else but in the air gap $0 \le y \le g$ and in the rotor conductor slab $-\Delta \le y \le 0$. In the gap $\overline{A}$ satisfies $\nabla^2 \overline{A} = \frac{\partial^2 \overline{A}}{\partial x^2} + \frac{\partial^2 \overline{A}}{\partial y^2} = 0$, as in our previous examples, and, as before, $\overline{A} = A\,\overline{a_z}$. In the conducting slab, $\overline{A}$ satisfies (1-33) with $\overline{J}$ given by (1-186), because the slab is moving relative to the stator, and with the time derivative term negligible at the operating frequency of interest (at 60 Hz, $\omega^2 \mu \epsilon L^2 \sim 10^{-8}$).

In addition, the electric field $\overline{E}$ follows from (1-26), upon setting the electrostatic potential equal to zero (as we can typically do in electromagnetic machines, as opposed to electrostatic ones) $\overline{E} = -\partial \overline{A}/\partial t$. Thus, the equation satisfied by $\overline{A}$ within the conducting slab is

$$(1\text{-}199) \qquad \nabla^2 \overline{A} = \mu\sigma \left[\partial \overline{A}/\partial t - \overline{v}_m \times \nabla \times \overline{A} \right].$$

Because we are concerned with sinusoidal, steady-state, phenomena we proceed using phasors. Hence, (1-197) becomes

$$(1\text{-}200) \qquad \overline{J}_{cs} = N' I_m e^{j(\omega t - kx)}\, \overline{a_z}$$

and (1-199)

$$(1\text{-}201) \qquad \nabla^2 \overline{A} = \mu\sigma \left(j\omega \overline{A} - \overline{v}_m \times \nabla \times \overline{A} \right).$$

If we postulate a traveling wave solution for $\overline{A}$ of the form

$$(1\text{-}202) \qquad \overline{A}(x,y,t) = A(y)\, e^{j(\omega t - kx)}\, \overline{a_z},$$

then $A(y)$ in the gap satisfies

$$(1\text{-}203) \qquad \frac{d^2 A(y)}{dy^2} - k^2 A(y) = 0, \qquad 0 \le y \le g$$

which has for its solution

(1-204) $$A(y) = A_1 \sinh ky + A_2 \cosh ky \ , \quad 0 \le y \le g.$$

Similarly, a proposed solution of the form (1-202) for $\bar{A}$ within the rotor conductor has A(y) satisfying (after carrying out the indicated vector operations in (1-199) or (1-201))

(1-205) $$\frac{d^2 A(y)}{dy^2} - k^2\left(1 + j\,\frac{\mu_0 \sigma S\, v_{syn}}{k}\right) A(y) = 0 \ , \quad -\Delta \le y \le 0.$$

This equation has for its solution

(1-206) $$A(y) = B_1 \sinh \alpha_1 y + B_2 \cosh \alpha_1 y \ , \quad -\Delta \le y \le 0$$

where

(1-207) $$\alpha_1 = k\,(1 + jSR_m)^{1/2}$$

$$S = \text{"slip"} = \frac{v_{syn} - v_m}{v_{syn}}$$

$$R_m = \text{"magnetic Reynolds number"} = \frac{\mu_0 \sigma\, v_{syn}}{k}.$$

The boundary conditions are that at the common boundary, the air-gap-conductor surface, y=0, the two values of vector potential must be equal. Thus, upon equating (1-204) and (1-206) we get

(1-208) $$A_2 = B_2.$$

At the air-gap-stator interface, y=g, the tangential component, H_x, of the air-gap field intensity equals the stator surface current density N'Im (see (1-197). But $H_x = \frac{1}{\mu_0}\cdot\frac{dA}{dy}$. Thus, upon substituting (1-204), we obtain

(1-209) $$\frac{1}{\mu_0}\left(kA_1 \cosh kg + kA_2 \sinh kg\right) = N' I_m$$

Similarly, at y = $-\Delta$, because there is no surface current flowing, we obtain by using (1-206)

(1-210) $$\frac{\alpha_1}{\mu_0}\left(B_1 \cosh \alpha_1 \Delta - B_2 \sinh \alpha_1 \Delta\right) = 0.$$

Finally, at the common boundary, y=0, the surface current vanishes (one can postulate a surface current only if σ, the conductivity, is infinite). Thus, the tangential components of $\bar{H}$, H_x must be continuous:

(1-211) $\quad [\text{illegible}] = \dfrac{[\text{illegible}]}{\mu_0} \, a_1$

Equations (1-208) - (1-211) constitute a system for the arbitrary constants A_1, A_2, B_1, B_2. The solutions are

$$A_1 = \frac{N'\mu_0 I_m}{k} \cdot \frac{\tanh \alpha_1 \Delta}{\cosh kg \tanh \alpha_1 \Delta + (k/\alpha_1) \sinh kg}$$

$$A_2 = \frac{N'\mu_0 I_m}{\alpha_1} \cdot \frac{1}{\cosh kg \tanh \alpha_1 \Delta + (k/\alpha_1) \sinh kg}$$

(1-212) $$B_1 = \frac{N'\mu_0 I_m}{\alpha_1} \cdot \frac{\tanh \alpha_1 \Delta}{\cosh kg \tanh \alpha_1 \Delta + (k/\alpha_1) \sinh kg}$$

$$B_2 = \frac{N'\mu_0 I_m}{\alpha_1} \cdot \frac{1}{\cosh kg \tanh \alpha_1 \Delta + (k/\alpha_1) \sinh kg}$$

The magnetic induction (flux density) is determined from the relation

$$\bar{B} = \nabla \times \bar{A} = \bar{a}_x \frac{\partial A_z}{\partial y} - \bar{a}_y \frac{\partial A_z}{\partial x}$$

$$= \left[\bar{a}_x \frac{dA(y)}{dy} + \bar{a}_y jk A(y) \right] e^{j(\omega t - kx)}$$

(1-213) $$= \bar{a}_x k \left[A_1 \cosh ky + A_2 \sinh ky \right] e^{j(\omega t - kx)}$$

$$+ \bar{a}_y jk \left[A_1 \sinh ky + A_2 \cosh ky \right] e^{j(\omega t - kx)}, \quad 0 < y < g.$$

Our interest now is in the air-gap induction, hence, we will be concerned only with the constants A_1 and A_2. In fact, a reasonable approximation to incorporate into our model is that $|\alpha_1 \Delta| << 1$, i.e., that the conducting sheet is electrically "thin". When this assumption is substituted into (1-212), and use is made of the fact that $\tanh \alpha_1 \Delta \simeq \alpha_1 \Delta$ for $|\alpha_1 \Delta| << 1$, there results, after some straight-forward algebra

(1-214) $$kA_1 = \frac{j s R_m' N' \mu_0 I_m}{\sinh kg + j s R_m' \cosh kg}$$

$$kA_1 = \frac{N' \mu_0 I_m}{\sinh kg + j s R_m' \cosh kg}$$

where $R_m' \equiv \mu_0 \sigma' v_{s_y}$ and $\sigma' = \sigma \Delta$ is the surface conductivity of the conducting sheet.

Thus, the phasor components of $B_x(x,y)$ and $B_y(x,y)$ are obtained by substituting (1-214) into (1-213)

$$B_x = \frac{N' \mu_0 I_m e^{-jkx}}{\sinh kg + jsR_m' \cosh kg}\left[jsR_m' \cosh ky + \sinh ky \right]$$

(1-215)

$$B_y = j\,\frac{N' \mu_0 I_m e^{-jkx}}{\sinh kg + jsR_m' \cosh kg}\left(jsR_m' \sinh ky + \cosh ky \right), \quad 0 \le y \le g.$$

In order to calculate the forces at the rotor (which is not really "rotating" but moves linearly) we must know the values of the stress-tensor components at the rotor surface, y=0. This necessitates setting y=0 in (1-215)

$$B_x(x,0) = \frac{N' \mu_0 I_m e^{-jkx} \cdot jsR_m'}{\sinh kg + jsR_m' \cosh kg}$$

(1-216)

$$B_y(x,0) = j\,\frac{N' \mu_0 I_m e^{-jkx}}{\sinh kg + jsR_m' \cosh kg}.$$

Because the outward normal to the rotor surface is $\bar{a}_y$, it follows from (1-154) that the stresses acting on the rotor are

(1-217) $$\bar{p} = \bar{\bar{T}}_M(x,0) \cdot \bar{a}_y = \frac{B_x(x,0)\,B_y(x,0)}{\mu_0}\,\bar{a}_x + \frac{B_y^2(x,0) - B_x^2(x,0)}{2\mu_0}\,\bar{a}_y.$$

Because all quantities are time-varying, we ask for the <u>time-averaged</u> values of the stresses. Because the stresses involve <u>products</u> of phasor quantities, it follows that the time-averaged values will be obtained by taking one-half of the real-part of the product of one phasor with the complex conjugate of the other. Thus, upon letting brackets, $\langle\,\rangle$, denote time-averaged values, we have, from (1-217) and (1-216)

(1-218) $\langle \bar{p} \rangle = \bar{a}_x \cdot \mathrm{Re}\left\{ \dfrac{b_y(z,\omega)\, b_y^*(z,\omega)}{2\mu_0} \ldots \right\}$ [illegible]

$= \dfrac{\mu_0 N'^2 I_m^2}{4}\left[\dfrac{2 s R_m' \, \bar{a}_x}{(\coth kg)^2 + (s R_m' \coth kg)^2} + \dfrac{[1-(s R_m')^2]}{(\coth kg)^2 + (s R_m' \coth kg)^2}\ \bar{a}_y\right]$

Thus, the time-averaged force acting on the rotor surface of area $\ell \cdot \lambda$ is

(1-219) $$\bar{F} = \frac{\mu_0 N'^2 I_m^2 \ell \lambda}{4}\left[\frac{2 s R_m' \bar{a}_x + [1-(s R_m')^2]\bar{a}_y}{(\coth kg)^2 + (s R_m' \coth kg)^2}\right]$$

The x-component of force is, of course, the tangential thrust that propels the rotor (its negative propels the stator, which is, in fact, the train) whereas the y-component is the normal levitation thrust. When this force component is positive, i.e., $s < 1/R_m'$, then there is an attractive force between rotor and stator. An example of this condition occurs when the slip, s, vanishes. Then there is no relative motion between the rotor and the stator traveling wave of sheet current. Hence, there is no induced rotor current, which means that the infinitely permeable rotor iron is not "shielded" from the stator current and there is a consequent force of attraction on the rotor iron due to the flux established by the stator current.

For other values of s, $0 \le s < 1/R_m'$, the induced rotor current is in space-phase with the stator current sheet, thereby producing a net magnetic field resulting in attraction between the two current sheets (Figure 1-18(a)). When $1/R_m' < s < 1$, the resultant normal force is negative, producing levitation. This can be explained on the basis of the rotor and stator current sheets being 180° out of space phase (Figure 1-18(b)).

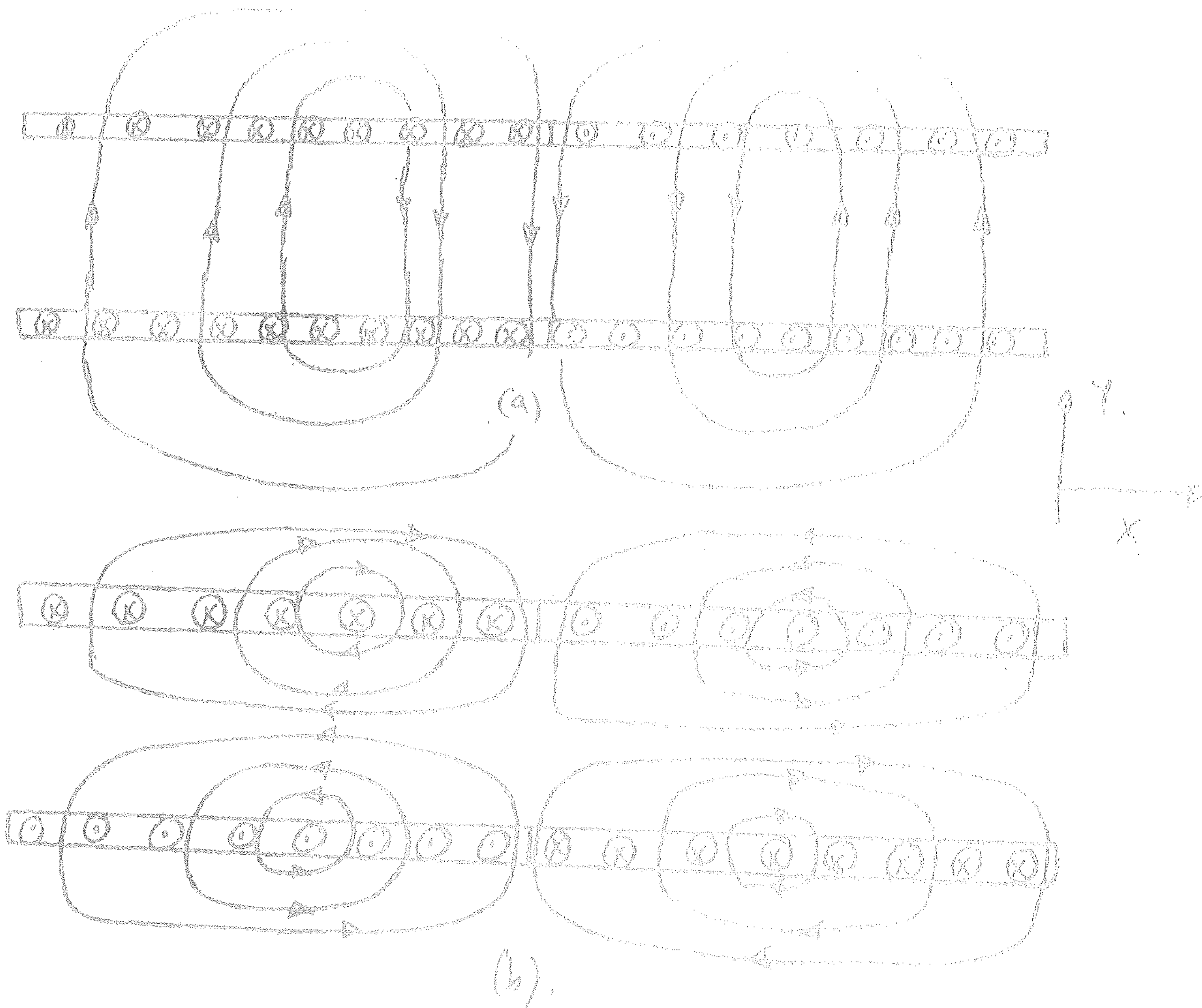


Figure 1-18. Normal forces between two sinusoidally distributed current sheets (representing the rotor and stator currents). (a). Mutual attraction results because the resulting flux lines are under tension in the y-direction when the currents are in space-phase. (b). Mutual repulsion (levitation) results because the flux lines are under compression in the y-direction when the currents are 180° out of space phase.

Normally, induction machines (or any other machine that we will study) are excited by constant voltage sources, whereas (1-219) gives the forces under the condition of constant current excitation. We remedy this defect by deriving an equivalent circuit which will enable us to calculate I_m, and hence, $\bar{F}$, in terms of the terminal voltage. The equivalent circuit will be derived by considering only one phase, hence, it is called a per-phase equivalent circuit [illegible] which is sketched from $-\pi/2$ to $\pi/2$ in Figure 1-19.

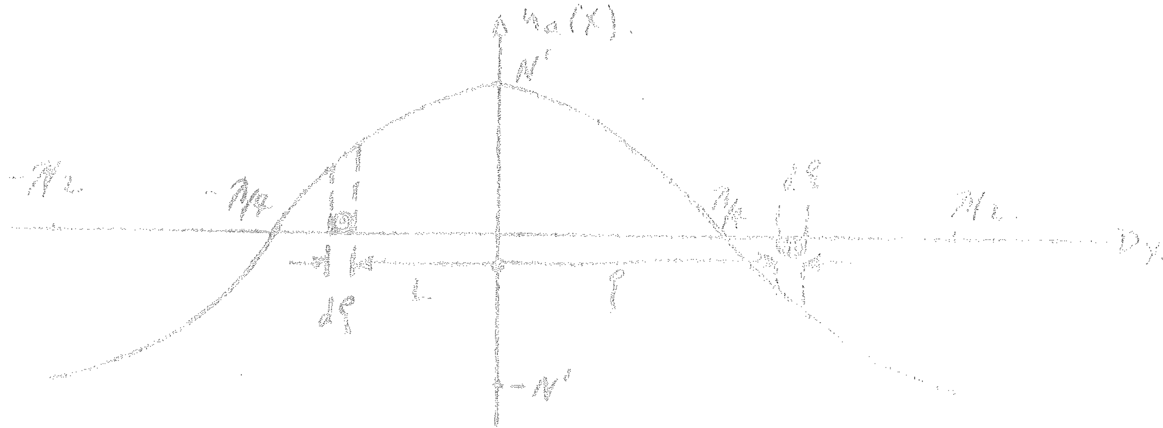


Figure 1-19. Showing the winding distribution for stator phase a. The cross-section of the flat coil considered in the text extends from $-L-d\xi$ to $-L$ and then from ξ to $\xi + d\xi$. Note that $L + \xi = \pi/2$, where $0 \le \xi \le \pi/2$.

Imagine a flat coil of depth l out of the page (the x-direction of Figure 1-17) and whose cross-section lies entirely within the dotted intervals of Figure 1-19. The total number of turns making up this coil are

(1-220) $\quad \#$ of turns $= N' \cos k\xi \, d\xi$.

The flux linkages enclosed by this coil are obtained by determining the flux enclosed and multiplying by the number of turns. The flux enclosed is equal to the integral of $By(x,g)$, where $By(x,g)$ is given in (1-215):

(1-221) $\quad$ Flux linkage between $-L$ and ξ $= lN'\cos k\xi \, d\xi \displaystyle\int_{\xi-\pi/2}^{\xi} jN'\mu_0 I_m \left\{ \frac{\cosh ky + jS\ell\ldots \sinh ky}{\sinh kg + jS\ell\ldots \cosh kg} \right\} e^{-jkx} dx$

$$= -\frac{2lN'^2\mu_0 I_m}{k} e^{-jk\xi} \cos k\xi \, d\xi \left[\frac{\cosh ky + jS\ell\ldots \sinh ky}{\sinh kg + jS\ell\ldots \cosh kg} \right].$$

Therefore, the total flux linkages enclosed by the continuously distributed coil extending from $X = -\pi/2$ to $X = \pi/2$ is obtained by integrating (1-221) over the range $0 \le \xi \le \pi/2$

(1-222) $\quad$ Total Flux Linkages $\Lambda = -\dfrac{2lN'^2\mu_0}{k} \left[\dfrac{\cosh ky + jS\ell\ldots \sinh ky}{\sinh kg + jS\ell\ldots \cosh kg} \right] I_m \displaystyle\int_0^{\pi/2} e^{-jk\xi} \cos k\xi \, d\xi$

$$= -\frac{lN'^2\mu_0 \pi}{2k} \left[\frac{\cosh ky + jS\ell\ldots \sinh ky}{\sinh kg + jS\ell\ldots \cosh kg} \right] I_m.$$

According to the Faraday [illegible] law, the [illegible] obtained by integrating the electric field along the path contained by the [illegible] and cross in Figure 1-19 is equal to the negative time rate of change of Λ

$$EMF = V = -j\omega\Lambda = j\frac{N'^2\mu_0 a l v_{syn}}{2}\left[\frac{\cosh kg + jSR_m' \sinh kg}{\sinh kg + jSR_m' \cosh kg}\right] I_m, \qquad (1\text{-}223)$$

where we have used the fact that $w/k = v_{syn}$.

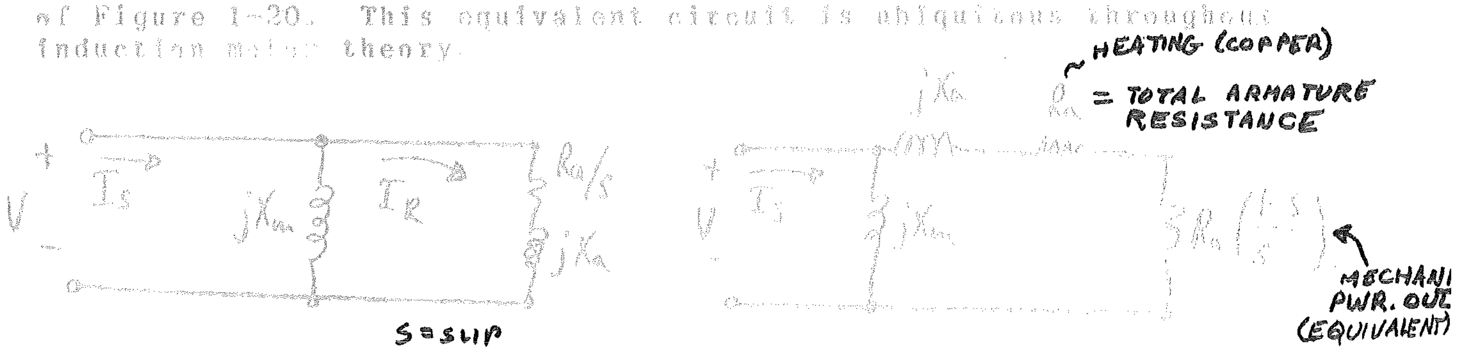
If we make the associations

$$(1\text{-}224) \qquad X_m = \frac{\mu_0 N'^2 a l v_{syn}}{2} \cdot \coth kg$$

$$X_a = \frac{\mu_0 N'^2 a l v_{syn}}{2} \cdot \sinh kg \cosh kg$$

$$R_a = \frac{\mu_0 N'^2 a l v_{syn}}{2} \cdot \frac{\cosh^2 kg}{R_m'}$$

then (1-223) can be represented by the simple equivalent circuit of Figure 1-20. This equivalent circuit is ubiquitous throughout induction motor theory.



$I_{s_{MAX}} = I_M$

Figure 1-20. (a) The equivalent circuit representation of (1-223). This is the per-phase, per-wavelength circuit for an infinitely long linear induction machine (or a finitely long machine neglecting any end or edge effects). The width of the machine in the z-direction is l-meters. In (1-223), I_M is the maximum value of the stator current (per-phase), I_S, and I_R is the rotor current "reflected" or "referred" to the stator. (b) Showing the total resistance Ra/s separated into the rotor resistance Ra and the "effective load resistance" Ra$\left(\frac{1-s}{s}\right)$. This latter resistor accounts for conversion of electrical energy into mechanical energy. Ra accounts for heating of the rotor

COPPER LOSSES = $|I_a|^2 R_a$

MECHANICAL PWR. = $|I_a|^2 R_a\left(\frac{1-S}{S}\right)$

$$\Rightarrow \frac{P_{MECH}}{P_{COPPER}} = \frac{1-S}{S}$$

From (1-224) we have

$$(1\text{-}225)\quad |T_n|^2 = \frac{4\,|N|^2}{\alpha^4 \mu_0^2 \ldots V_{syn}^2}\left[\frac{(\sinh ky)^2 + (S R_m' \cosh ky)^2}{(\cosh ky)^2 + (S R_m' \sinh ky)^2}\right],$$

which when substituted into (1-219) yields

$$(1\text{-}226)\quad F_x = \frac{2}{\mu_0 A \ell}\left(\frac{|N|}{N' v_{syn}}\right)^2 \frac{S R_m'}{(\cosh ky)^2 + (S R_m' \sinh ky)^2}$$

$$F_y = \frac{1}{\mu_0 A \ell}\left(\frac{|N|}{N' v_{syn}}\right)^2 \frac{1 - (S R_m')^2}{(\cosh ky)^2 + (S R_m' \sinh ky)^2}$$

Qualitative sketches of the normalized (dimensionless) forces versus speed characteristics are presented in Figures 1-21-1-23. In all of these figures the levitation force is $-F_y$.

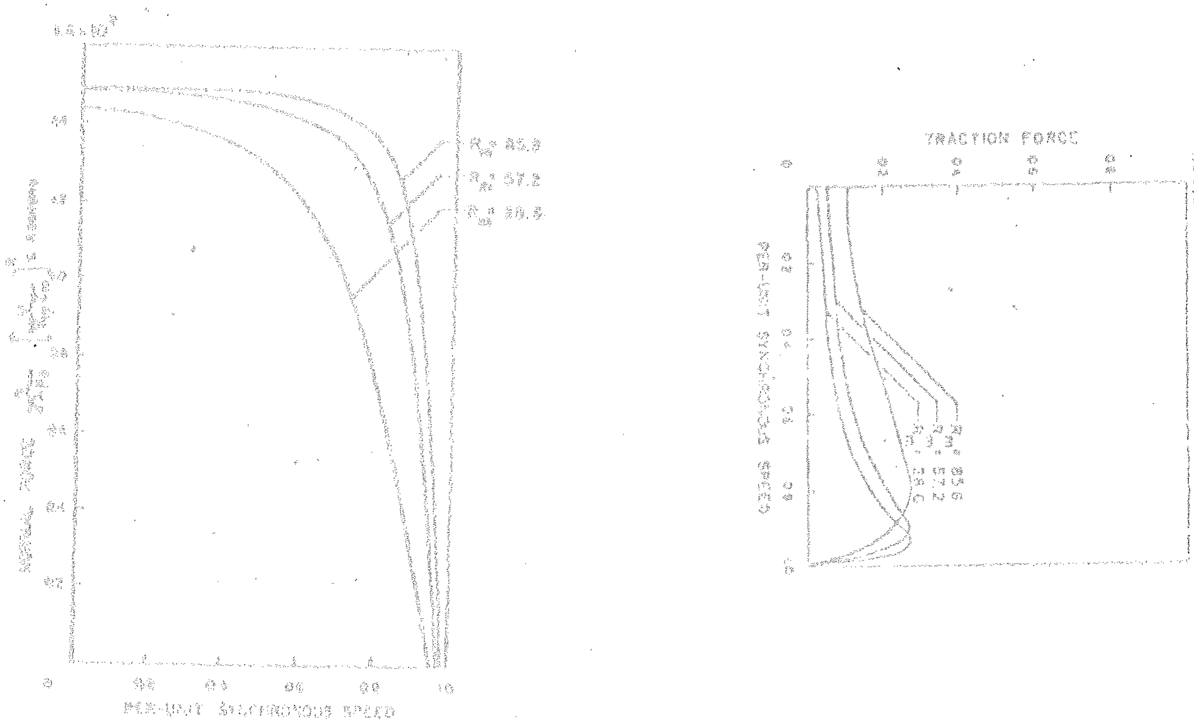


Figure 1-21. Normalized levitation (a) and traction forces (b) versus per-unit synchronous speed, V_m/V_{syn}, and slip, S. The effect of varying R_m' while keeping ko is shown. Note in (b) that the maximum value of traction force, F_x, is independent of [illegible]

APPENDIX 4A. APPLICATION OF MATRICES

The results of Chapter 4, especially section 4-[illegible] may be [illegible] simply using matrix algebra. Consider, first, a doubly excited system. If a system has two coils, with self and mutual inductances, the flux linkages may be written

$$(4A\text{-}1)\qquad [\lambda] = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} L_{11} i_1 + L_{12} i_2 \\ L_{21} i_1 + L_{22} i_2 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = [L][i]$$

The voltage equation for the terminal voltages in the absence of resistance, is

$$(4A\text{-}2)\qquad [v] = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{d\lambda_1}{dt} \\ \frac{d\lambda_2}{dt} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{d}{dt}[\lambda]$$

From (4A-1), we solve for the currents in terms of the flux linkages

$$(4A\text{-}3)\qquad [i] = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \frac{L_{22}}{\Delta} - \lambda_2 \frac{L_{12}}{\Delta} \\ -\lambda_1 \frac{L_{21}}{\Delta} + \lambda_2 \frac{L_{11}}{\Delta} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = [L]^{-1}[\lambda]$$

where $\Delta = L_{11}L_{22} - L_{12}L_{21}$ is the determinant of the [illegible]

$$[L] = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}.$$

Equation (4A-3) defines the reciprocal $[L]^{-1}$ of $[L]$

$$(4A\text{-}4)\qquad [L]^{-1} = \frac{1}{\Delta}\begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{bmatrix}.$$

The computation of $[L]^{-1}$ proceeds first by [illegible] $[L]_t$, the transpose of $[L]$,

$$(4A\text{-}5)\qquad [L]_t = \begin{bmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{bmatrix},$$

[illegible] for finding the inverse of any [illegible]

[illegible] connected in the two coils [illegible]

[illegible] scalar product of two column matrices [illegible] transpose of one matrix with the other [illegible] product of two matrices to be [illegible] left multiplicand must equal the [illegible]

[illegible] of two matrices is equal to the [illegible] reverse order. Thus, from (4A-3)

[illegible]

Letting

$$[\Gamma] = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = [L]^{-1} = \frac{1}{\Delta}\begin{bmatrix} \text{[illegible]} \end{bmatrix}, \qquad \text{(4A-13)}$$

W_m may be put in the scalar-product form

$$\boxed{W_m = \frac{1}{2}[\lambda]_t [\Gamma][\lambda]}. \qquad \text{(4A-14)}$$

The coenergy is obtained similarly

$$[v] = [L]\frac{d}{dt}[i]. \qquad \text{(4A-15)}$$

The power is equal to

$$\begin{aligned} p &= [i]_t [v] \\ &= [i]_t [L]\frac{d}{dt}[i]. \end{aligned} \qquad \text{(4A-16)}$$

Thus, the increment of energy (and hence coenergy, because the two are equal in a linear system) in time dt becomes

$$dW_m = dW_m' = p\,dt = [i]_t [L]\, d[i], \qquad \text{(4A-17)}$$

which corresponds to (4A-11). Finally, the coenergy [illegible]

$$\boxed{W_m' = \frac{1}{2}[i]_t [L][i] = \frac{1}{2}\{\text{[illegible]}\}} \qquad \text{(4A-18)}$$

Equations (4A-14) and (4A-18), though derived assuming a doubly excited system, are valid, <u>in their matrix forms</u>, for a multiply excited system and will be used later in the text.

As an application of these results to a multiply excited system, consider the example on p. 122; the inductance matrix is given by

$$[L] = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & -2 \\ 3 & -2 & 6 \end{bmatrix}. \qquad \text{(4A-19)}$$

Following the procedure in (4A-7), the inverse matrix is found to be

$$[L]^{-1} = [\Gamma] = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0.3 & 0.1 \\ -1 & 0.1 & 0.7 \end{bmatrix} \qquad \text{(4A-20)}$$

Thus from (4A-14)

(4A-21)
$$W_m = \frac{1}{2}\begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}\begin{bmatrix}2 & 0 & -1\\ 0 & 0.3 & -0.1\\ -1 & 0.1 & [illegible]\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}$$

$$= \lambda_1^2 - \lambda_1\lambda_3 + 0.15\,\lambda_2^2 + 0.1\,\lambda_2\lambda_3 + [illegible]\,\lambda_3^2$$

as shown on p. 124.

The coenergy is

(4A-22)
$$W_m' = \frac{1}{2}\begin{bmatrix}i_1 & i_2 & i_3\end{bmatrix}\begin{bmatrix}2 & -1 & [illegible]\\ -1 & [illegible] & [illegible]\\ 3 & [illegible] & 4\end{bmatrix}\begin{bmatrix}i_1\\ i_2\\ i_3\end{bmatrix} = [illegible]$$

To show that $W_m = W_m'$ for a linear system, we start with

(4A-23)
$$W_m + W_m' = i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 = [i]_t[\lambda]$$

But $[\lambda] = [L][i]$, so that

(4A-24)
$$[i]_t[\lambda] = [i]_t[L][i] = 2W_m'$$

Upon substitution of this result back into (4A-23) we obtain

(4A-25)
$$W_m + W_m' = 2W_m'$$

which concludes our proof.

APPENDIX [illegible] [illegible]
CONSIDERATIONS [illegible]

In many systems [illegible] ... [illegible] by applying [illegible] ... where the magnetic [illegible] ... [illegible] linked, and by what [illegible]

This is the case for [illegible] ... rotor configuration [illegible] ... [illegible] calculate the self induc[tance] [illegible] ... applying stored magnetic energy considerations [illegible] ... obtain the total stored magnetic energy [illegible] ... and equate this result to [illegible]

We proceed to demonstrate this approach [illegible] ... 2.8, all of the magnetic energy is stored [illegible] ... vanishes in the iron [illegible] ... the gap [illegible] is given by [illegible]. Hence

[illegible]

When we square each [illegible] and [illegible] ... obtain

[illegible]

[illegible] total magnetic [illegible] ...

(B-3) $$W_m = \int_{R_a}^{R_f} r\,dr \int_0^{2\pi} d\theta \cdot \ldots(r,\theta)$$

We carry out the θ integration first and utilize the [illegible]

(B-4) $$\int_0^{2\pi} \cos m\theta \cdot \cos n\theta \, d\theta = \int_0^{2\pi} \sin m\theta \, \sin n\theta \, d\theta = 0 \quad \text{if } n \neq m$$

(B-5) $$\int_0^{2\pi} \cos^2 n\theta \, d\theta = \int_0^{2\pi} \sin^2 n\theta \, d\theta = \pi .$$

Upon substituting (B-2) into (B-3), [illegible] there results

(B-6) $$W_m = \frac{8 \mu_0 R_a^2 T_s^2}{\pi} \sum_{n \text{ odd}} \left([illegible] \right) \int_{R_a}^{R_f} \left(\frac{f_n^2(r) + g_n^2(r)}{r} \right) dr$$

Recalling the definitions of $f_n(r)$, $g_n(r)$ the [illegible] is evaluated easily to give

(B-7) $$\int_{R_a}^{R_f} \left(\frac{f_n^2(r) + g_n^2(r)}{r} \right) dr = \frac{1}{n} \; \frac{1 + \left(\frac{R_a}{R_f}\right)^{2n}}{1 - \left(\frac{R_a}{R_f}\right)^{2n}}$$

The result for W_m becomes, therefore

(B-8) $$W_m = 8 \mu_0 \frac{R_a^2 T_s^2}{\pi} \sum_{n \text{ odd}} \frac{1}{n^3} \; \frac{1 + \left(\frac{R_a}{R_f}\right)^{2n}}{1 - \left(\frac{R_a}{R_f}\right)^{2n}}$$

If there are N_a turns per unit length on the armature winding each turn carrying the armature current, [illegible] in terms of I, b_n becomes

(8-8) $W_{m} = \frac{8\mu_0}{\pi}(R_a N_a^{\prime})^2$ [illegible] $\sum_{n\ odd}$ [illegible]

Equating this result to the stored magnetic [illegible] armature inductance (per meter of axial length) [illegible] we readily determine L_a to be

(8-10) $$L_a = \frac{16\mu_0}{\pi}(R_a N_a^{\prime})^2 \sum_{n\ odd}$$ [illegible]

The infinite series can be summed on a digital computer to give L_a in terms of the air-gap ratio $\left(\frac{R_a}{R_f}\right)$ [illegible]

then L_a takes on its minimum value

(8-11) $$L_{a_{min}} = \frac{16\mu_0}{\pi}(R_a N_a^{\prime})^2 \sum_{n\ odd}$$ [illegible]

as we expect because the air gap in each [illegible]

Conversely, if R_f [illegible] R_a the air gap [illegible] armature inductance.

Appendix C. CALCULATION OF ARMATURE-FIELD MUTUAL INDUCTANCE

On the rotor-stator configuration of section [illegible] [illegible] two stator windings essentially concentrated at the stator surface S_f. The "coil sides" of each winding are located at [illegible] and are separated from each other by a thin slot. When the coils are connected in series and carry currents in the directions shown, they form a stator "north" and "south" pole. These windings are called the *field windings*. The distributed winding located on the surface S_a, is, of course, the armature winding.

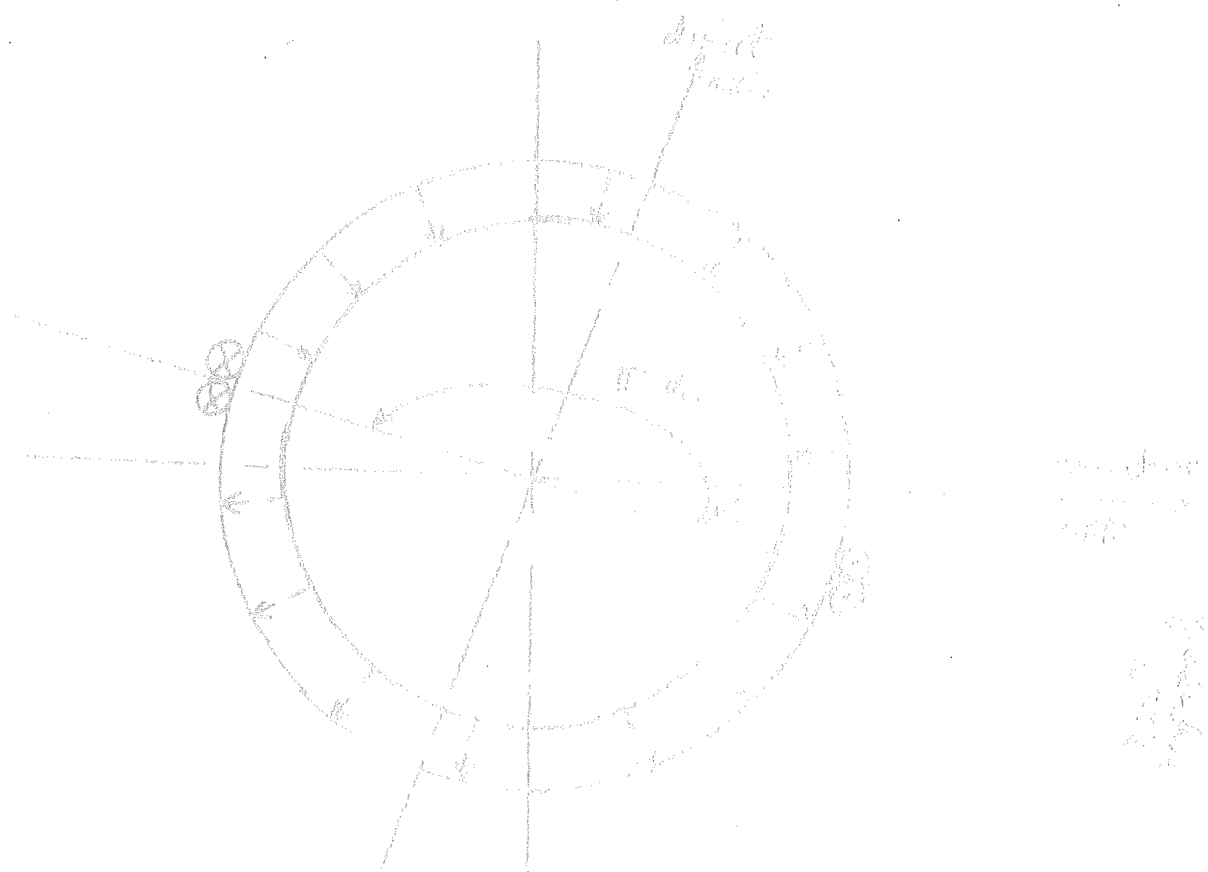


Fig. C.1. [illegible]

Of the flux density $\vec{B}$ established by the armature [illegible], only the radial component $B_r(R_f, \theta)$ at [illegible] links the field windings. The total armature flux linking the upper field winding is (per unit axial length)

$$\lambda_{fa} = N_f \int_{\alpha}^{\pi-\alpha} B_r(R_f, \theta) R_f \, d\theta \quad [illegible] \tag{C-1}$$

[illegible]

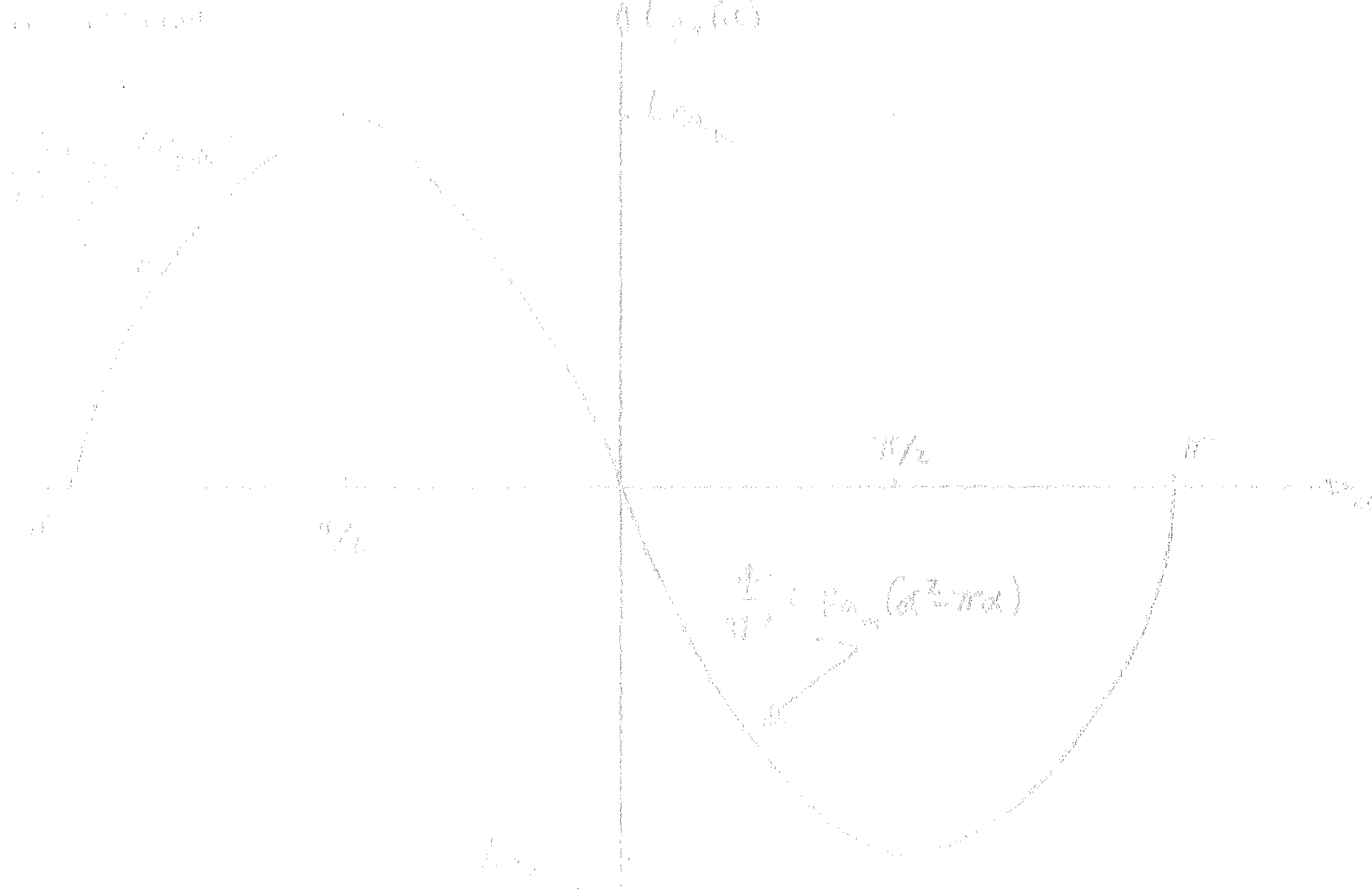
[illegible] and q_f for the [illegible]

$$[illegible]$$

[illegible] than [illegible] when [illegible] armature and field winding (because the field [illegible] infinity).

The [illegible] from (C-2) is that the mutual coupling [illegible] and field depends on the relative orientation of the [illegible]. We define the armature axis to be along [illegible] axis (also called the direct axis) to be along [illegible]. The quadrature axis lies along θ [illegible]

[illegible] maximum coupling occurs for $\alpha = \frac{\pi}{2}$, i.e., when the [illegible] aligned with the field direct axis. Minimum coupling [illegible] when the field quadrature axis and armature axis [illegible]



[illegible] mutual [illegible]

[illegible]

Appendix D. CALCUL[illegible]

In this appendix, we shall calculate [illegible] [illegible] [illegible] ature) of the system in Appendix "C", assuming [illegible] armature [illegible] field coils are excited. We shall perform [illegible] [illegible] ng the results of Section 3.11 (equation [illegible]) [illegible] and [illegible] ng the stress-tensor.

Calculation of torque using mutual inductance

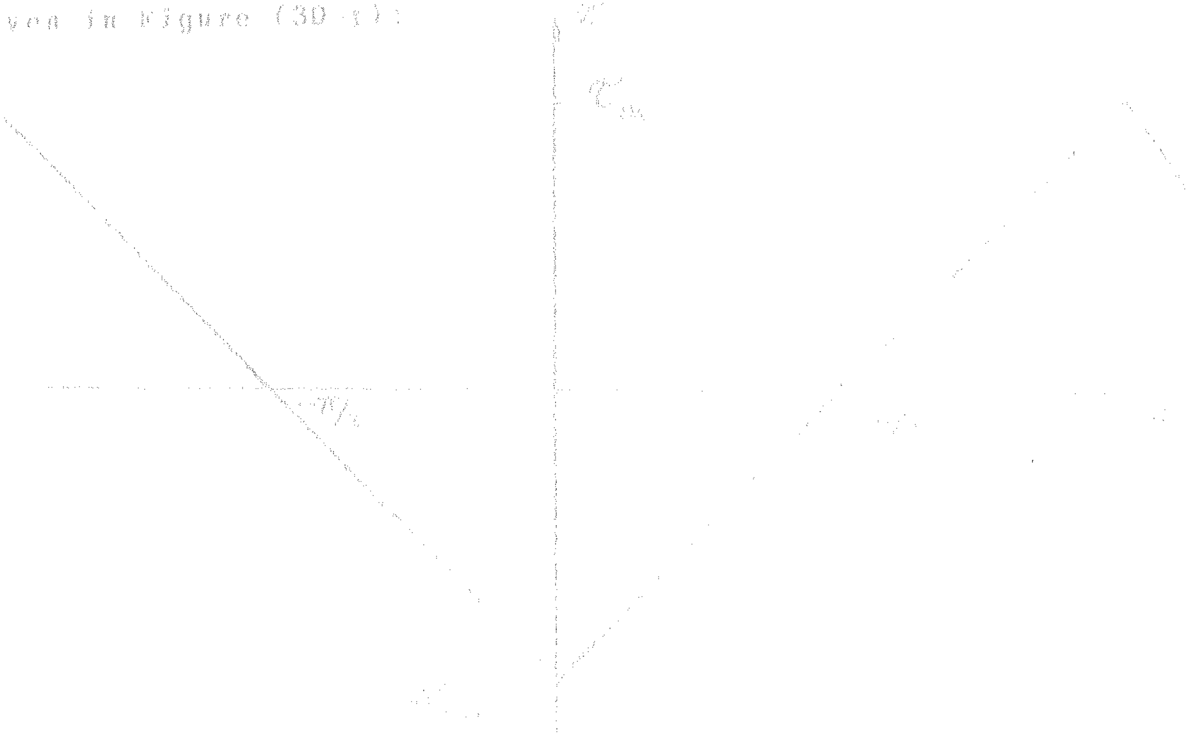
In order to apply (3.147) we must determine the [illegible] the self and mutual inductances with angle. From [illegible] it follows the armature self inductance is independent of the position of the rotor, while a study of Figure 30-1 indicates that the field self inductance is independent of α because the symmetry [illegible] variation of reluctance of the field flux path [illegible]. Then, [illegible] ves only the mutual inductance term for torque. [illegible]

$$\tau = \lambda_a \lambda_f \frac{dL_{fa}}{d\alpha} \quad (\text{[illegible]-1})$$

$$= \frac{\lambda_a \lambda_f \, 8 \mu_0 N_a N_f}{\pi^2} \text{ [illegible]}$$

e we have written N_a [illegible] for the [illegible] armature pole.

The Fourier series in (30-[illegible]) is identical to the [illegible] ter 2, for $B_r(R_f, \theta)$ (see equation (2-30[illegible]) [illegible] given in Figure (30-[illegible]):



D-1 [illegible] Torque as [illegible] [illegible] axis and field quadrature axis [illegible] [illegible] in the [illegible]

From this figure we see that [illegible] when the quadrature axis of the field is aligned with the armature [illegible] maximum torque [illegible], and when the direct axis of the field is aligned with the armature axis, the torque vanishes.

2. Calculation of torque using the stress [illegible]

Because there is no air gap flux density in the z direction (out of the page) in Fig. (B-1) it follows that the magnetic stress tensor at $r = R_a$ is given by

$$\text{(B-2)} \quad \bar{\bar{T}}_M(R_a, \theta) = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & \frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix}$$

The net traction (or stress) acting on the armature (whose outward normal is $\bar{a}_R$) is given by

$$\text{(B-3)} \quad \bar{P} = \bar{\bar{T}}_M(R_a, \theta) \cdot \bar{a}_R = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & \frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left(\frac{B_r^2 - B_\theta^2}{2\mu_0} \right) \bar{a}_R + \frac{B_r B_\theta}{\mu_0} \bar{a}_\theta$$

Upon taking the cross-product of (B-3) with $R_a \bar{a}_R$, the radius vector from the origin to any point on the armature, we get the torque-per-unit area established by the air gap flux

$$\bar{\tau}' = R_a \bar{a}_R \times \bar{P}$$

$$\text{(B-4)} \quad = \frac{R_a B_r B_\theta}{\mu_0} \bar{a}_R \times \bar{a}_\theta$$

$$= \frac{R_a B_r B_\theta}{\mu_0} \bar{a}_z .$$

The torque-per-unit length of rotor (and [illegible]) is given by (dropping the unit vector)

$$\text{(B-5)} \quad \tau = \frac{R_a}{\mu_0} \int_0^{2\pi} B_r B_\theta R_a \, d\theta$$

In order to carry out the integration of (D-5) we must know the total (armature plus field) contributions to B_r and B_θ.

A common approximation is to say that the field winding (Fig. [illegible]) establishes a purely radial flux density which is uniform at the rotor, and

$$\bar{B}_f(R_a,\theta) = -B_{rf}\,\bar{a}_R \quad , \quad -\alpha < \theta < \pi - \alpha \tag{D-6}$$

$$= B_{rf}\,\bar{a}_R \quad , \quad \pi - \alpha < \theta < 2\pi - \alpha$$

The constant value of B_{rf} is easily determined to be $\frac{\mu_0 N_f i_f}{g}$

where g is the gap length, $R_f - R_a$.

The integrand of (D-5) is written

$$B_r(R_a,\theta)\,B_\theta(R_a,\theta) = (B_{ra} + B_{rf})\,B_{\theta a} \tag{D-7}$$

where B_{ra} and $B_{\theta a}$ are the armature components given by ([illegible] 50) and B_{rf} is given by (3D-6).

It follows immediately from the expressions for B_{ra} and $B_{\theta a}$ that the integral of $B_{ra} \cdot B_{\theta a}$ over $0 \leq \theta \leq 2\pi$ vanishes because the integral of $\sin m\theta \cdot \cos n\theta$ vanishes over this range. Thus we are left with the integral of $B_{rf} \cdot B_{\theta a}$

From the analysis of section 2-8, we found that

$$B_{\theta a}(R_a,\theta) = -\mu_0 J_s(\theta) \tag{D-8}$$

with $J_s(\theta)$ shown in Figure 2-10. Hence, utilizing Figure 2-10 and (D-6) we sketch $B_{rf} \cdot B_{\theta a}$ in Figure D-2

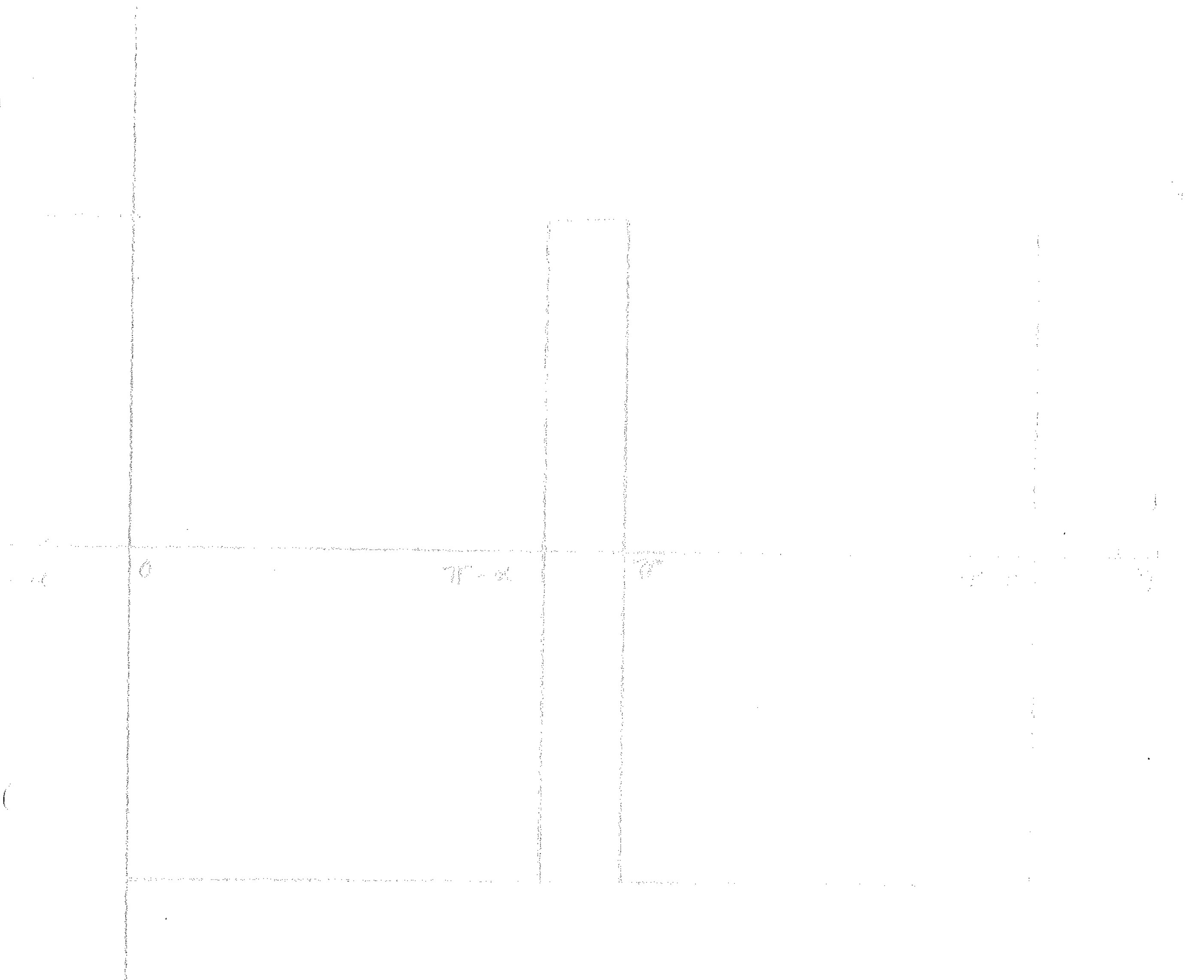


Figure D-2. Sketch of the integrand for [illegible]

The integral of this function, in terms of [illegible] is precisely of the form shown in Figure D-1 [illegible] case being $\frac{\mu_0 R_a}{2}$, $N_a N_f i_a i_f$. Thus, [illegible] consistency of the two distinct methods of [illegible] inductance and stress tensor

The negative sign implies a torque input into the generator. Upon substitution of the matrix from (3A-16) for W_m', we obtain

$$\tau = -\frac{1}{2}[i']_t \frac{\partial [L]}{\partial \alpha_m}[i'] \qquad (5\text{-}25)$$

where $[L]$ is given by (5-5) and $[i']$ in (5-9). The subscript "t", of course, implies "transpose."

We recall that $\partial[L]/\partial\alpha_m$ was determined in (5-8) as the matrix coefficient of ω_m:

$$\frac{\partial [L]}{\partial \alpha_m} = -P M_{afm}\begin{bmatrix} 0 & \sin P\alpha_m & \sin(P\alpha_m-120^\circ) & \sin(P\alpha_m-240^\circ) \\ \sin P\alpha_m & 0 & 0 & 0 \\ \sin(P\alpha_m-120^\circ) & 0 & 0 & 0 \\ \sin(P\alpha_m-240^\circ) & 0 & 0 & 0 \end{bmatrix} \qquad (5\text{-}26)$$

When the scalar product of (5-25) is evaluated as described in Appendix 3A, there results for the torque expression

$$\tau = P M_{afm}\left[i_a' i_f' \sin P\alpha_m + i_b' i_f' \sin(P\alpha_m - 120^\circ) + i_c' i_f' \sin(P\alpha_m - 240^\circ)\right] \qquad (5\text{-}27)$$

For balanced three phase currents, (5-27), thus becomes

$$\begin{aligned} \tau &= P I_m I_f M_{afm}\left[\sin(P\alpha_m-\theta)\sin P\alpha_m + \sin(P\alpha_m-\theta-120^\circ)\sin(P\alpha_m-120^\circ) \right. \\ &\quad \left. + \sin(P\alpha_m-\theta-240^\circ)\sin(P\alpha_m-240^\circ)\right] \qquad (5\text{-}28) \\ &= \frac{P I_m I_f M_{afm}}{2}\left[\cos(-\theta) - \cos(2P\alpha_m-\theta) + \cos(-\theta) - \cos(2P\alpha_m-\theta-240^\circ) \right. \\ &\quad \left. + \cos(-\theta) - \cos(2P\alpha_m-\theta-120^\circ)\right] \\ &= \boxed{\frac{3}{2} P I_m I_f M_{afm}\cos\theta} \end{aligned}$$

We note in the above that because the currents are balanced the instantaneous torque is constant in time (and therefore equal to the average torque). If the currents were not balanced a second harmonic torque would also appear.

The mechanical power input is

$$P_m = \tau\omega_m = \frac{3}{2} P I_m I_f \omega_m M_{af_m} \cos\theta \tag{5-29}$$

But $E_g = P\omega_m M_{af_m} I_f$ (Equation (5-20)), so that (5-29) becomes

$$P_m = \frac{3}{2} E_g I_m \cos\theta \tag{5-30}$$

and this agrees exactly with the generated electrical power given in (5-22), as it should.

5-3. Torque Angle

Let us neglect the rotor resistance of the machine. The average power output will then be equal to the average generated power and thus to the mechanical power input. The phasor diagram for the case of zero rotor resistance is shown in Figure 5-5.

Figure 5-5. Phasor diagram (phase a) corresponding to zero rotor resistance. δ is the torque angle. This phasor diagram is the same as Figure 5-4 with $R_a = 0$.

The power output from Figure 5-5 is

$$P = \frac{3}{2} V_a I_a \cos\varphi \tag{5-31}$$

As can be seen from Figure 5-5, $I_a X_s \cos\phi = E_g \sin\delta$,

so that

(5-32) $$P = \frac{3}{2} \frac{V_a E_g \sin\delta}{X_s}$$

and

(5-33) $$\tau = \frac{P}{\omega_m} = \frac{3}{2} \frac{V_a E_g \sin\delta}{\omega_m X_s}$$

$\delta > 0$ → τ INPUT

The angle δ is called the torque angle. At no load $\delta = 0$ and there is no torque. If δ is negative the machine acts as a motor supplying mechanical torque and power to the load connected on its shaft.

5-4. Equivalent Circuit Diagram.

The equivalent circuit, per phase, of a synchronous generator delivering balanced, three phase, sinusoidal currents follows from (5-23) and is shown in Figure 5-6.

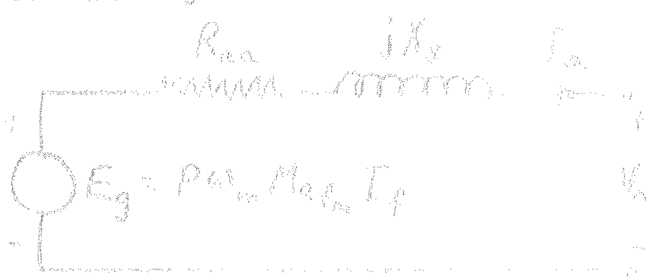


Figure 5-6. Per phase equivalent circuit of synchronous machine delivering balanced, three phase, sinusoidal currents.

The per cent regulation of a generator is given by

(5-34) $$\% \, Reg = \frac{|E_g| - |V_a|}{|V_a|} \cdot 100 .$$

Notice that $I_a X_s$ may have a value equal to or greater than V_a, in which case the regulation will be poor. In order to maintain uniform terminal voltages relatively independent of terminal currents, one uses voltage regulators, employing solid state circuitry [illegible] Machine

The [illegible] differs from the constant gap machine [illegible] in that the gap between stator and rotor [illegible] not constant. We shall assume that the saliency is on the stator, thereby giving the [illegible] view of Figure 4A-1 (Appendix 4A). [illegible] figure [illegible] phase rotor winding. We shall continue [illegible] machine. The modifications are that we have [illegible] windings on the rotor rather than two and these windings are [illegible] 120 electrical degrees rather than 90 electrical degrees.

The general [illegible] of Appendix 4A is applicable but we shall [illegible] matrices here and discuss the performance in more detail.

With the saliency on the field, the <u>self inductance</u> L_{aa} of phase a [illegible] function of the electrical angle of rotation $P\alpha_m$. When α_m is zero, i.e., when the center of coil a [illegible] the center of the stator pole, the self inductance is maximum. When the center of phase a is aligned with the center of the <u>quadrature axis</u> of the stator pole, i.e., 90 electrical degrees from the center (or <u>direct axis</u>), the self inductance is minimum.

It follows from the symmetry of Figure 4A-1 that the self inductance may be written as

$$L_{aa} = L_{s} + L_{v_2} \cos 2P\alpha_m + L_{v_4} \cos 4P\alpha_m + L_{v_6} \cos 6P\alpha_m + \cdots \quad (5\text{-}35)$$

Neglecting the harmonics higher than the second and dropping the subscript 2 from L_{v_2}, we get

$$L_{aa} = L_{s} + L_{v} \cos 2P\alpha_m \quad (5\text{-}36)\ (a)$$

The self inductance of the field, however, is a constant equal to L_{ff}.

Similarly, the inductances of phases b and c are also written

$$L_{bb} = L_{s} + L_{v} \cos 2(P\alpha_m - 120^\circ) \quad (5\text{-}36)\ (b)$$

$$L_{cc} = L_{s} + L_{v} \cos 2(P\alpha_m - 240^\circ) \quad (5\text{-}36)\ (c)$$

The <u>mutual inductances between the phases</u> are functions of α_m, as well.

$$(5\text{-}37) \quad (a) \quad L_{ab} = -\left[M_s - L_v \cos (2P\alpha_m - 120^\circ) \right]$$

$$(b) \quad L_{bc} = -\left[M_s - L_v \cos 2P\alpha_m \right]$$

$$(c) \quad L_{ca} = -\left[M_s - L_v \cos (2P\alpha_m + 120^\circ) \right].$$

[illegible] the mutual inductances between the three phases and the field [illegible] given by

$$[illegible] \; \cos (P\alpha_m - [illegible]^\circ)$$

The analysis of section 5-1 applies except that the mutual inductance matrix (5-8) becomes

$$(5\text{-}39) \qquad \frac{d[L]}{d\theta_m}\cdot\frac{d\theta_m}{dt} = -P\omega_m \begin{bmatrix} 0 & M_{afm}\sin P\theta_m & M_{afm}\sin(P\theta_m-120^\circ) & M_{afm}\sin(P\theta_m-240^\circ) \\ M_{afm}\sin P\theta_m & 2L_y\sin 2P\theta_m & 2L_y\sin(2P\theta_m-120^\circ) & 2L_y\sin(2P\theta_m+120^\circ) \\ M_{afm}\sin(P\theta_m-120^\circ) & 2L_y\sin(2P\theta_m-120^\circ) & 2L_y\sin 2(P\theta_m-120^\circ) & 2L_y\sin 2P\theta_m \\ M_{afm}\sin(P\theta_m-240^\circ) & 2L_y\sin(2P\theta_m-240^\circ) & 2L_y\sin 2P\theta_m & 2L_y\sin 2(P\theta_m-240^\circ) \end{bmatrix}$$

Again, because of the complexity of the general analysis, we return to the two cases considered before.

Case I. <u>No load</u> $(i_a = i_b = i_c = 0)$ <u>and</u> v_f is d.c.

After steady state has been established we have $\frac{d}{dt}[i] = 0$, so that (5-7) becomes

$$(5\text{-}40)\quad (a) \qquad v_f = V_f = I_f R_{ff}$$

$$(b) \qquad v_a = P\omega_m M_{afm} I_f \sin P\theta_m$$

$$(5\text{-}40)\quad (c) \qquad v_b = P\omega_m M_{afm} I_f \sin(P\theta_m - 120^\circ)$$

$$(d) \qquad v_c = P\omega_m M_{afm} I_f \sin(P\theta_m - 240^\circ)$$

Note the similarity between (5-40) and (5-10). This indicates that an unloaded salient pole machine behaves as an unloaded constant gap machine. Incidentally, in (5-40) we have made the signs of v_a, v_b and v_c negative when the machine acts as a generator.

<u>Case II. Balanced three phase load and v_f as before.</u>

As far as the flux linkages in the field are concerned, they are given by (5-12) when the three phase currents are equal to (5-11). Therefore the field current under balanced load currents is not changed from its value V_f/R_{ff}.

In order to determine the flux linkages in phase a we substitute the expressions for the self and mutual inductances (5-36) - (5-38), into (5-5) and then use (5-4). The result is

$$(5\text{-}41)\qquad \begin{aligned} \lambda_a = {} & M_{afm} I_f \cos p\theta_m + (L_K + L_V \cos 2p\theta_m) I_m \sin(p\theta_m - \theta) \\ & -(M_K - L_V \cos(2p\theta_m - 120^\circ)) I_m \sin(p\theta_m - \theta - 120^\circ) \\ & -(M_K - L_V \cos(2p\theta_m + 120^\circ)) I_m \sin(p\theta_m - \theta + 120^\circ). \end{aligned}$$

After some algebra, and the use of the trigonometric identity

$$(5\text{-}42)\qquad 2 \sin\alpha \cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta),$$

equation (5-41) reduces to

$$(5\text{-}43)\qquad \begin{aligned} \lambda_a = {} & (L_K + M_K - \tfrac{3}{2} L_V) I_m \sin p\theta_m \cos\theta \\ & -(L_K + M_K + \tfrac{3}{2} L_V) I_m \cos p\theta_m \sin\theta + M_{afm} I_f \cos p\theta_m \end{aligned}$$

If we define the <u>quadrature axis inductance</u>, Lq, and the <u>direct axis inductance</u>, Ld, by

$$(5\text{-}44)\qquad \begin{aligned} &(a) \quad L_q = L_K + M_K - \tfrac{3}{2} L_V \\ &(b) \quad L_d = L_K + M_K + \tfrac{3}{2} L_V , \end{aligned}$$

(5-43) becomes

$$(5\text{-}45).\qquad \lambda_a = L_q I_m \cos\theta \sin p\theta_m - L_d I_m \sin\theta \cos p\theta_m + M_{afm} I_f \cos p\theta_m .$$

Now that we have computed λ_a we can easily determine the terminal voltage for phase a when the machine is operated as a generator.

$$(5\text{-}46)\qquad \begin{aligned} v_a = {} & -i_a R_{aa} - \frac{d\lambda_a}{dt} \\ = {} & -R_{aa} I_m \sin(p\theta_m - \theta) - p\omega_m L_q I_m \cos\theta \cos p\theta_m \\ & - p\omega_m L_d I_m \sin\theta \sin p\theta_m + M_{afm} I_f \, p\omega_m \sin p\theta_m \\ = {} & -R_{aa} I_m \cos\theta \sin p\theta_m + R_{aa} I_m \sin\theta \cos p\theta_m \\ & - p\omega_m L_q I_m \cos\theta \cos p\theta_m - p\omega_m L_d I_m \sin\theta \sin p\theta_m + p\omega_m M_{afm} I_f \sin p\theta_m . \end{aligned}$$

where we recall the [illegible]

$$\text{(a)}\quad I_q = I_m \cos\theta$$

$$\text{(5-47)}\quad \text{(b)}\quad I_d = I_m \sin\theta$$

$$\text{(c)}\quad X_d = p\omega_m L_d$$

$$\text{(d)}\quad X_q = p\omega_m L_q$$

Iq is called the quadrature axis current, Id the direct axis current, Xd the direct axis reactance and Xq the quadrature axis reactance.

If we use $\sin p\alpha_m$ as the reference phasor, then the term $E_g \sin p\alpha_m$, the generated voltage, is represented by a pure real number in phasor rotation, as are all other $\sin p\alpha_m$ terms in (5-46). All $\cos p\alpha_m$ terms are ninety degrees out of phase with the reference phasor. Hence, in phasor rotation (5-46) becomes

$$\text{(5-48)}\quad \boxed{\dot{V}_a = \dot{E}_g - \dot{I}_q R_{aa} - \dot{X}_d \dot{I}_d + j(\dot{R}_{aa}\dot{I}_d - \dot{X}_q \dot{I}_q)}$$

The interpretation of the direct and quadrature axis currents may be made using (5-45). In that equation we see that the direct axis current acts along the field, or direct axis, in producing flux linkages in phase a. The quadrature axis current, however, acts in the quadrature axis, i.e., ninety electrical degrees ahead or behind the direct axis. The direct axis space is along a pole axis in Figure 4A-1, whereas the quadrature axis space is between adjacent poles.

This decomposition of armature current into currents producing direct and quadrature axis mmf's will be useful later in d.c. machines as well.

5-6. Torque in a Salient Pole Machine.

The torque in a salient pole machine is obtained in the same manner as for a constant gap machine. Here, however, because the inductances, with the exception of L_{ff}, are functions of α_m, the angle of rotation, the value of the torque is not the same as for a constant gap machine.

We start the calculations with (5-25), except now [illegible] use $\frac{dL}{d\alpha_m}$ as given in (5-39). When the scalar product of (5-25) is evaluated as described in Appendix 3A, [illegible] for the torque

$$\tau = p L_y \{ I_m^2 \sin 2p\alpha_m + 2 I_m I_f \sin(2p\alpha_m \ldots) [illegible]$$

$$\text{(5-49)}\quad + I_b^2 \sin 2(p\alpha_m \ldots) + 2 I_b I_c \ldots [illegible]$$

$$+ p M_{af} I_f [\ldots] [illegible]$$

If the currents are balanced, as in ([illegible]), [illegible], after trigonometric manipulations, is

$$(5\text{-}50) \qquad \tau = p\left[\frac{3}{2} M_{af_m} I_f I_a \cos\theta - \frac{3}{2} L_g I_a^2 \sin 2\theta\right]$$

From (5-49), however, we obtain $L_g = \frac{1}{2}(L_d - L_q)$ so that (5-50) becomes

$$\tau = p\left[\frac{3}{2} M_{af_m} I_f I_q - \frac{3}{4}(L_d - L_q) I_m^2 \sin 2\theta\right]$$

$$(5\text{-}51) \qquad = p\left[\frac{3}{2} M_{af_m} I_f I_q - \frac{3}{2}(L_d - L_q) I_m \sin\theta \, I_m \cos\theta\right]$$

$$= p\left[\frac{3}{2} M_{af_m} I_f I_q - \frac{3}{2}(L_d - L_q) I_d I_q\right],$$

where we have employed the definitions of I_d, I_q given in (5-47). I_d and I_q may be determined in terms of V_a, E_g and the torque angle, δ, between V_a and E_g by the following equations, whose proof is left as an exercise.

$$\text{(a)} \qquad I_q = \frac{V_a \sin\delta}{X_q}$$

$$(5\text{-}52) \qquad \text{(b)} \qquad I_d = \frac{E_g - V_a \cos\delta}{X_d}$$

When (5-52) is substituted into (5-51), we obtain

$$(5\text{-}53) \qquad \tau = p\left[\frac{3}{2} M_{af_m} I_f \frac{V_a \sin\delta}{X_q} - \frac{3}{2}(L_d - L_q)\left(\frac{E_g - V_a \cos\delta}{X_d}\right)\frac{V_a \sin\delta}{X_q}\right].$$

Recalling that $p M_{af_m} I_f = E_g/\omega_m$, we obtain our final result

$$(5\text{-}54) \qquad \tau = \frac{3}{2}\left[\frac{E_g V_a}{\omega_m X_q} \sin\delta - \frac{p\omega_m (L_d - L_q)(E_g - V_a \cos\delta) V_a \sin\delta}{\omega_m^2 X_d X_q}\right]$$

$$= \boxed{\frac{3}{2}\left[\frac{E_g V_a}{\omega_m X_d} \sin\delta + \frac{(X_d - X_q) V_a^2 \sin 2\delta}{2\omega_m X_d X_q}\right]}.$$

The first term is identical to the torque calculated in (5-33) for a constant gap machine if we replace X_s by X_d. The second term, depending as it does on the variations of self inductance of the rotor coils with angle, is called the reluctance torque. It is of the form calculated for the reluctance motor (equation 4-86) in that it varies as sin 2 . Thus, we may interpret in sections 4-5 and 4-7 also as a torque angle.

5-7. Power Input to Machine

The mechanical power input to the machine is

(5-55) $$P_m = \tau\omega_m = \frac{3}{2}\left[\frac{E_g V_a}{X_d}\sin\delta + \frac{(X_d - X_q)V_a^2}{2X_d X_q}\sin 2\delta\right]$$

Let us interpret this result electrically, neglecting copper losses in the machine (i.e., letting $R_{aa} = R_{bb} = R_{cc} = 0$). Letting the load impedance angle be ϕ (i.e., the angle between the terminal voltage V_a and terminal current I_a), the total time average electrical power output is given by (5-31). ϕ, however, is equal to $\theta - \delta$, because δ is the angle between E_g and V_a. Thus, (5-31) becomes

(5-56) $$P = \frac{3}{2}V_a I_a \cos(\theta - \delta)$$

$$= \frac{3}{2}V_a I_a (\cos\theta\cos\delta + \sin\theta\sin\delta).$$

But $I_q = I_a\cos\theta$, $I_d = I_a\sin\theta$, so that (5-56) becomes

(5-57) $$P = \frac{3}{2}V_a(I_q\cos\delta + I_d\sin\delta).$$

Upon substitution of (5-52), we obtain

(5-58) $$P = \frac{3}{2}V_a\left(\frac{V_a\sin\delta\cos\delta}{X_q} + \frac{(E_g - V_a\cos\delta)\sin\delta}{X_d}\right)$$

$$= \frac{3}{2}\left(\frac{E_g V_a}{X_d}\sin\delta + \frac{V_a^2}{2}\sin 2\delta\left(\frac{1}{X_q} - \frac{1}{X_d}\right)\right)$$

$$= \boxed{\frac{3}{2}\left(\frac{E_g V_a}{X_d}\sin\delta + \frac{(X_d - X_q)}{2X_d X_q}V_a^2\sin 2\delta\right)}$$

Thus, the average electrical power output is equal to the mechanical power input, as we expect in the absence of electrical (and mechanical friction) losses.

For an isolated machine (not connected to other power sources), if the excitation, I_f, is reduced to zero then E_0 by virtue of (5-27) also reduces to zero, as does V_a. Hence, the machine delivers no power.

On the other hand, if the machine is connected to a "bus bar" which is always excited by other alternators so as to keep the terminal voltage V_a constant, when excitation (and hence E_0) is reduced to zero, the machine will still deliver reluctance power given by the second term in (5-58)

5-8. The Three Phase Synchronous Motor.

The three phase alternator can be used as a three phase synchronous motor if electric power is supplied to the machine and mechanical power is taken from the shaft. This reversibility permits us to carry over the expressions for torque and power, as well as phasor diagrams appropriately modified, which were derived in the preceding sections and now interpret them in terms of motor behavior.

Analytically, all that is required is to change the signs of voltage, or currents appropriately in making the transition from generator to motor. Thus if the synchronous machine is a generator the voltage equations become

(5-59)

(a) $v_f = i_f R_{ff} + d\lambda_f/dt$

(b) $-v_a = i_a R_{aa} + d\lambda_a/dt$

(c) $-v_b = i_b R_{bb} + d\lambda_b/dt$

(d) $-v_c = i_c R_{cc} + d\lambda_c/dt$.

These are the equations previously used in this chapter.

If the synchronous machine is operating as a motor the voltage equations are

(5-60)

(a) $+v_f = i_f R_{ff} + d\lambda_f/dt$

(b) $v_a = i_a R_{aa} + d\lambda_a/dt$

(c) $v_b = i_b R_{bb} + d\lambda_b/dt$

(d) $v_c = i_c R_{cc} + d\lambda_c/dt$.

In either system the left-hand voltages refer to the terminal voltages.

A little thought reveals that because the flux linkages of all the coils are linearly dependent on the currents, the transition from generator to motor implied by (5-59) and (5-60), is equivalent to change the direction of all currents [illegible]

Thus, in Figure (5-6), we need change only the direction of I_a relative to V_a (i.e., let I_a enter the (+) terminal) in order to arrive at the equivalent circuit (per phase) of a constant gap synchronous motor. The corresponding phasor diagram is shown in Figure 5-7.

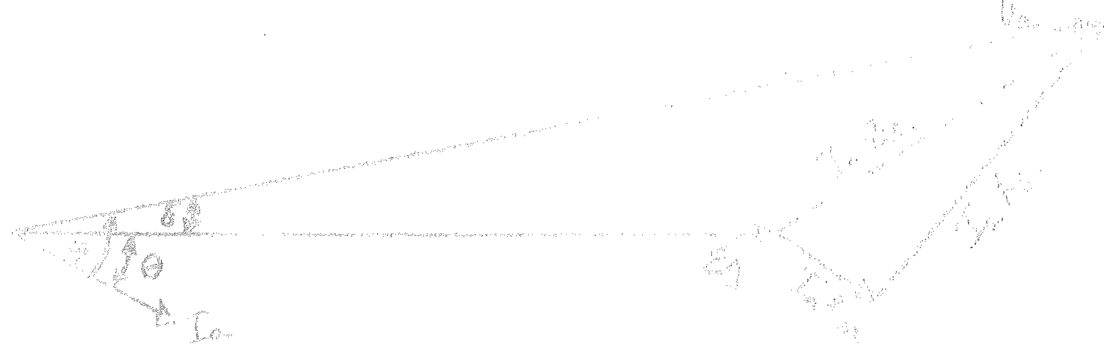


Figure 5-7. Per phase phasor diagram of constant gap synchronous motor.

The most important thing to glean from this figure is that V_a now leads the generated (or counter) voltage, E_f, which means that δ'', the torque angle, is negative. According to (5-32), therefore the electrical power output is negative as is the torque also. These are reasonable inferences for a motor. Identical conclusions hold for a salient pole motor whose developed torque is given by (5-54) (with positive δ'') and whose electrical power input (in the absence of I^2R losses in the rotor) is given by (5-58) (also with positive δ'').

The synchronous motor develops an average torque at only one mechanical angular velocity, ω_m, which is given by

(5-61) $$\boxed{\omega = P\omega_m}$$

where ω is the electrical angular frequency of the (balanced) three phase currents. This is easily shown for a constant gap rotor by referring to (5-27) and substituting

(5-62)
$$\text{(a)} \quad \lambda'_a = I_m \sin(\omega t - \theta)$$
$$\text{(b)} \quad \lambda'_b = I_m \sin(\omega t - \theta - 120^\circ)$$
$$\text{(c)} \quad \lambda'_c = I_m \sin(\omega t - \theta - 240^\circ)$$

Then (5-27) becomes

(5-63)
$$\tau = P I_m I_f M_{afm} [\sin(\omega t - \theta) \sin P\theta_m + \sin(\omega t - \theta - 120^\circ) \sin (P\theta_m - 120^\circ) + \sin(\omega t - \theta - 240^\circ) \sin(P\theta_m - 240^\circ)]$$

The (infinite) time-average value [illegible] $\omega = P\omega_m$, in which case the time-average torque output is given by (5-20)

$$\tau = \frac{3}{2} P I_m I_f [\text{illegible}] \tag{5-64}$$

Because the value of ω_m, (5-61), at which average torque is developed is, in a sense, synchronized to the line frequency, (5-61) defines the <u>synchronous speed</u> of the motor. This fact also indicates the origin of the name "synchronous machine".

There is an often used formula for the synchronous speed, N_s:

$$
\begin{aligned}
N_s &= \frac{\omega}{P} && (\text{rad/sec}) \\
&= \frac{\omega}{2\pi P} && (\text{rev/sec}) \\
&= \frac{f}{P} && (\text{rev/sec}) \\
&= \frac{60 f}{P} && (\text{rev/min}) \\
&= \frac{120 f}{p} && (\text{rev/min})
\end{aligned}
\tag{5-65}
$$

where p is the number of poles on the machine.

The fact that an average torque is developed at only the synchronous speed implies that a synchronous motor cannot be self-starting unless it is endowed with auxiliary windings called <u>amortisseur windings</u>, which enable the machine to start as an <u>induction motor</u>.

In Chapter 4 we saw an earlier manifestation of this non-self-starting condition in connection with the doubly excited reluctance motor which also developed torque at only one speed.

Before ending this discussion of the synchronous motor, let us return to Figure 5-7, and neglect copper losses. Then the electrical power input, $\frac{3}{2} V_a I_a \cos\theta$, equals the mechanical power output. If the mechanical load (i.e., mechanical power output) remains constant as the electrical excitation (I_f) is varied, then, clearly the component, $I_a \cos\theta$, of rotor current in phase with the line voltage V_a, remains constant (we assume, of course, that V_a is fixed).

If I_f varies, however, so does E_g [illegible] (5-20), and therefore I_a. Thus, for a constant mechanical load, but varying field excitation, we obtain the current locus and phasor diagrams of Figure 5-[illegible]

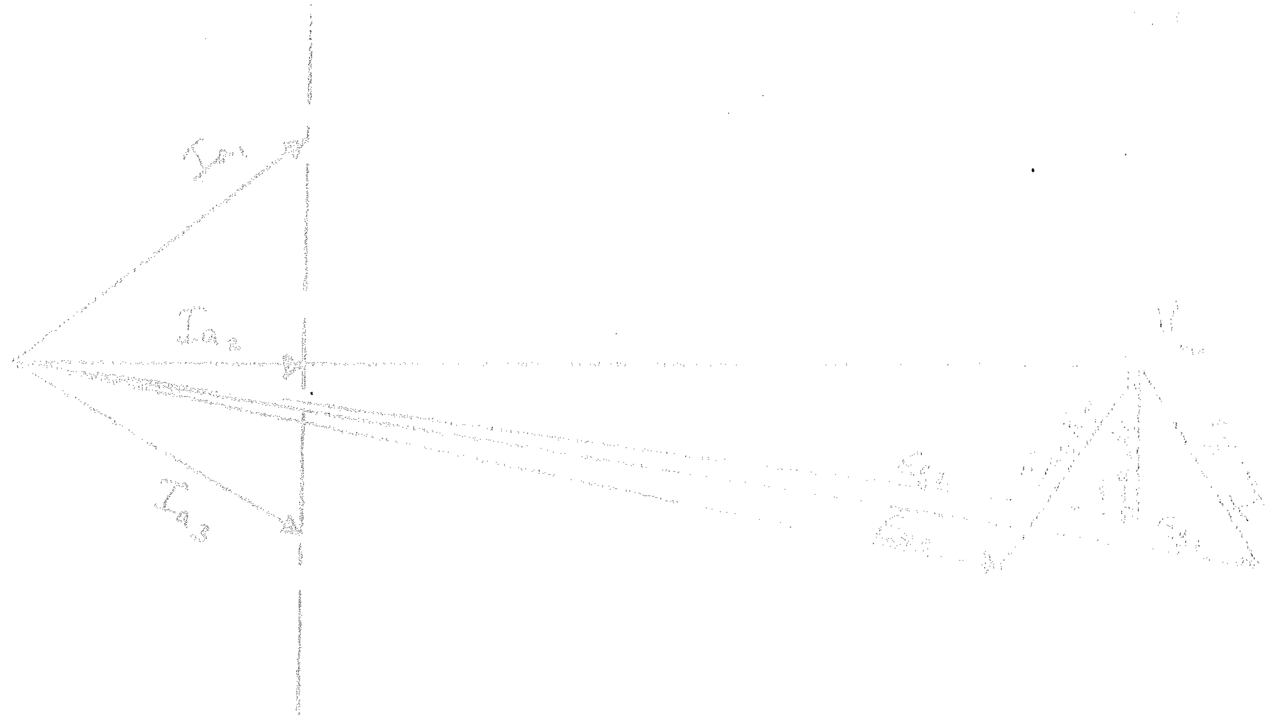
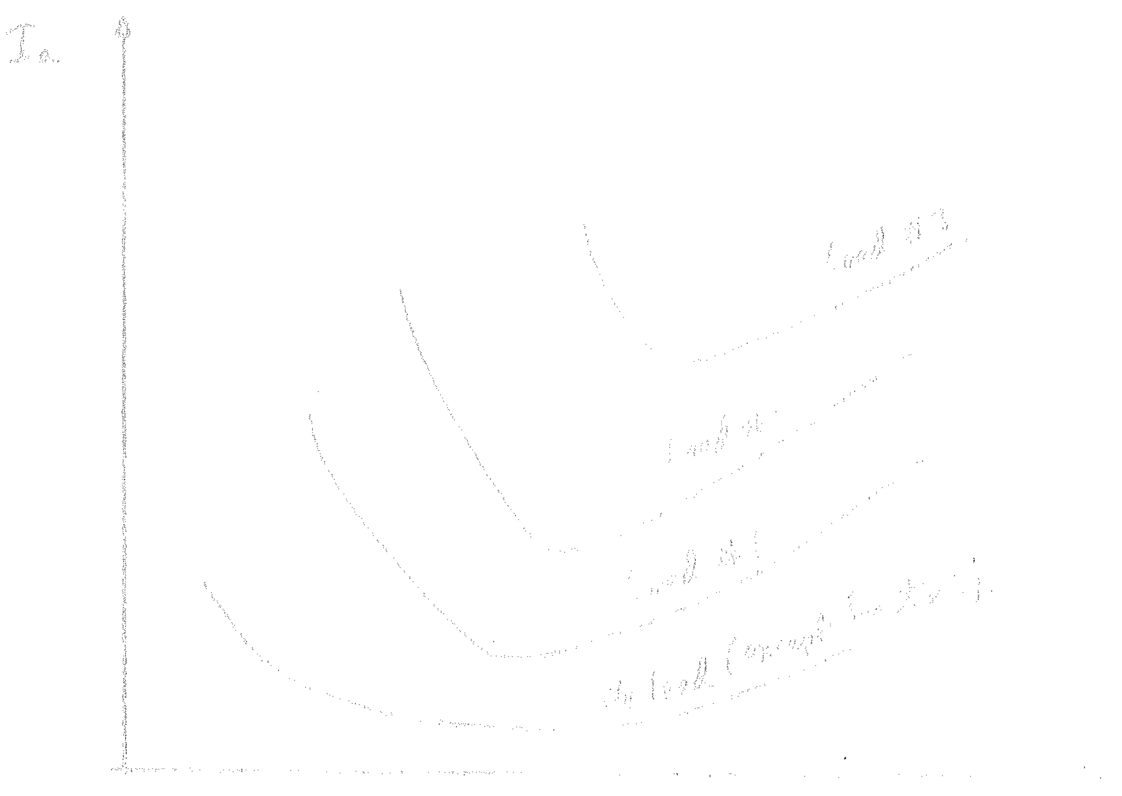


Figure 5-8. Current locus and the associated phasor diagrams for a constant mechanical load and varying field excitation.

The armature current can lead V_a, as I_{a1}, be in phase, as I_{a2}, or lag, as I_{a3}. E_{g1} corresponds to an overexcited field, E_{g2} to the correct excitation and E_{g3} to an underexcited field. We thus see that for a motor, overexcitation causes a leading power factor and underexcitation gives a lagging power factor. If the armature current is plotted versus the field current for a constant mechanical load, the curves obtained will be the "V curves" of Figure 5-9.

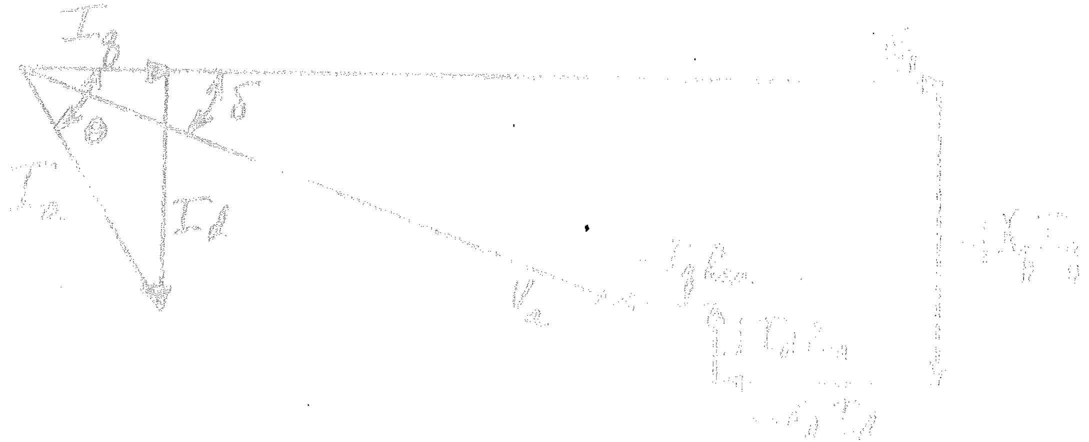


The fact that an overexcited synchronous motor gives a leading power factor, enables this machine to be [illegible] used as a capacitor to correct the power factors of lagging systems (such as an industrial plant using many motors). [illegible] called a "synchronous capacitor."

Problems for Chapter 5

5-1. Draw a phasor diagram showing V_a, I_a (lagging V_a by θ degrees), I_d and I_q (see Equation (5-47)).

5-2. Show that the phasor diagram representing (5-48) is



5-3. From the phasor diagram of problem 5-2, show that, [illegible] of copper losses,

$$I_q = \frac{V_a \sin\delta}{X_q}, \quad I_d = \frac{E_f - V_a \cos\delta}{X_d},$$

5-4. Assume that a constant gap alternator has a constant value of field current, I_f, and that only phase a has a load. Derive the equations for voltages V_a, V_b, and V_c.

[illegible]

[illegible]

Consider the system of current sheets [illegible] by [illegible], distributed windings as described in [illegible] and [illegible] in Figure 5A-1.

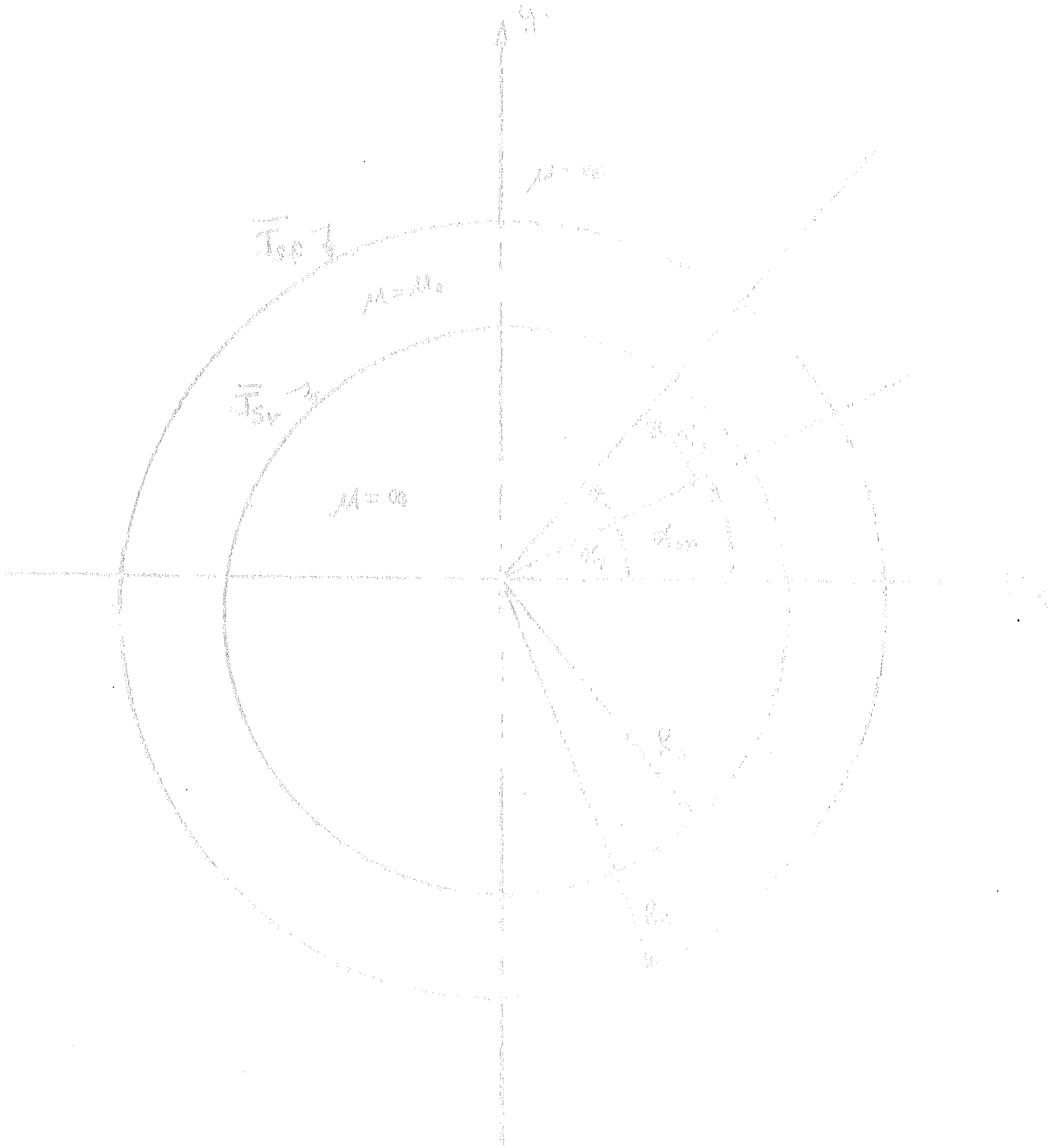


Figure 5A-1 Illustrating equivalent [illegible] gap machine. The [illegible] the three phase [illegible] angle of the [illegible]

For the current sheets, we take

(5A-1)

(a) $$\bar{J}_{sf}(\alpha_2) = J_f \sin P\alpha_2 \; \bar{a}_z$$

(b) $$\bar{J}_{sr}(\alpha_2) = \left[J_a \sin P\alpha_1 + J_b \sin(P\alpha_1 - 120^\circ) + J_c \sin(P\alpha_1 - 240^\circ) \right] \bar{a}_z$$

$$= \left[J_a \sin(P\alpha_2 - P\alpha_m) + J_b \sin(P\alpha_2 - 120^\circ - P\alpha_m) + J_c \sin(P\alpha_2 - 270^\circ - P\alpha_m) \right] \bar{a}_z$$

α_1 is the angle measured with respect to a fixed point on the rotor. α_m is the angle that point makes with the stator referance. J_f, J_a, J_b, J_c are independent of angles.

Within the air gap, A_z, the z-component of the vector potential, satisfies Laplace's equation (2-30). From the analysis of section 2-8 we know that the most general solution is the Fourier series

(5A-2)

$$A_z(r, \alpha_2) = \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n})(A_n \sin n\alpha_2 + B_n \cos n\alpha_2)$$

$$= \sum_{n=1}^{\infty} (a_{1n} r^n \sin n\alpha_2 + a_{2n} r^n \cos n\alpha_2 + a_{3n} r^{-n} \sin n\alpha_2 + a_{4n} r^{-n} \cos n\alpha_2)$$

At $r = R_f$, the boundary condition on A_z is

(5A-3) $$-\frac{1}{\mu_0} \cdot \frac{\partial A_z}{\partial r}\bigg|_{r=R_f} = H_{tang}\bigg|_{r=R_f} = -J_f \sin P\alpha_2 ,$$

while that at $r = R_a$ is

(5A-4)

$$-\frac{1}{\mu_0} \cdot \frac{\partial A_z}{\partial r}\bigg|_{r=R_a} = H_{tang}\bigg|_{r=R_a} = J_a \sin(P\alpha_2 - P\alpha_m) + J_b \sin(P\alpha_2 - 120^\circ - P\alpha_m) + J_c \sin(P\alpha_2 - 240^\circ - P\alpha_m) .$$

Upon substitution of (5A-2) into (5A-3), we get

(5A-5) $$\sum_{n=1}^{\infty} n \left(a_{1n} R_f^{(n-1)} \sin n\alpha_2 + a_{2n} R_f^{(n-1)} \cos n\alpha_2 - a_{3n} R_f^{-(n+1)} \sin n\alpha_2 - a_{4n} R_f^{-(n+1)} \cos n\alpha_2 \right) = \mu_0 J_f \sin P\alpha_2 .$$

Since P is an integer we obtain

(5A-6)

$$(a)\quad P\left(a_{1P} R_f^{(P-1)} - a_{3P} R_f^{-(P+1)}\right) = \mu_0 J_f$$

$$(b)\quad a_{1n} R_f^{(n-1)} - a_{3n} R_f^{-(n+1)} = 0 \qquad \text{for all } n \neq P.$$

$$(c)\quad a_{2n} R_f^{(n-1)} - a_{4n} R_f^{-(n+1)} = 0, \qquad \text{for all } n,$$

Substitution of (5A-2) into (5A-4) yields

$$\sum_{n=1}^{\infty} n\left(a_{1n} R_a^{(n-1)} - a_{3n} R_a^{-(n+1)}\right)\sin n\alpha_z + \sum_{n=1}^{\infty} n\left(a_{2n} R_a^{(n-1)} - a_{4n} R_a^{-(n+1)}\right)\cos n\alpha_z =$$

$$= -\mu_0 J_a \sin(P\alpha_z - P\alpha_m) - \mu_0 J_b \sin(P\alpha_z - 120^\circ - P\alpha_m) - \mu_0 J_c \sin(P\alpha_z - 240^\circ - P\alpha_m)$$

(5A-6)

$$= \sin P\alpha_z \cdot \left[-\mu_0 J_a \cos P\alpha_m - \mu_0 J_b \cos(120^\circ + P\alpha_m) - \mu_0 J_c \cos(240^\circ + P\alpha_m)\right]$$

$$+ \cos P\alpha_z \cdot \left[\mu_0 J_a \sin P\alpha_m + \mu_0 J_b \sin(120^\circ + P\alpha_m) + \mu_0 J_c \sin(240^\circ + P\alpha_m)\right].$$

Equating the sin $P\alpha_z$ and cos $P\alpha_z$ ~~terms~~ [terms] either side of the equal sign yields

(5A-7)

$$(a)\quad P\left[a_{1P} R_a^{(P-1)} - a_{3P} R_a^{-(P+1)}\right] = -\mu_0 J_a \cos P\alpha_m - \mu_0 J_b \cos(120^\circ + P\alpha_m) - \mu_0 J_c \cos(240^\circ + P\alpha_m),$$

$$(b)\quad P\left[a_{2P} R_a^{(P-1)} - a_{4P} R_a^{-(P+1)}\right] = \mu_0 J_a \sin P\alpha_m + \mu_0 J_b \sin(120^\circ + P\alpha_m) + \mu_0 J_c \sin(240^\circ + P\alpha_m).$$

$$(c)\quad n\left[a_{1n} R_a^{(n-1)} - a_{3n} R_a^{-(n+1)}\right] = 0 \qquad \text{for all } n \neq P$$

$$(d)\quad n\left[a_{2n} R_a^{(n-1)} - a_{4n} R_a^{-(n+1)}\right] = 0 \qquad \text{" " "}$$

Equations (5A-6) (b), (c), together with (5A-7) (c), (d), imply that $a_{1n} = a_{3n} = a_{3n} = a_{4n} = 0$ for all $n \neq P$. Equations (5A-6) (a), (c) (for n = P), together with (5A-7) (a), (b), yield the system of equations from which a_{1P}, a_{3P}, a_{2P} and a_{4P} are to be determined

(5A-8)

$$(a)\quad a_{1P} - a_{3P} R_f^{-2P} = \mu_0 J_f / P R_f^{(P-1)} = A$$

$$(b)\quad a_{1P} - a_{3P} R_a^{-2P} = \frac{-\mu_0 J_a \cos P\alpha_m - \mu_0 J_b \cos(120^\circ + P\alpha_m) - \mu_0 J_c \cos(240^\circ + P\alpha_m)}{P R_a^{(P-1)}} = B$$

$$(c)\quad a_{2P} - a_{4P} R_f^{-2P} = 0$$

$$(d)\quad a_{2P} - a_{4P} R_a^{-2P} = \frac{\mu_0 J_a \sin P\alpha_m + \mu_0 J_b \sin(120^\circ + P\alpha_m) + \mu_0 J_c \sin(240^\circ + P\alpha_m)}{P R_a^{(P-1)}}$$

$$= C.$$

The solutions of these two systems of two equations each are easily found, by the use of determinants, to be

$$\text{(a)}\quad a_{1p} = \frac{-AR_a^{-2p} + BR_f^{-2p}}{-R_a^{-2p} + R_f^{-2p}}$$

(5A-9)

$$\text{(b)}\quad a_{3p} = \frac{B - A}{-R_a^{-2p} + R_f^{-2p}}$$

$$\text{(c)}\quad a_{2p} = \frac{CR_f^{-2p}}{-R_a^{-2p} + R_f^{-2p}}$$

$$\text{(d)}\quad a_{4p} = \frac{C}{-R_a^{-2p} + R_f^{-2p}}.$$

Therefore, by (5A-2), $A_z(r,\alpha_z)$ becomes

$$A_z(r,\alpha_z) = a_{1p}\, r^p \sin p\alpha_z + a_{2p}\, r^p \cos p\alpha_z + a_{3p}\, r^{-p} \sin p\alpha_z + a_{4p}\, r^{-p} \cos p\alpha_z$$

(5A-10)

$$= \frac{1}{-R_a^{-2p} + R_f^{-2p}} \Big[\left(-Ar^p R_a^{-2p} + Br^p R_f^{-2p} - Ar^{-p} + Br^{-p}\right) \sin p\alpha_z$$

$$+ \left(Cr^p R_f^{-2p} + Cr^{-p}\right) \cos p\alpha_z \Big]$$

From the definition of $\bar{A}$, $\bar{B} = \nabla \times \bar{A}$, which in cylindrical symmetry implies

(5A-11)
$$\bar{B} = \bar{a}_r \cdot \frac{1}{r} \frac{\partial A_z}{\partial \alpha_z} - \bar{a}_\theta \frac{\partial A_z}{\partial r},$$

we obtain

$$\text{(a)}\quad B_r = \frac{1}{r} \frac{p}{R_f^{-2p} - R_a^{-2p}} \Big[\left(-Ar^p R_a^{-2p} + Br^p R_f^{-2p} - Ar^{-p} + Br^{-p}\right) \cos p\alpha_z$$

$$- \left(Cr^p R_f^{-2p} + Cr^{-p}\right) \sin p\alpha_z \Big]$$

(5A-12)

$$\text{(b)}\quad B_\theta = \frac{-p}{R_f^{-2p} - R_a^{-2p}} \Big[\left(-Ar^{(p-1)} R_a^{-2p} + Br^{(p-1)} R_f^{-2p} + Ar^{-(p+1)} - Br^{-(p+1)}\right)$$

$$\cdot \sin p\alpha_z + \left(Cr^{(p-1)} R_f^{-2p} - Cr^{-(p+1)}\right) \cos p\alpha_z \Big].$$

The [illegible] per [illegible] axial length within the air gap is

$$W_m = \frac{1}{2\mu_0}\int_0^{2\pi} d\theta_s \int_{R_a}^{R_f} r\,dr\,\left(B_r^2 + B_\theta^2\right) \qquad \text{(5A-13)}$$

When (5A-12) is substituted into (5A-13), and use is made of the definitions of α, β and γ (equation (5A-8)), there results

$$W_m = \frac{\ell\mu_0\pi}{2}\Bigg[\left(\frac{1+\left(\frac{R_f}{R_a}\right)^{-2p}}{1-\left(\frac{R_f}{R_a}\right)^{-2p}}\right)\left(\lambda_a'^2 + \lambda_b'^2 + \lambda_c'^2 + \frac{\lambda_f'^2}{\left(\frac{N_f}{N}\right)^2} - \lambda_a'\lambda_b' - \lambda_a'\lambda_c' - \lambda_b'\lambda_c'\right)$$

$$+ \frac{2NN_f}{\left(\left(\frac{R_f}{R_a}\right)^p + \left(\frac{R_a}{R_f}\right)^p\right)}\left(\lambda_a'\lambda_f'\cos p\theta_m + \lambda_b'\lambda_f'\cos(120° + p\theta_m) + \lambda_c'\lambda_f'\cos(240° + p\theta_m)\right)\Bigg]. \qquad \text{(5A-14)}$$

In obtaining this form, we have made the definitions (N = # of turns on rotor windings, N_f = # of turns on stator winding):

$$N_f\lambda_f' = \frac{I_f}{R_f^{-1}}, \quad \lambda_a' = \frac{I_a}{NR_a^{-1}}, \quad \lambda_b' = \frac{I_b}{NR_a^{-1}}, \quad \lambda_c' = \frac{I_c}{NR_a^{-1}}. \qquad \text{(5A-15)}$$

The results of chapter 3, for multiply excited systems, imply that we may make the following identifications

(5A-16)

(a) $$L_{aa} = L_{bb} = L_{cc} = \mu_0\pi N^2\left(\frac{1+\left(\frac{R_f}{R_a}\right)^{-2p}}{1-\left(\frac{R_f}{R_a}\right)^{-2p}}\right)$$

(b) $$L_{ff} = N_f^2\mu_0\pi\left(\frac{1+\left(\frac{R_f}{R_a}\right)^{-2p}}{1-\left(\frac{R_f}{R_a}\right)^{-2p}}\right)$$

(c) $$L_{ab} = L_{ac} = L_{bc} = -\frac{N^2\mu_0\pi}{2}\left(\frac{1+\left(\frac{R_f}{R_a}\right)^{-2p}}{1-\left(\frac{R_f}{R_a}\right)^{-2p}}\right)$$

(d) $$L_{af} = \frac{2\mu_0\pi\, NN_f}{\left(R_f/R_a\right)^p + \left(\frac{R_a}{R_f}\right)^p}\cos p\theta_m$$

(e) $$L_{bf} = \frac{2\mu_0\pi\, NN_f}{\left(R_f/R_a\right)^p + \left(R_a/R_f\right)^p}\cos(p\theta_m + 120°)$$

(f) $$L_{cf} = \frac{2\mu_0\pi\, NN_f}{\left(R_f/R_a\right)^p + \left(R_a/R_f\right)^p}\cos(p\theta_m + 240°).$$

In particular, we conclude that for a constant gap machine with sinusoidally distributed windings on both the field and rotor, the self inductances are constant and equal (if the turns per-unit-length around the circumference are equal), the mutual inductances between the phases are constant with respect to rotor position θ_m, are negative (implying "bucking" polarity when the currents in the coils are positive), and, in magnitude, are half the self inductances. Finally, the mutual between the field and phase windings vary sinusoidally and are 120° out of phase with each other.

Appendix 5B. Rotating Magnetic Field

The current sheet on the rotor was given in (5A-1) by

$$J_{Sr}(\alpha_1) = J_a \sin(p\alpha_1) + J_b \sin(p\alpha_1 - 120^\circ) + J_c \sin(p\alpha_1 - 240^\circ) \tag{5B-1}$$

where α_1 is the angle measured relative to the rotor (see Figure 5A-1).

Let

$$\begin{aligned} &(a) \quad J_a = N' I_m \sin \omega t \\ &(b) \quad J_b = N' I_m \sin(\omega t - 120^\circ) \\ &(c) \quad J_c = N' I_m \sin(\omega t - 240^\circ), \end{aligned} \tag{5B-2}$$

where N' is a constant (independent of α_1 or t) distribution factor for the coil windings, and is measured in turns-per-meter. I_m is the maximum value of the sinusoidal currents in the windings.

Upon substitution of (5B-2) into (5B-1), we obtain

$$\begin{aligned} J_{Sr}(\alpha_1) &= N' I_m \left[\sin p\alpha_1 \sin \omega t + \sin(p\alpha_1 - 120^\circ) \sin(\omega t - 120^\circ) + \sin(p\alpha_1 - 240^\circ) \sin(\omega t - 240^\circ) \right] \\ &= \frac{N' I_m}{2} \left[\cos(\omega t - p\alpha_1) - \cos(\omega t + p\alpha_1) + \cos(\omega t - p\alpha_1) - \cos(\omega t + p\alpha_1 - 240^\circ) + \cos(\omega t - p\alpha_1) - \cos(\omega t + p\alpha_1 - 120^\circ) \right] \\ &= \frac{3N'}{2} I_m \cos \omega\left(t - \frac{p}{\omega} \alpha_1\right). \end{aligned} \tag{5B-3}$$

Therefore, we conclude that balanced three phase currents flowing in balanced, sinusoidally distributed windings produce a traveling wave of sheet current whose phase velocity, relative to the rotor, is

$$\begin{aligned} v_p &= \frac{\omega}{p} \quad (\text{rad/sec}) \\ &= \frac{f}{p} \quad (\text{rev/sec}) \\ &= \frac{120 f}{\mathrm{P}} \quad (\text{rev/min}) \quad (\mathrm{P} = \#\text{ of poles}) \\ &= n_s. \end{aligned} \tag{5B-4}$$

Thus, the phase velocity of the rotating magnetic field produced by the current sheet is equal to the synchronous speed of the machine. This gives us an alternative definition of synchronous speed.

In order to calculate the rotating magnetic field, at the surface of the rotor, we return to (5A-8) and substitute (5B-2). The expressions for B and C then become

(5B-5)

$$\text{(a)} \qquad B = -\frac{3}{2}\,\frac{\mu_0 N' I_m}{p R_a^{(p-1)}} \sin(\omega t + p\alpha_m)$$

$$\text{(b)} \qquad C = -\frac{3}{2}\cdot\frac{\mu_0 N' I_m}{p R_a^{(p-1)}} \cos(\omega t - p\alpha_m)$$

Thus, from (5A-12)

(5B-6)

$$\text{(a)} \qquad B_r(R_a,\alpha_2) = \frac{1}{R_a(R_a^{-2p} - R_f^{-2p})}\left\{\frac{\mu_0 I_f}{(R_a R_f)^p}\cos p\alpha_2 + \frac{3}{2}\mu_0 N I_m (R_f^{-2p} + R_a^{-2p}) \sin(\omega t + p\alpha_m - p\alpha_2)\right\}$$

$$\text{(b)} \qquad B_\theta(R_a,\alpha_2) = \mu_0 J_{sp}(\alpha_2) - \frac{3}{2}\mu_0 N' I_m \cos(\omega t + p\alpha_m - \alpha_2)$$

where, in (a), $N = R_a N'$.

From these two expressions we see that the tangential component of $\bar{B}$, at the surface of the rotor, consists of a traveling (or rotating) wave with phase velocity ω/p rad/sec. The radial component also has a traveling portion with the same phase velocity, but this component is 90° out of space and time phase with the traveling component of B_θ. The remaining portion of B_r is established by the d.c. field current and consists of a non-traveling sinusoidal, spatial distribution.

Now that we have the magnetic field distributions we are in a position to calculate the stress tensor, $\bar{\bar{T}}_M$, and the resulting torque on the rotor. Following the treatment in Appendix 3D, we write

$$\text{(5B-7)} \qquad \bar{\bar{T}}_M(R_a,\alpha_2) = \begin{bmatrix} \dfrac{B_r^2 - B_\theta^2}{2\mu_0} & \dfrac{B_r B_\theta}{\mu_0} & 0 \\ \dfrac{B_r B_\theta}{\mu_0} & \dfrac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & \dfrac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix}$$

The traction acting on the armature (whose outward normal is $\bar{a}_R$) is given by

$$\bar{p} = \bar{\bar{T}}_M(R_a, \alpha_2) \cdot \bar{a}_R = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5B-8}$$

$$= \left(\frac{B_r^2 - B_\theta^2}{2\mu_0}\right)\bar{a}_R + \frac{B_r B_\theta}{\mu_0}\bar{a}_\theta$$

The torque-per-unit area is given by

$$\bar{\tau}' = R_a(\bar{a}_R \times \bar{p}) \tag{5B-9}$$

$$= R_a \frac{B_r B_\theta}{\mu_0} \bar{a}_z \; .$$

The torque-per-unit length of rotor is then given by

$$\tau = \frac{R_a}{\mu_0}\int_0^{2\pi} B_r B_\theta \, R_a \, d\alpha \, . \tag{5B-10}$$

$$= \frac{R_a}{\mu_0} \cdot \frac{1}{\left(R_a^{-2p} - R_f^{-2p}\right)} \int_0^{2\pi} \left[\frac{3\mu_0^2}{(R_a R_f)^p} N' I_m I_f \cos p\alpha_2 \cdot \cos(\omega t + p\alpha_m - p\alpha_2) + \frac{9}{4}\mu_0^2 N N' I_m^2 \left(R_f^{-2p} + R_a^{-2p}\right) \sin(\omega t + p\alpha_m - p\alpha_2) \cdot \cos(\omega t + p\alpha_m - p\alpha_2) \right] d\alpha_2$$

$$= \frac{3}{2} \cdot \frac{\mu_0 N I_m I_f}{\left(\frac{R_f}{R_a}\right)^p - \left(\frac{R_a}{R_f}\right)^p} \cos(\omega t + p\alpha_m) \; .$$

The time-average torque vanishes unless $\dot{\alpha}_m = \omega_m = -\omega/p = -\Omega_s$. Under this condition $\alpha_m = -\frac{\omega}{p}t + \theta/p$, where θ is an arbitrary reference phase angle measured in electrical degrees. Note that the above requirement implies that the machine must run in the negative direction relative to the stator reference, as indicated in Figure 5B-1. This is of no significance, implying only that with reference to the hypothetical current sheets established the machine does not run in the direction of positive α_m.

When the above expression is substituted into (5B-10), we obtain

$$\tau = \frac{3}{2} \frac{\mu_0 N I_m I_f}{\left(\frac{R_f}{R_a}\right)^p - \left(\frac{R_a}{R_f}\right)^p} \cos\theta, \tag{5B-11}$$

This result if of the same form as that previously derived (5-22) based on the motional inductance matrix, and again demonstrates the consistency of the stress-tensor, inductance approaches.

Appendix 5C. Three-Phase Circuits*

By a three-phase balanced system of voltages (or currents) is meant three voltages (or currents) of equal magnitude and displaced from each other by 120° ($\frac{2\pi}{3}$ radians). The positive sequence of voltages is shown in Figure 5C-1:

Figure 5C-1. Illustrating the positive sequence of three phase voltages (a) and their phasor representation when balanced (b).

A Y-connected load excited by a three-phase source is diagrammed in Fig. 5C-2.

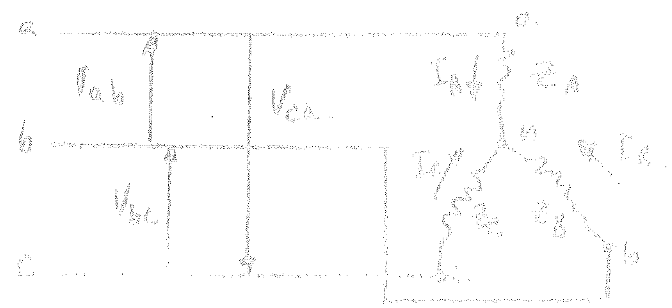


Fig. 5C-2. A Y-connected load

Kirchoff's current law gives us

$$I_A + I_B + I_C = 0 \qquad \text{(5C-1)}$$

The line voltages satisfy

(5C-2)

(a) $V_{ab} = I_A Z_A - I_B Z_B$

(b) $V_{bc} = I_B Z_B - I_C Z_C$

(c) $V_{ca} = I_C Z_C - I_A Z_A$

---

* E. M. Sabbagh, Circuit Analysis, Chapt. 23, The Ronald Press Co., New York, 1961.

The sum of (5C-2)(a),(b) and (c) [illegible]

(5C-3) $\quad V_{ab} + V_{bc} + V_{ca} = 0.$

Using (5C-1), (5C-2) (a), (b), we [illegible] I_a

(5C-4) $$I_a = \frac{V_{ab}(z_b + z_c) + V_{bc} z_b}{z_b z_c + z_a z_b + z_a z_c}$$

So far, no restrictions have been [illegible] on [illegible] impedances or the line voltages V_{ab}, V_{bc}. Let us [illegible] the impedances are equal (i.e., the load is balanced): $z_a = z_b = z_c = z$. Then

(5C-5) $$I_a = \frac{2V_{ab} + V_{bc}}{3z} = \frac{V_{ab} - V_{ca}}{3z}$$

If the line voltages are balanced, [illegible] V_{bc} [illegible] then $V_{ab} - V_{ca} = \sqrt{3}\, V_{ab} e^{-j30°}$, as shown in Figure [illegible]

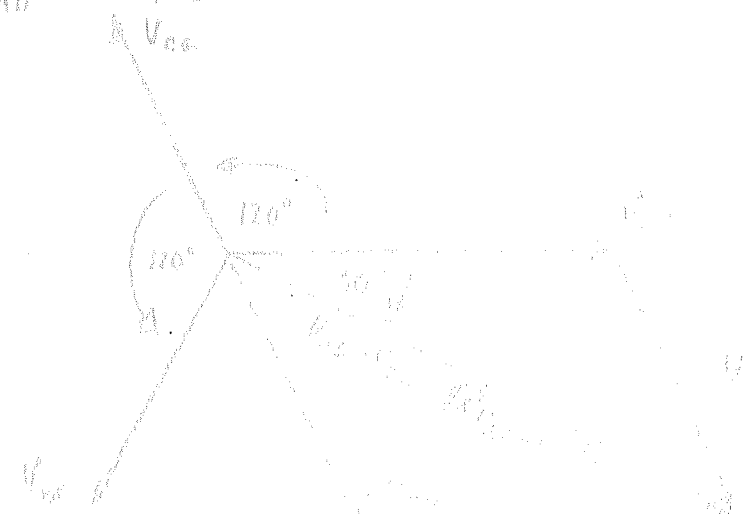


Figure 5C-3. Illustrating the [illegible] balanced system of [illegible]

Thus,

(5C-6) (a) $$I_a = \frac{\sqrt{3}}{3}\frac{V_{ab}}{z} e^{-j30°} = \frac{V_{ab}}{\sqrt{3}\, z} e^{-j30°}$$

and

(5C-6) (b) $$I_b = \frac{V_{bc}}{\sqrt{3}\, z} e^{-j30°}$$

(c) $$I_c = \frac{V_{ca}}{\sqrt{3}\, z} e^{-j30°}$$

and we see that the line currents are balanced.

If we define V_A to be the voltage across Z_A, [illegible]

which, upon substitution into (5C-6)(a), implies that

(5C-7) (a) $$V_A = \frac{V_{ab}}{\sqrt{3}} e^{-j30^\circ}.$$

Similarly,

(5C-7) (b) $$V_B = V_A e^{j240^\circ} = \frac{V_{ab}}{\sqrt{3}} e^{[illegible]}$$

(c) $$V_C = V_A e^{j120^\circ} = \frac{V_{ab}}{\sqrt{3}} e^{[illegible]}.$$

In other words, for a balanced Y-connected load and balanced line voltages the phase voltage (voltage across an impedance) is given in magnitude by

(5C-8) $$|V_\varphi| = \frac{1}{\sqrt{3}} |V_L|,$$

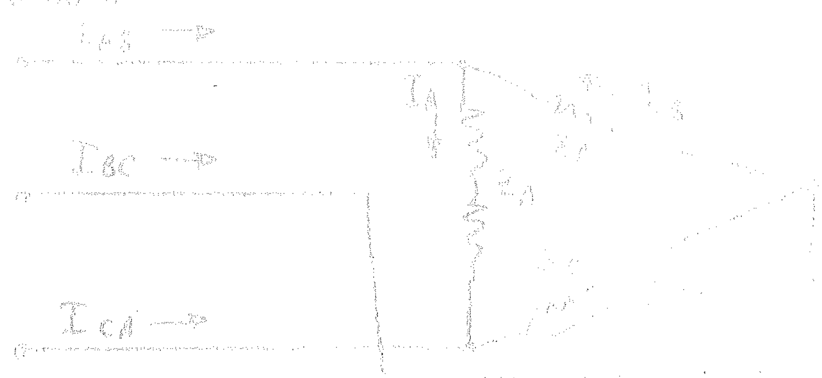
and the line current is given by

(5C-9) $$I_L = I_\varphi = V_\varphi / Z,$$

where V_φ is the phase voltage, V_L the line voltage, I_L the line current, and I_φ is the phase current. The phase and line voltages are shown in Figure 5C-4.

Figure 5C-4. Phase and line voltages in a balanced three-phase system

Another common load connection is the Δ-connected load shown in Figure 5C-5



For a Δ connected, balanced, load an analysis similar to that of the Y connected load yields the following results

(5C-10)

$$(a)\quad V_A = \left(I_{AB}/\sqrt{3}\,Y\right) e^{-j30°}$$

$$(b)\quad V_B = \left(I_{BC}/\sqrt{3}\,Y\right) e^{-j30°}$$

$$(c)\quad V_C = \left(I_{CA}/\sqrt{3}\,Y\right) e^{-j30°}$$

$$(d)\quad V_A = I_A/Y$$

$$(e)\quad V_B = I_B/Y$$

$$(f)\quad V_C = I_C/Y$$

$$(g)\quad I_A = \left(I_{AB}/\sqrt{3}\right) e^{-j30°}$$

$$(h)\quad I_B = \left(I_{BC}/\sqrt{3}\right) e^{-j30°}$$

$$(i)\quad I_C = \left(I_{CA}/\sqrt{3}\right) e^{-j30°},$$

where Y is the admittance of each phase load.

The results (g) - (i) may be restated as the phase current is equal to the line current divided by $\sqrt{3}$ and shifted 30° in the negative direction (see Figure 5C-6).

(5C-11) $$I_\varphi = \left(I_L/\sqrt{3}\right) e^{-j30°}.$$

Figure 5C-6. Phase and line currents in a balanced three-phase Δ-connected load

Power Measurement in a Balanced System

The power dissipated in a balanced [illegible] is

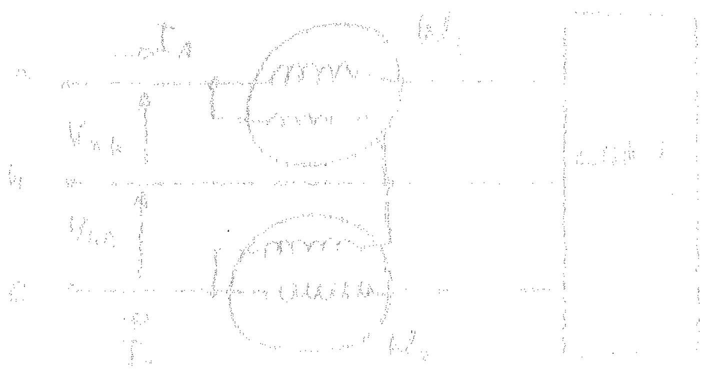
$$P = 3 V_\varphi I_\varphi \cos\theta \qquad (5C\text{-}12)$$

where V_φ is the effective value of the [illegible] value of phase current, and θ is the [illegible] voltage and phase current.

For a Y-connected load, $I_\varphi = I_L$ and [illegible] Δ-connected load $I_\varphi = I_L/\sqrt{3}$, $V_\varphi = V_L$ [illegible] yields

$$P = \sqrt{3}\, V_L I_L \cos\theta \qquad (5C\text{-}13)$$

The power in a three-phase, three-wire [illegible] by means of two wattmeters connected as shown in Fig. [illegible]



To prove the above assertion, we note that [illegible]

$$\begin{aligned} P_1 &= |V_{ab}|\,|I_a| \cos\theta_1 \\ &= V_L I_L \cos(30° + \theta) \\ &= V_L I_L (\cos 30° [illegible]) \end{aligned} \qquad (5C\text{-}14)$$

where $\theta_1 = 30° + \theta$ is the angle between [illegible]

W_2 reads

$$P_2 = |V_{cb}||I_c| \cos\theta_2$$

(5C-15) $$= V_L I_L \cos(30° - \theta)$$

$$= V_L I_L \left(\cos\theta \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\sin\theta\right)$$

Thus, the sum of the two readings is

(5C-16) $$\boxed{P = P_1 + P_2 = \sqrt{3}\, V_L I_L \cos\theta}$$

This result holds for either a Δ- or Y-connected load

[illegible]

When [illegible] electromagnet [illegible] that a rotor [illegible] pole, and [illegible]

Obvious [illegible] that is, the [illegible] and stator [illegible] depending [illegible]

The [illegible] which [illegible] Before [illegible] synchronous machine [illegible] stationary in space. [illegible] are energized with alternating [illegible] however, in that the synchronous machine [illegible] only, and produces no starting torque.

The armature winding [illegible] closing on itself, with taps to the [illegible] of the coil lying under a stator pole [illegible] and are placed in slots on the rotor.

The coil is wound [illegible] drum winding (imagine the [illegible] a child's drum). [illegible] two types. [illegible] poles on [illegible] regardless of the number of poles.

6-1. Motor Windings and [illegible]

To show the two types of drum windings [illegible] with twelve slots on the armature. In accordance [illegible] shall place two coil sides in a slot, one on top [illegible] the bottom. The top coil side is shown solid and the [illegible] dashed. The first three slots are under a north pole, the next [illegible] the next three under a north and the last three [illegible] is shown in Fig. 6-1.

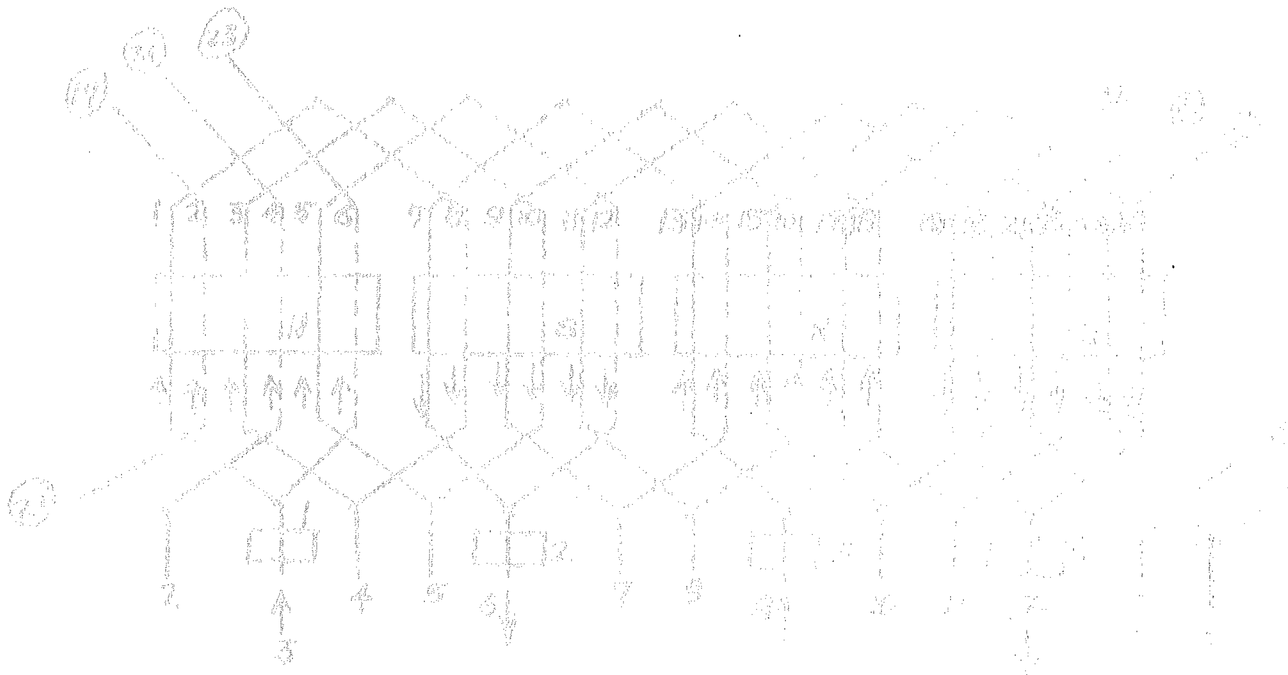


Fig. 6-1. Lap Wound Machine

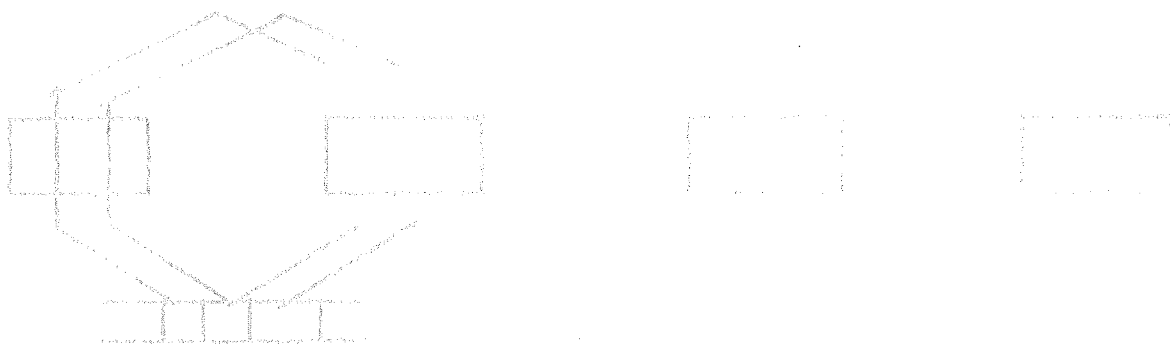
Assume the armature to be rotating clockwise. The direction of the generated voltage in the conductors is as shown by the arrows. Start with commutator segment 3 and connect it to coil side 1. The current will enter segment 3 from the line through brush #1. The other end of coil side 1, which is under a north pole, is connected to the back of coil side 8 and this to commutator segment 4. Segment 4 is connected to coil side 3 and this to coil side 10, then to commutator segment 5, to coil side 5, to coil side 12 and to commutator segment 6, in which there is brush #2, and current flows out to the line. Between commutator segments 3-6 (or brushes 1 and 2) there is a voltage equal to 6 times the voltage per one coil side.

So far one path for the flow of current is established. From segment 6 we go to coil side 7. Notice the direction of [illegible] Continue following the circuit until commutator segment 9 and brush (#3) are reached. Current flows [illegible] segment 9 (through brush #3) and comes out [illegible]. Thus far half the coil sides have been used and [illegible] established. Brushes 1 and 3 are connected [illegible].

Continuing to follow the circuit [illegible] 12 we establish another path for the flow of current [illegible] connected to brush 2. Commutator segment [illegible] 4, to coil side 21, to commutator segment [illegible] to commutator segment 12. From commutator segment [illegible] coil side 6, to coil side 23, to commutator segment [illegible] parallel path.

In the lap winding a conductor under one pole [illegible] to a conductor which occupies almost a corresponding position under the next pole. The second conductor is connected back to the [illegible] the slot next to the original one under the first pole and so on.

It does not make any difference, insofar as [illegible] the second conductor, instead of being connected back to a conductor under the original pole, is connected to a conductor under the next pole and so on, as shown in Fig. 6-2b. After going once around the



(a) Lap Winding

(b) Wave Winding

Fig. 6-2 Drum Windings

armature the end connection comes to a [illegible] the from which we started. When the winding [illegible] is called "Wave Winding."

Assume a 4 pole 17 slots, 2 coil sides per slot machine [illegible] commutator segments. Starting with commutator segment [illegible] 9, go to coil side 18, then 33, 34 and come back to commutator segment 17 adjacent

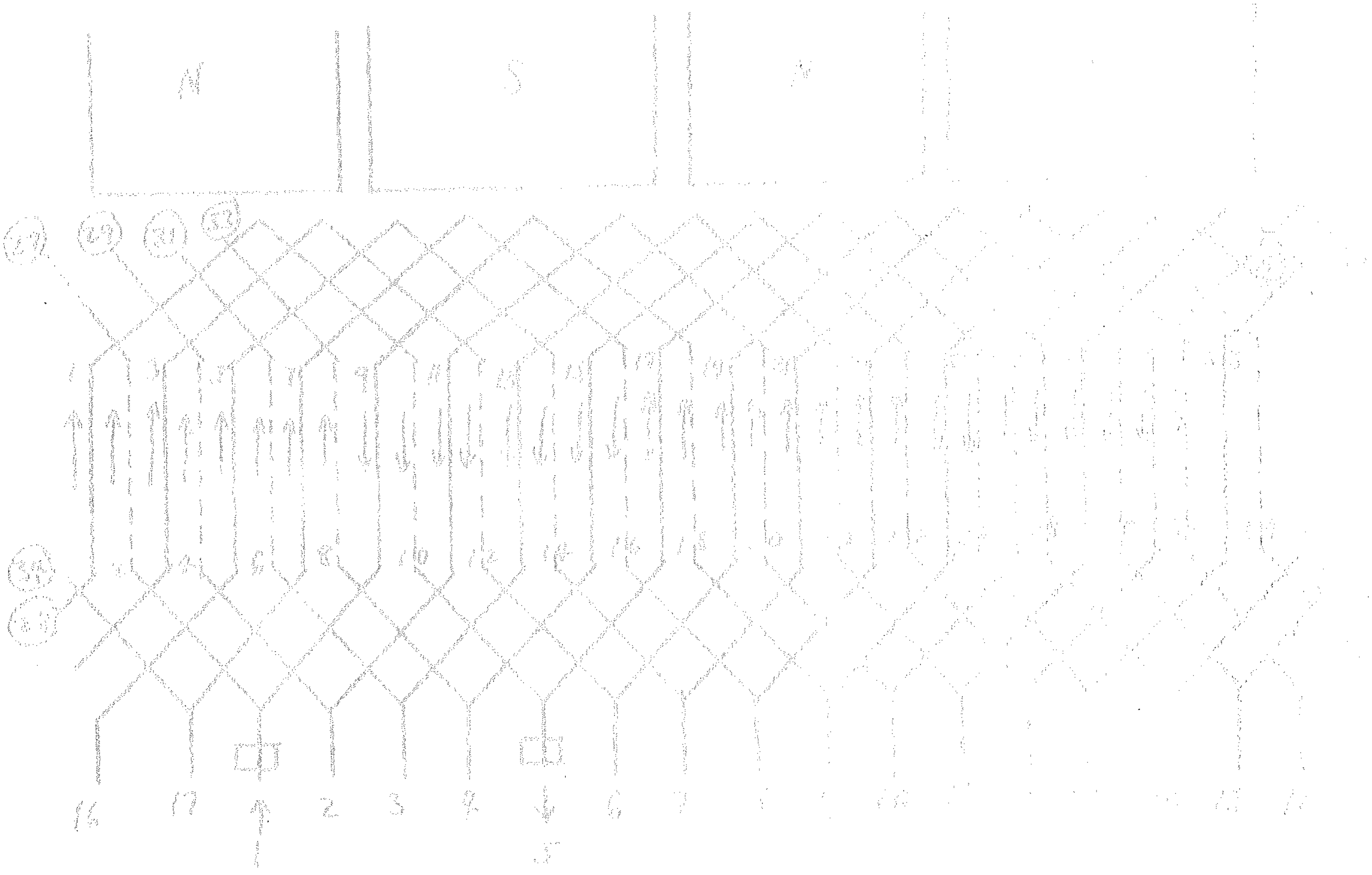


Fig. [illegible] Wave Winding

to segment 1. Follow coil side 7 to 15, [illegible] 17, [illegible] 19, 28, 1, 10 to commutator segment 5 [illegible] the direction of the voltage in all [illegible] 1 to 5 is additive. We also notice that so far we [illegible] coil sides on the machine. We also notice that [illegible] side 7 is towards the brush in commutator segment [illegible] have already formed one path for the flow of current [illegible] and are beginning to form another path for [illegible] As we follow the circuit through the [illegible] half of the coil sides we come back to the brush on commutator segment 1, thus forming another path between the brushes on segments 1 and 5. We notice that [illegible] machine we have two parallel paths, only, between the brushes.

As the armature turns one segment, the brushes [illegible] still have two parallel paths between them.

For a constant speed and constant polarity of the poles we get constant d.c. voltage between the stationary brushes.

We see from the preceding discussion that the action of the commutator on either type of drum winding is to [illegible] current sheet which is fixed in space even though the [illegible] conductors that are carrying the current rotate.

In addition, the position of the brushes [illegible] field poles determines the orientation of this [illegible] to the stator poles. We have calculated [illegible] field distribution of a current sheet which [illegible] except for a change in sign, around the periphery [illegible] (Figure 2-10). Then in Appendix 3C we showed that [illegible] the rotor and stator current sheets depend [illegible] the armature (rotor) axis and the quadrature axis [illegible]

Clearly, in Figures 6-1 or 6-3 the position of the brushes determines the direction of the current sheet relative to the stator poles. Upon comparing Figures 6-1 and 6-3 with Figure 3C [illegible] arrangement of the brushes in the former figures [illegible] axis which lies along the field quadrature axis. [illegible] an α equal to zero in Figure 3C-1, and, according to Figure 3C-2, produces a minimum mutual inductance between rotor and stator, but a maximum space-rate-of-change (slope) of mutual inductance.

If, in either Figures 6-1 or 6-3, we were to move the brushes in either direction 90 electrical degrees [illegible] then the rotor current sheet axis (what we have [illegible] the armature axis) lies along the field direct axis [illegible] This corresponds to an α = 90° in Figure 3C-1 and [illegible] maximum magnitude) value of armature-field mutual [illegible] slope, as is indicated in Figure 3C-2.

In the former case, the brushes [illegible] current, whereas in the latter direct [illegible] brush current is a combination of both [illegible] currents.

This is [illegible] equivalence [illegible] axis current. In the [illegible] of the [illegible] equation (5-43) we found it useful to define direct and quadrature axis currents similarly to the above.

By virtue of Figure 3D-1 we see that quadrature axis [illegible] produces maximum torque on the rotor [illegible] when the rotor carries direct axis current, $\delta = 90°$ and the torque vanishes. Hence in the conventional mode of operation of a d.c. machine [illegible] are arranged as shown in Figures 6-1 and 6-2 so that [illegible] is in the quadrature axis.

6-2. Terminal Equations of the d.c. Machine

Consider the one-field, two winding [illegible] of Figure 6-4.

Figure 6-4 Schematic illustration of a [illegible] with two windings on the rotor, displaced [illegible] degrees from each other, and a fixed field coil. Also shown are the direct and quadrature axes.

Let P be the number of pairs of poles on the field and α_m the mechanical angle turned through by the rotor [illegible] Figure 6-4.

Consistent with the model of a d.c. machine [illegible] studied in section 2-8 and Appendices 3B-3D, we take the self inductances of the two rotor coils, L_{aa}, L_{bb}, to be constant with [illegible] α_m (see Appendix 3B) and equal. L_{ff}, the self inductance of the field coil, is also constant (which means that there is no [illegible]). We also put the resistances of the two rotor windings equal to each other $R_{aa} = R_{bb} = R_a$.

As for the mutual inductances, that between the rotor coils vanishes, but those between the rotor and field windings will be taken to vary sinusoidally

(6-1) (a) $L_{af} = M \cos P\alpha_m$

(b) $L_{bf} = M \sin P\alpha_m$

The minus sign simply refers to the [illegible]

Referring to Figure 3C-7, we see that [illegible] our axis in that figure to the left 90 [illegible] angles of rotation from the new axis. [illegible] ductance between coils a and f is a maximum [illegible] which corresponds to the coil a axis being along the [illegible]

In addition to the above described [illegible] Figure 3C-2, (6-1) implies that the curve for $L_{af}(\theta)$ in Figure [illegible] by the <u>fundamental harmonic</u> in its Fourier series [illegible]. Upon retaining only the fundamental harmonic, (6-1) results:

The equations of the system are

(6-2)

(a) $v_f = i_f R_f + d\lambda_f/dt$

(b) $v_a = i_a R_a + d\lambda_a/dt$

(c) $v_b = i_b R_b + d\lambda_b/dt$ [illegible]

We have, from Figure 6-4:

(a) $\lambda_f = L_{ff} i_f + M (i_a \cos\theta_m$ [illegible]

(b) $\lambda_a = L_{aa} i_a + M i_f \cos\theta_m$

(6-3)

(c) $\lambda_b = L_{bb} i_b - M i_f \sin\theta_m$ [illegible]

Note that $i_a \cos\theta_m - i_b \sin\theta_m$ is the [illegible] of the currents i_a and i_b in the field or direct axis [illegible]. Hence we define a direct axis current i_d by:

(6-4) $i_d = i_a \cos\theta_m - i_b \sin\theta_m$

A study of Figure 6-4 shows that i_a and i_b may [illegible] along the quadrature axis, as well, so as to produce a quadrature axis current.

Thus

(6-5) $i_q = -(i_a \sin\theta_m + i_b \cos\theta_m)$

Similarly,

(a) $\lambda_d = \lambda_a \cos\theta_m - \lambda_b \sin\theta_m$

(6-6)

(b) $\lambda_q = -(\lambda_a \sin\theta_m + \lambda_b \cos\theta_m)$

(c) $v_d = v_a \cos\theta_m - v_b \sin\theta_m$

(d) $v_q = -(v_a \sin\theta_m + v_b \cos\theta_m)$.

We wish to derive expressions [illegible]

(6-7) $\psi_d = $ [illegible]

$= $ [illegible]

$= $ [illegible]

$= $ [illegible]

where $\omega_m = \dfrac{d\alpha_m}{dt}$.

A virtually identical process yields

(6-8) $v_q = $ [illegible]

Finally

(6-9) $v_f = $ [illegible]

because [illegible]

Because the equation for V_f involves [illegible] it is complete. What we must now do [illegible] pressions in (6-7) and (6-8). From [illegible]

(6-10) $\lambda_q = $ [illegible]

$= $ [illegible]

$= $ [illegible]

Thus, i_f does not produce quadrature [illegible] all along, because the quadrature axis is [illegible] axis.

Also

(6-11) $\lambda_d = $ [illegible]

$= $ [illegible]

$= $ [illegible]

Upon substituting (6-10) and [illegible]

(6-12) $v_d = $ [illegible]

(6-13) $v_q = $ [illegible]

(6-14) $v_f = $ [illegible]

These are the basic equations [illegible] (commutator) machines. Usually, the [illegible] of two brushes per pair of poles and [illegible] $i_d = 0$. Hence, the appropriate equations [illegible]

(6-15) $$v_q = i_q R_a + L_{aa} \, di_q/dt + [illegible]$$

(6-16) $$v_f = i_f R_f + L_{ff} \, di_f/dt$$

Multiply (6-15) by i_q

(6-17) $$v_q i_q = i_q^2 R_a + \frac{L_{aa}}{2} \frac{d i_q^2}{dt} + p\omega_m M i_f i_q$$

The left hand term is the electrical power delivered to the armature. The first two terms on the right represent, respectively, heat loss in the armature and the rate at which energy is being stored in the magnetic field of the armature self-inductance. The last term represents the power converted to mechanical power.

(6-18) $$p\omega_m M i_q i_f = T_d \omega_m$$

or, the developed torque is given by

(6-19) $$T_d = p M i_q i_f$$
$$= K_t i_f i_q$$

where K_t, the torque constant, has dimensions of [illegible] (recall that torque $= i_1 i_2 \, \partial L_{12}/\partial\theta$). Note that this torque is [illegible] with respect to θ or θ_m because M is constant. Hence, the developed torque is constant with respect to position of the rotor. [illegible] that (6-19) is consistent with our [illegible] in Appendix [illegible].

Next, consider what happens if we change the position of the brushes so that $i_q = 0$. Then the appropriate equations are

(6-20) $$v_d = i_d R_a + L_{aa} \, di_d/dt + [illegible] \, di_f/dt$$
$$v_f = i_f R_f + L_{ff} \, di_f/dt + [illegible] \, di_d/dt$$

Because the equation for v_d does not [illegible] (the [illegible] variable) there can be no developed motion [illegible] developed torque. As a matter of fact (6-20) [illegible] two stationary, coupled coils. Thus, if the coils [illegible] that the resulting armature flux linkages [illegible] the field flux we get a reduction in [illegible]

A final point to make is that [illegible] for V_f, V_d, V_q and I_d, I_f, I_q are differential equations with constant coefficients, whereas those for V_a, [illegible] have [illegible] coefficients involving $\cos P\alpha_m(t)$ or $\sin P\alpha_m(t)$.

Before leaving our fundamental [illegible] (6-[illegible]) – (6-[illegible]), let us re-examine the energy inputs. The total [illegible] input into the direct and quadrature axes, as well as the field, is given by $v_d i_d + v_q i_q + v_f i_f$, which by virtue of the fundamental equations becomes

$$(6\text{-}21) \qquad P_{in}(t) = [illegible] + i_d^2 R_a + i_q^2 R_a + i_f^2 R_f$$
$$+ \frac{d}{dt}\left(\frac{1}{2} L_{aa} i_d^2 + \frac{1}{2} L_{aa} i_q^2 + [illegible]\right) + \frac{d}{dt}\left([illegible]\right)$$
$$+ \omega M i_q i_f$$

The first three terms are the [illegible] is the rate of change of stored energy [illegible] third is the rate of change of [illegible] axis-field system. Finally, the last [illegible] power, given by the product of [illegible] (equation (6-18)).

6-3. Equivalent Circuit Models for the [illegible]

We shall develop equivalent circuits [illegible] the same lines as the development in Chapter [illegible] given in (6-19), of a motor is equal to [illegible] friction and windage, the torque to accelerate [illegible] mechanical torque supplied by the shaft. [illegible] or load torque. Thus,

$$(6\text{-}22) \qquad T_d = K_f i_f i_q = J \frac{d\omega_m}{dt} + [illegible]$$

where J is the rotor moment of inertia [illegible] T_m [illegible] input to the shaft. For simplicity [illegible] friction and windage torque by a [illegible]

$$(6\text{-}23) \qquad T_{friction} = D\omega_m$$

where D is the viscous friction coefficient. [illegible]

$$(6\text{-}24) \qquad K_f i_f i_q = J \frac{d\omega_m}{dt} + D\omega_m [illegible]$$

[illegible]

$$
(6\text{-}25)\quad
\begin{cases}
(a)\quad v_q = i_q R_a + L_{aa}\, di_q/dt \\
(b)\quad v_f = i_f R_f + L_{ff}\, di_f/dt \\
(c)\quad K_f\, i_f\, i_q + T_m = J\dfrac{d\omega_m}{dt} + \ldots
\end{cases}
$$

Note that K_f [illegible] has been used [illegible]. The coupled system of equations, (6-25), has [illegible] representation shown in Figure 6-5:

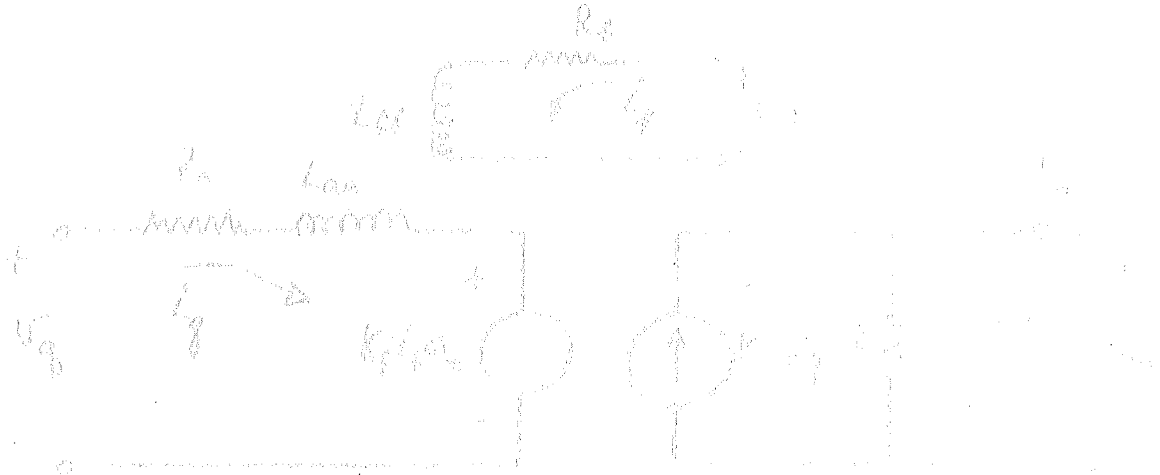


Fig. 6-5. A three-port electrical equivalent [illegible] the electromechanical [illegible]

We represent $K_f\, i_f\, \omega_m$ as a controlled [illegible] emf [illegible] a controlled torque source. [illegible] torques are analogous to currents and [illegible] voltages [illegible] sources may be replaced by an "ideal [illegible]" whose "turns ratio" is given by $K_f i_f$ [illegible]. The [illegible] is obvious—$K_f i_f$ is not dimensionless, but [illegible] per angle, and therefore, can hardly be [illegible] proceed, nevertheless, to obtain an [illegible]

Using the ideal transformer, Figure [illegible] that the "turns ratio" is controlled by the [illegible]

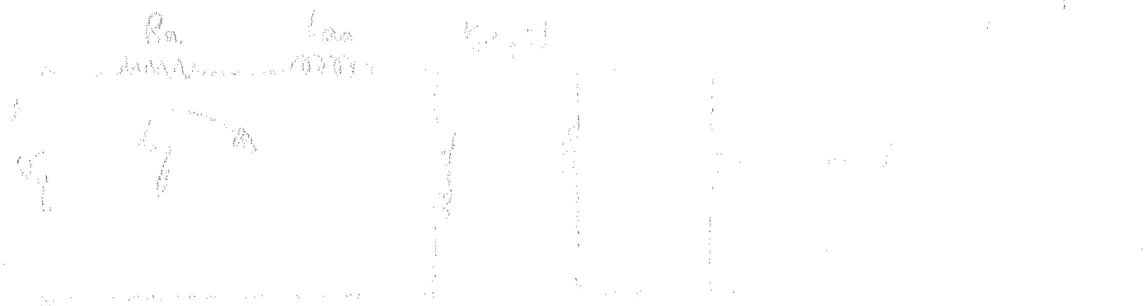


Fig. 6-6. [illegible] sources replaced by an "ideal transformer" [illegible]

[illegible] [illegible] factor of turns ratio N [illegible] equivalent primary impedance is $N^2 Z_L$.

In Figure 6-6, the load impedance [illegible]

(6-26) $Z_L = sT \ldots D$ [illegible]

where s is the Laplace transform complex [illegible] "reflected" (equivalent) load impedance [illegible] electrical side, is

(6-27) $Z_L' = N^2 Z_L$ [illegible] $\frac{(K_f l_f)^2}{sT + D}$

The equivalent reflected admittance [illegible]

(6-28) $Y_L' = \frac{sT}{(K_f l_f)^2} + \frac{D}{(K_f l_f)^2}$

Upon making use of the additional [illegible] are transformed to NV_L and currents [illegible] primary, Figure 6-6, together with (6-28) [illegible] electric circuit of Figure 6-7.

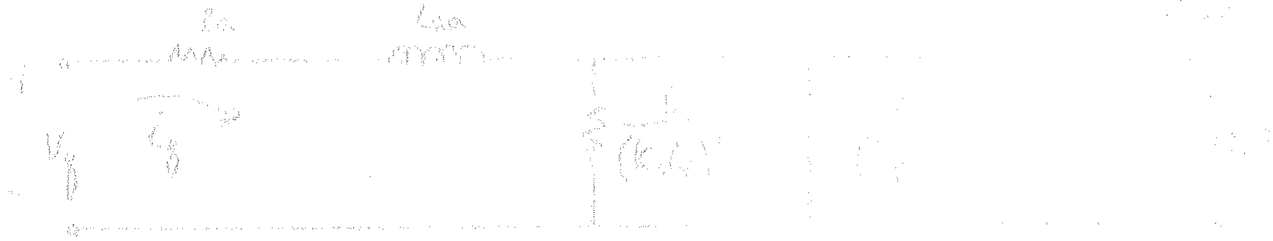


Figure 6-7 An all-electric equivalent [illegible] hybrid model [illegible] of Figure 6-6.

Figure 6-7 should be compared with Figure [illegible] be drawn from such a comparison is that [illegible] electromechanical transducer, so that [illegible] analyzed in chapter 4 is one class of [illegible] identical.

If we refer the electrical side [illegible] ary (or mechanical) side, we get the [illegible] mechanical system. This is shown in [illegible]

Figure 6-8 The [illegible] side. Note that [illegible] torque [illegible]

a) If $\dfrac{V_g}{K_f i_f} > \omega_m$ the machine [illegible]

and T_m is negative, implying a torque output [illegible] ω_m gives the mechanical power output.

b) If $\dfrac{V_g}{K_f i_f} < \omega_m$ the machine [illegible]

i_g reverses direction and there will be [illegible] mechanical power input.

c) If the machine is running as an unloaded [illegible] the [illegible] torque ($T_m = 0$), it still draws a line current [illegible] torque to overcome friction will be supplied [illegible] is being accelerated, there is an additional [illegible]

d) If the machine is running as an unloaded [illegible] an input torque is required to supply the [illegible] again.

Under all conditions notice that the field [illegible] factor k_f gives the ratio of transformation of the [illegible] of Figure 6-6. In an ideal transformer, the power [illegible] equal to power output, regardless of the [illegible] too, the power input to the controlled [illegible] put. We thus see that the energy-conversion [illegible] is likened to a conservative transducer [illegible] D and R_a. This result holds also for the [illegible] we conclude that the field current does not [illegible] machine.

6-4. A Block Diagram for the d.c. Mach[illegible]

It is often useful in systems [illegible] of equations (6-25) by a block diagram. [illegible] manner those aspects of the [illegible] analysis problem.

If we assume that the [illegible] a block diagram describing (6-25)(a) and [illegible]

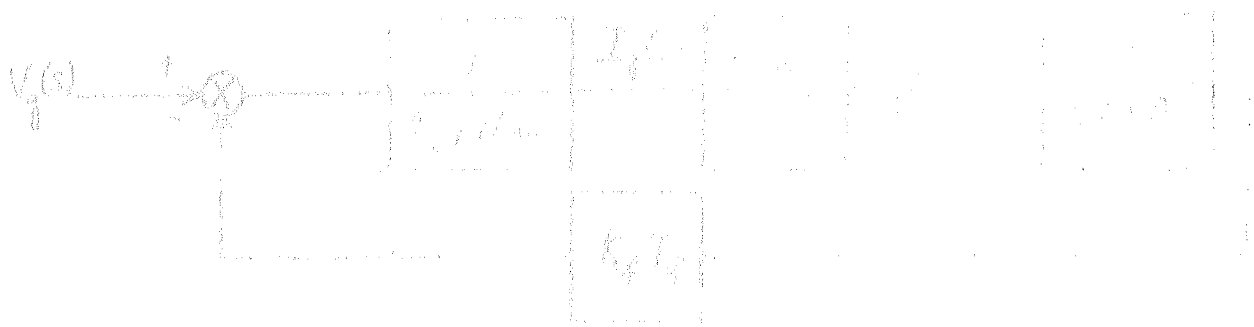


Figure 6-9. A block diagram involv[illegible] ables, equivalent to (6-25). For [illegible] also equivalent to Figure 6-7.

6-5. Transient and Steady State Analyses

We may use either of Figures 6-7 or [illegible] of the machine. Let us use the equivalent [illegible] Figure 6-7. Using Kirchoff's current [illegible] we obtain

$$(6\text{-}29) \qquad K_g I_f \, \Omega_m(s)\left(\frac{sJ}{(K_g I_f)^2} + \frac{D}{(K_g I_f)^2}\right) [illegible]$$

where $\Omega_m(s)$ [illegible]

In the problems you will be asked to [illegible] from Figure 6-9. Algebraic rearrangement [illegible]

$$(6\text{-}30) \qquad \Omega_m(s) = V_f(s)\left[\frac{K_g I_f}{sJ L_{am}(s + \ldots)}\right] [illegible] + T_L(s)\left[\frac{R_a + s L_{am}}{J L_{am}(s^2 + \ldots)}\right] [illegible]$$

In obtaining (6-30) from Figure 6-7, we [illegible] that there were no electro-mechanical [illegible]. That [illegible] Figure 6-7 we assumed all capacitors and inductors [illegible] uncharged. We shall illustrate the [illegible] as follow.

Suppose now that we let the armature voltage be [illegible] $T_m(t) = -T_L$, where T_L is a constant torque. [illegible] a torque output, or motor action. Under [illegible]

(6-31) $$V_g(s) = \frac{V}{s}, \quad T_m(s) = -\frac{T_L}{s}$$

Then, (6-30), becomes

(6-32) $$\Omega_m(s) = \frac{V}{s}\left[\frac{k_f I_f}{J L_a\left(s^2 + \left(\frac{D}{J} + \frac{R_a}{L_a}\right)s + \frac{DR_a + (k_f I_f)^2}{J L_a}\right)}\right] - \frac{T_L}{s}\left[\frac{R_a + s L_a}{J L_a\left(s^2 + \left(\frac{D}{J} + \frac{R_a}{L_a}\right)s + \frac{DR_a + (k_f I_f)^2}{J L_a}\right)}\right]$$

In order to find the steady-state response, we [illegible] by S and let $S \to 0$ in the resulting expression. [illegible] final value theorem of the Laplace transform,[1] [illegible]

(6-33) $$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s),$$

where $F(s) = \mathcal{L}[f(t)]$,

and conclude that the final, or steady-state [illegible] under steady armature excitation, v_g, and load torque, [illegible]

(6-34) $$\omega_m = \frac{V k_f I_f}{DR_a + (k_f I_f)^2} - \frac{T_L R_a}{DR_a + \text{[illegible]}}$$

(1) Cooper and McGillem, p. 164

Upon taking D=0, (6-34) yields the d.c. torque-speed curve of Figure 6-10, for a separately excited machine [illegible]

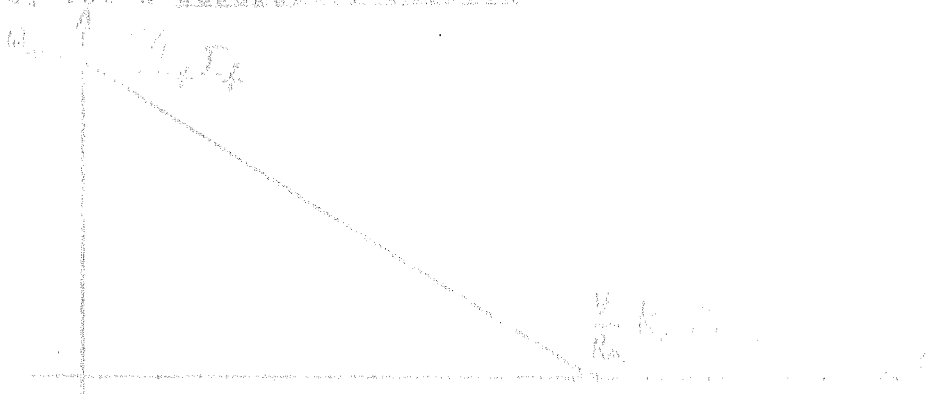


Figure 6-10. Torque (T_L) vs. speed [illegible] for a [illegible] excited machine in the steady-state. T_L and V are d.c. values of load torque and armature voltage, respectively.

We shall return to a fuller discussion of speed control and torque-speed curves, but first let us illustrate [illegible] of Figure 6-7 to transient analysis of a d.c. machine.

Example 1:

A 20 H.P., [illegible] volt, 600 RPM d.c. shunt motor [illegible] constants: R_f = 115, L_{ff} = 50h, R_a = 0.133 [illegible] K_f = 1.76. The motor is running at no load [illegible] when the rated torque is applied to the shaft.

Find

(a) The current and speed as a function of [illegible]
(b) When the motor was running at no load [illegible] J of 1.5 kg-m² is applied to the rotor [illegible] clutch. Find the current and speed of [illegible]

Solution:

(a) With the field and armature circuits [illegible] 230 volt, d.c. supply, we have I_f = 2 amperes [illegible] any transients), and $K_f I_f$ = 3.52. Hence [illegible] circuit (Figure 6-7) is, with T_m = 0, [illegible]

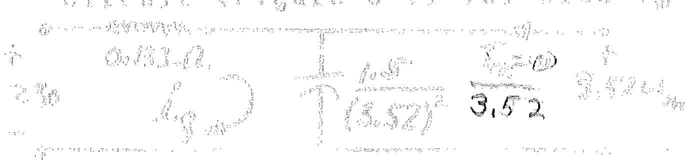


Figure 6-11

Therefore [illegible] K_f [illegible] and [illegible] Because [illegible] 600 RPM and [illegible] the rated torque is found to be $T_{rated} = \frac{(20)(746)}{62.8}$ [illegible]

(1) By separately excited we mean that the field [illegible] from the armature. Hence, in the [illegible]

A shunt motor has its field connected [illegible]

When rated torque is applied in the [illegible] 157.6 N·m, the equivalent circuit becomes [illegible]

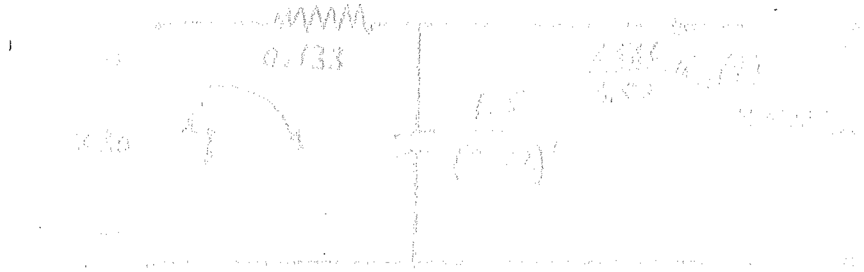


Figure 6.[illegible]

The equation governing Figure 6.12 is derived [illegible] current law:

(1) $$\left(\frac{1.5}{(3.72)^2}\right)\frac{d\omega_m}{dt} = \frac{230 - 3.72\omega_m}{0.133} - \left(\frac{157.6}{3.72}\right)u_{-1}(t)$$

or

(2) $$\frac{d\omega_m}{dt} + 61.5\,\omega_m = 3860\,u_{-1}(t)$$ [illegible]

The solution of (2) is

(a) $$\omega_m(t) = 62.7 + 1.6\,e^{-61.5t}$$

(3)

(b) $$i(t) = i_g + i_p = \frac{230 - 3.72\omega_m}{0.133} \; [illegible] \; e^{-61.5t}$$

(c) If the additional mass is stationary when added, [illegible] is equivalent to adding an additional uncharged capacitor [illegible]. The equivalent circuit is shown in Figure 6.[illegible]

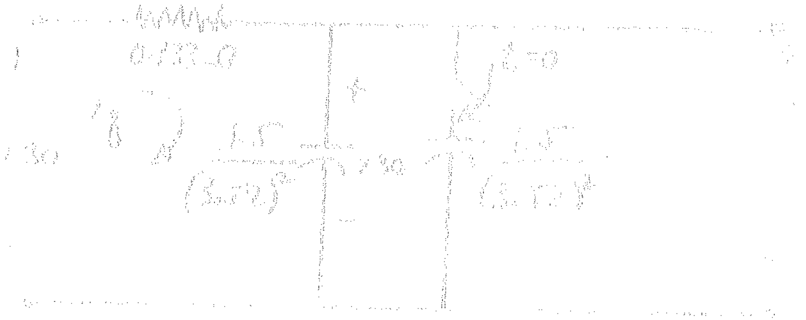
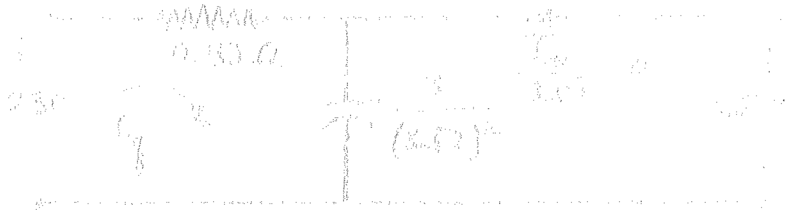


Figure 6.[illegible]

The initially charged capacitor [illegible] charge to the initially uncharged capacitor [illegible] voltage to instantly [illegible] [illegible] [illegible] ratio is $\frac{115}{3.52} = 32.7$ rad/sec., and the [illegible]



i_q instantly increases to $\frac{230-115}{0.133}$ [illegible]

From Figure 6-14, we see that

(4) $$i_q(t) = 865\,e^{-97t}$$

so that

(5) $$i(t) = i_q(t) + i_p = [illegible]$$

ω_m satisfies the differential equation

(6) $$\frac{d\omega_m}{dt} + 97\,\omega_m = [illegible]$$

whose solution is

(7) $$\omega_m(t) = 65.[illegible]\,e^{-97t}$$

Note that V_a and i_q regain their original [illegible] was expected. Observe that it [illegible] a current impulse) to instantly change [illegible] to add mass from 0 to its new value [illegible]

Example 2.

The motor in the example above [illegible] the steady-state with rated voltage [illegible] machine is suddenly disconnected from the [illegible] resistance of 4 [illegible] ohm [illegible]

[illegible]

(c) The energy dissipated in the armature [illegible] before the machine comes to rest.

(d) the value of resistance to be [illegible] to stop the machine in the shortest time

Solution:

(a) After the switching, the field current remains [illegible], and the equivalent circuit becomes that of Figure [illegible]

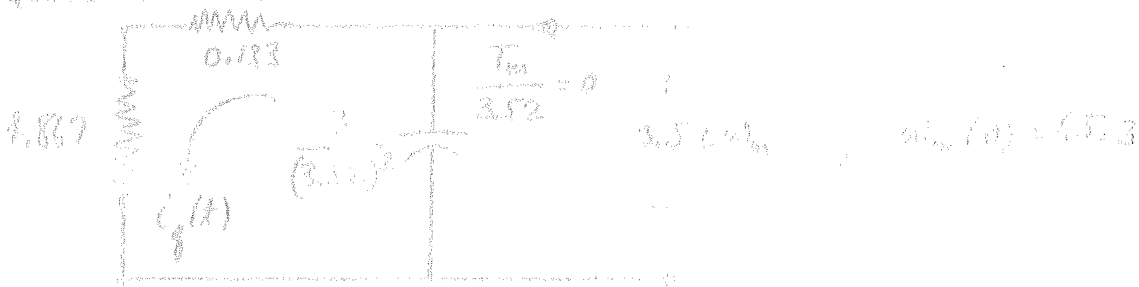


Figure 6-15.

Noting the change of current direction [illegible] machine now acts as a generator), we easily calculate i_g [illegible]

(a) $$i_g(t) = 46\, e^{-0.183t}$$

(b) $$v_{cap}(t) = 5\, i_g(t) = 230\, e^{-0.183t}$$ [illegible]

(c) $$\omega_m(t) = 65.3\, e^{-0.183t}$$

(c) $$W_{diss} = \int_0^\infty i_g^2(t)\, R\, dt = 6380 \text{ Joules}$$ [illegible]

(d) The time constant for discharging the capacitor [illegible]. Reducing the external resistance to 0 (i.e., shortcircuiting the armature terminals) minimizes the braking time. This method of [illegible] the machine is called <u>dynamic braking</u>. We can [illegible] by applying a <u>reversed polarity</u> voltage source across the armature terminals. This latter method is called [illegible].

CHAPTER 15.

15-1. Calculation of Self- and Mutual-Inductances for a Constant Gap, Polyphase Synchronous Machine.

In example 1-3-3, Chapter 1, we calculated the flux density within the airgap of a polyphase synchronous machine with constant gap, utilizing the "idealized current sheet model." Let us extend the results obtained there, (1-114), and determine the self and mutual inductances. We will use the energy-coenergy concepts developed in Chapter 4.

The energy stored per unit axial length within the air gap of the machine in Figure 1-8 is given from field theory by

$$W_m = \frac{1}{2\mu_0}\int_0^{2\pi} d\alpha_2 \int_{R_a}^{R_f} r\,dr\left[b_r^2 + b_\theta^2\right] \tag{15-1}$$

Upon substituing (1-114) into (15-1) and using the definitions of A, B and C given below (1-113) in the above result, we obtain

$$W_m' = \frac{\mu_0\pi}{2}\Big[N^2\left(\frac{1+(R_a/R_f)^{2p}}{1-(R_a/R_f)^{2p}}\right)\left(i_a'^2 + i_b'^2 + i_c'^2 + \left(\frac{N_f}{N}\right)^2 i_f'^2 - i_a' i_b' - i_a' i_c' - i_b' i_c'\right) + \frac{4NN_f}{\left((R_f/R_a)^p + (R_a/R_f)^p\right)}\Big(i_a' i_f' \cos p\alpha_m + i_b' i_f' \cos(p\alpha_m + 120^\circ) + i_c' i_f' \cos(p\alpha_m + 240^\circ)\Big)\Big]. \tag{15-2}$$

In obtaining the final form shown in (15-2), we have introduced the definitions

$$N_f i_f' = J_f R_f, \quad N i_a' = J_a R_a, \quad N i_b' = J_b R_a, \quad N i_c' = J_c R_a, \tag{15-3}$$

where N = no. of turns on the three identical rotor windings and N_f = no. of turns on the stator winding.

According to the theory of Chapter 4, the stored magnetic energy in an electrically linear system is equal to the stored coenergy

$$W_m = W_m' = \frac{1}{2}\Big(L_{aa} i_a^2 + L_{bb} i_b^2 + L_{cc} i_c^2 + L_{ff} i_f^2 + 2L_{ab} i_a i_b + 2L_{ac} i_a i_c + 2L_{bc} i_b i_c + 2L_{af} i_a i_f + 2L_{bf} i_b i_f + 2L_{cf} i_c i_f \Big) \tag{15-4}$$

We recall that (15-4) is the appropriate expression for a multiply excited system. Upon comparing (15-4) with (15-2), we can make the following identifications

$$\text{(a)}\quad L_{aa} = L_{bb} = L_{cc} = \mu_0 N^2 \pi \left(\frac{1 + (R_a/R_f)^{2p}}{1 - (R_a/R_f)^{2p}} \right)$$

$$\text{(15-5)}\quad \text{(b)}\quad L_{ff} = \mu_0 N_f^2 \pi \left(\frac{1 + (R_a/R_f)^{2p}}{1 - (R_a/R_f)^{2p}} \right)$$

$$\text{(c)}\quad L_{ab} = L_{ac} = L_{bc} = -\mu_0 \frac{\pi}{2} N^2 \left(\frac{1 + (R_a/R_f)^{2p}}{1 - (R_a/R_f)^{2p}} \right)$$

$$\text{(d)}\quad L_{af} = \frac{2\mu_0 \pi N N_f}{(R_f/R_a)^p + (R_a/R_f)^p} \cos p\alpha_m$$

$$\text{(e)}\quad L_{bf} = \frac{2\mu_0 \pi N N_f}{(R_f/R_a)^p + (R_a/R_f)^p} \cos (p\alpha_m + 120^\circ)$$

$$\text{(f)}\quad L_{cf} = \frac{2\mu_0 \pi N N_f}{(R_f/R_a)^p + (R_a/R_f)^p} \cos (p\alpha_m + 240^\circ).$$

In particular, we conclude that for a constant gap machine with sinusoidally distributed windings on both the field and rotor, the self inductances are constant and equal (except for the difference between N_f and N), the mutual inductances between the phases are constant with respect to rotor position, α_m, are negative (implying "bucking" polarity when the currents in the coils are positive), and, in magnitude, are one-half the self-inductances. Finally, the mutual inductances between the field and phase windings vary sinusoidally and are 120° out of electrical phase with each other.

CALCULATION OF LEAKAGE FLUX AND LEAKAGE INDUCTANCE.*

Consider the "core" type transformer with sandwiched primary and secondary windings shown in Figure 1.

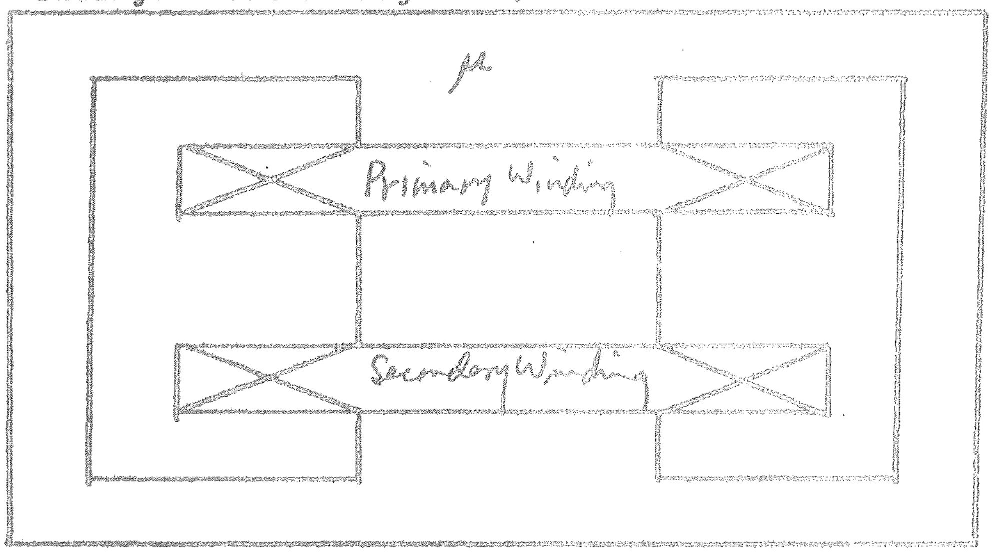


Figure 1. Core type transformer with sandwiched primary and secondary windings.

There are two systems of flux established by the currents in the windings. The first, or main, flux system is that which links the two windings through the closed iron paths. This flux system is called the core or magnetizing flux. Its calculation is easily made by using the usual approximate theory of magnetic circuits and requires knowledge only of the reluctance of the magnetic circuit. Because the circuit lies entirely within the iron (whose permeability, μ, is much greater than μ_0) the reluctance is easily determined in terms of the length and cross section of the iron path.

The leakage flux is the second system alluded to above. The principal path of this flux lies outside the iron and does not link both coils. The calculation of this flux, to which we now turn our attention, requires a more extensive use of electromagnetic field theory. In order that the resulting calculations be as simple as possible, however, we replace Figure 1 by the idealized model of Figure 2.

*See B. Hague, The Principles of Electromagnetism Applied to Electrical Machines, Dover, 1962, Ch. XII.

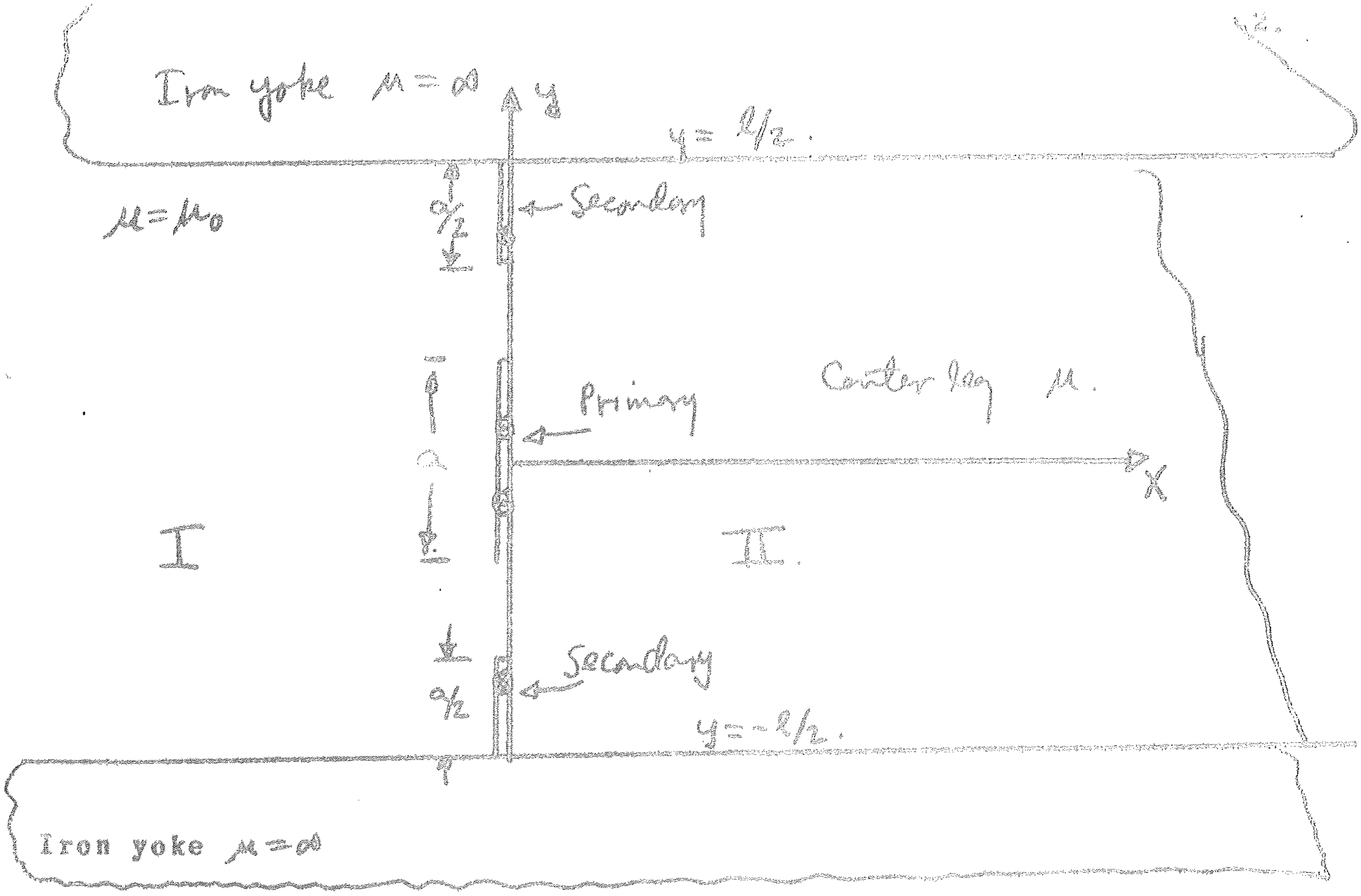


Figure 2. Idealized core transformer with infinitely wide center leg capped by a semi-infinite yoke of $\mu = \infty$. The windings are very fine, flat "current sheets".

In order that the problem be rendered two-dimensional, we assume all dimensions into the page to be infinite. Hence, our results should be applicable to transformers with thick cores (many laminations) or to the median plane of thinner stacked cores.

In addition, the current carry coils of Figure 1 are compressed into current sheets arranged as shown in Figure 2. This latter arrangement increases the symmetry of the model and facilitates the analysis.

Finally, note that the yoke now consists of infinitely permeable iron. This imposes certain conditions on the field variables at $y = \pm l/2$.

The analysis starts with the magnetic field relations

(1) (a) $\nabla \cdot \bar{B} = 0$
(b) $\nabla \times \bar{A} = \bar{B}$
(c) $B = \mu \bar{H}$;

where $\bar{B}$ is the magnetic induction (or flux density), $\bar{A}$ the magnetic vector potential and $\bar{H}$ the magnetic field intensity.

Because all currents lie along the z-axis, the vector potential will consist of a single component, the z-component

(2) $\bar{A} = A\,\bar{a}_z$.

and this component satisfies Laplace's equation in regions I and II:

(3) (a) $\nabla^2 A_I = \frac{\partial^2 A_I}{\partial x^2} + \frac{\partial^2 A_I}{\partial y^2} = 0$ Region I, $x < 0$, $-\ell/2 \le y \le \ell/2$.

(b) $\nabla^2 A_{II} = \frac{\partial^2 A_{II}}{\partial x^2} + \frac{\partial^2 A_{II}}{\partial y^2} = 0$ Region II, $x > 0$, $-\ell/2 \le y \le \ell/2$.

The boundary conditions to be satisfied by A_I and A_{II} are obtained by appealing to (1)(b),(c). The former equation becomes

(4) $$\bar{B} = \nabla \times \bar{A} = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & A \end{bmatrix} = \bar{a}_x \frac{\partial A}{\partial y} - \bar{a}_y \frac{\partial A}{\partial x}.$$

Thus,

(5) $$B_x = \frac{\partial A}{\partial y}, \quad B_y = -\frac{\partial A}{\partial x}.$$

At the interface with infinitely permeable iron, the tangential component of H must vanish. Hence, at $y = \pm \ell/2$, we must have

(6) (a) $H_{x_I} = B_{x_I}/\mu_0 = \frac{1}{\mu_0}\frac{\partial A_I}{\partial y} = 0$ for $x < 0$ (Region I)

(b) $H_{x_{II}} = B_{x_{II}}/\mu = \frac{1}{\mu}\frac{\partial A_{II}}{\partial y} = 0$ for $x > 0$ (Region II).

In either case, (b) becomes

(7) $$\frac{\partial A}{\partial y} = 0 \quad \text{for } y = \pm \ell/2, \ -\infty < x < \infty.$$

At any surfaces containing surface (sheet) currents, the tangential components of $\bar{H}$, as $\bar{H}$ approaches the surface from either side, must be discontinuous by the amount of the surface current at the point on the surface in question. Upon reference to Figure 2 we obtain, therefore,

(8) $$H_{y_{II}} - H_{y_I} = J_s(y) \quad \text{for } x = 0, \ -\ell/2 \le y \le \ell/2,$$

where $J_s(y)$ is the value of the surface current at point y. The order of the terms on the left-hand side of (8) follow upon consideration of the "right-hand" rule for magnetic fields.

Equation 8 can be rewritten

$$(9)\quad (a)\quad -\frac{1}{\mu}\frac{\partial A_{II}}{\partial x} + \frac{1}{\mu_0}\frac{\partial A_I}{\partial x} = J_s(y), \quad \text{for } x=0,\ -\ell/2 \le y \le \ell/2.$$

In addition, the normal components of $\bar{B}$ must be continuous as the boundary, x=0, is approached from Regions I and II. Hence,

$$(9)\quad (b)\quad \frac{\partial A_I}{\partial y} = \frac{\partial A_{II}}{\partial y}, \quad \text{for } x=0,\ -\ell/2 \le y \le \ell/2.$$

{This is actually equivalent to requiring continuity of $\bar{A}$ at $x=0$, i.e., $A_I = A_{II}$

Thus, (7) and (9), together with the vanishing of all fields at $x = \pm\infty$, comprise our boundary values. These boundary values, together with the partial differential equations (3) constitute the boundary-value problem which we will now solve.

Before doing this, however, we should direct our attention to the primary and secondary current sheets which produce $J_s(y)$. The integral of $J_s(y)$ with respect to y gives the total current (or ampere-turns) in that sheet (or coil). In the usual transformer with highly permeable iron, the ampere-turns of the primary are equal but opposite to those of the secondary. Hence, the surface current densities plotted versus y are of opposite sign, but have equal areas under their graphs. If we "continue" the actual currents into their "image space", $y < -\ell/2$ and $y > \ell/2$, as an even, periodic function, we are led to graph $J_s(y)$ as given in Figure 3.

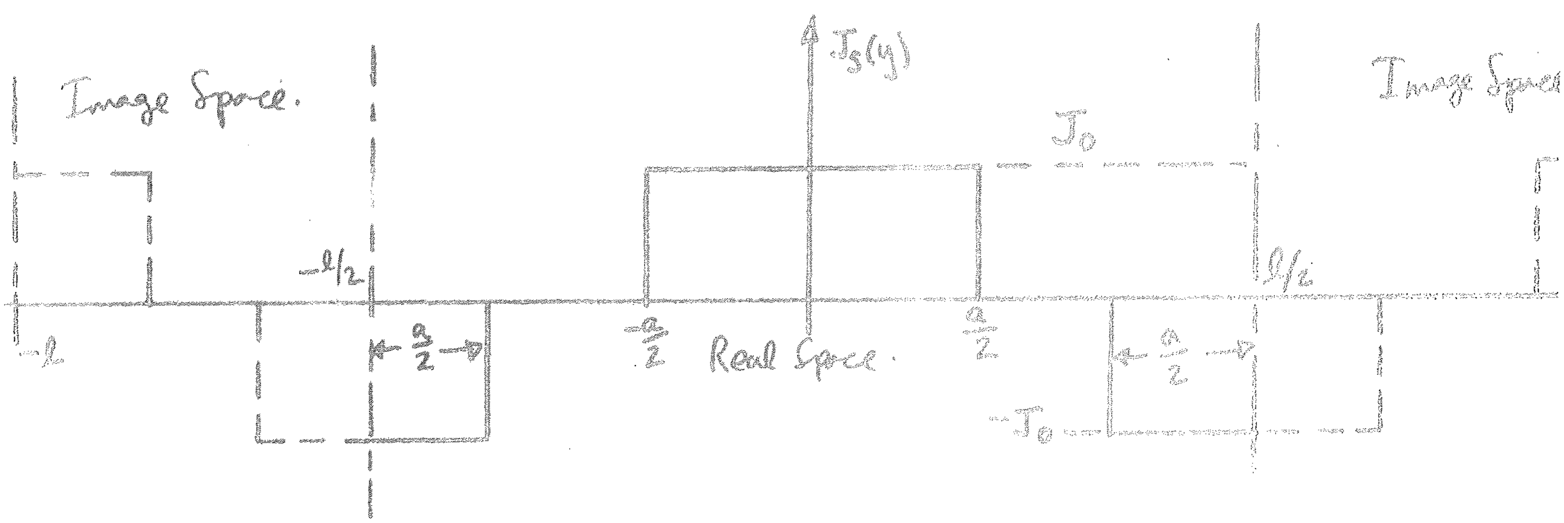


Figure 3. $J_s(y)$ in real and image space.

Because $J_s(y)$ is an even periodic function, of period l, it follows that its Fourier series expansion is

$$(10)\qquad J_s(y) = \sum_{n=1}^{\infty} a_n \cos nky, \quad k = \frac{2\pi}{\ell}.$$

We look for solutions A_I, A_{II}, in the regions I and II, respectively, of the same form as (10). We are led to do this because we must match conditions at $x=0, -\ell/2 \leq y \leq \ell/2$ in accordance with (9). Thus

(11) (a) $$A_I(x,y) = \sum_{n=1}^{\infty} A_n f_n(x) \cos nky$$

(b) $$A_{II}(x,y) = \sum_{n=1}^{\infty} B_n g_n(x) \cos nky.$$

The form of solution in (11) is called a <u>product solution</u>, wherein the variables x, y are <u>separated</u> into functions fn(x), gn(x) and cos n ky.

Let us determine the forms of fn(x), gn(x). Substitute the <u>nth</u> term fn(x) cos n ky into (3) (a)

$$\frac{\partial^2}{\partial x^2}\left(f_n(x) \cos nky\right) + \frac{\partial^2}{\partial y^2}\left(f_n(x) \cos nky\right) = \cos nky \frac{d^2 f_n(x)}{dx^2} - (nk)^2 f_n(x) \cos nky$$

$$= 0.$$

Thus, fn(x) satisfies the <u>ordinary</u> differential equation

(12) $$\frac{d^2 f_n(x)}{dx^2} - (nk)^2 f_n(x) = 0,$$

which has for its two linearly independent solutions e^{nkx}, e^{-nkx}.

Region I, however, extends to $x = -\infty$, and we want all fields to vanish as $x \to -\infty$. This precludes our using e^{-nkx} (recall nk is positive) because $e^{-nkx} \to \infty$ as $x \to -\infty$. We conclude, therefore, that $fn(x) = e^{+nkx}$.

A similar analysis holds for gn(x), except that the term e^{nkx} must be rejected because it explodes as $x \to +\infty$, which is in Region II.

We have, therefore, the results

(13) (a) $$A_I(x,y) = \sum_{n=1}^{\infty} A_n e^{nkx} \cos nky, \qquad -\infty < x \leq 0.$$

(b) $$A_{II}(x,y) = \sum_{n=1}^{\infty} B_n e^{-nkx} \cos nky \qquad 0 \leq x < \infty.$$

It should be noted that boundary condition ([illegible]) is automatically satisfied by our choice of the expansion functions $\{\cos nky\}$.

In order to satisfy (9) (a), we have

$$\left[\sum_{n=1}^{\infty} \frac{nk A_n e^{nkx} \cos nky}{\mu_0} - \sum_{n=1}^{\infty} \frac{-nk B_n e^{-nkx} \cos nky}{\mu} \right] \Bigg|_{x=0} = \sum_{n=1}^{\infty} a_n \cos nky,$$

which simplifies to

$$\sum_{n=1}^{\infty} nk \left(\frac{A_n}{\mu_0} + \frac{B_n}{\mu} \right) \cos nky = \sum_{n=1}^{\infty} a_n \cos nky .$$

Upon equating the nth coefficient of each side, we have

(14) (a) $\frac{nk}{\mu_0} A_n + \frac{nk}{\mu} B_n = a_n$.

Boundary condition (9) (b) applied to (13) yields $\left(-\sum_{n=1}^{\infty} nk A_n \sin nky\right) = \left(-\sum_{n=1}^{\infty} nk B_n \sin nky\right)$ or, upon equating coefficients:

(14) (b) An = Bn.

Equations (14) (a), (b), together, yield

(15) $A_n = B_n = \frac{a_n}{nk} \left(\frac{\mu \mu_0}{\mu + \mu_0} \right)$.

Note that the two permeabilities combine as a "parallel combination of permeabilities".

Hence

(16) (a) $A_I(x,y) = \left(\frac{\mu \mu_0}{\mu + \mu_0} \right) \sum_{n=1}^{\infty} \frac{a_n}{nk} e^{nkx} \cos nky$

(b) $A_{II}(x,y) = \left(\frac{\mu \mu_0}{\mu + \mu_0} \right) \sum_{n=1}^{\infty} \frac{a_n}{nk} e^{-nkx} \cos nky,$

and

(17) (a) $B_{x_I}(x,y) = -\left(\frac{\mu \mu_0}{\mu + \mu_0} \right) \sum_{n=1}^{\infty} a_n e^{nkx} \sin nky$

(b) $B_{y_I}(x,y) = -\left(\frac{\mu \mu_0}{\mu + \mu_0} \right) \sum_{n=1}^{\infty} a_n e^{nkx} \cos nky$

(c) $B_{x_{II}}(x,y) = -\left(\frac{\mu \mu_0}{\mu + \mu_0} \right) \sum_{n=1}^{\infty} a_n e^{-nkx} \sin nky$

(d) $B_{y_{II}}(x,y) = +\left(\frac{\mu \mu_0}{\mu + \mu_0} \right) \sum_{n=1}^{\infty} a_n e^{-nkx} \cos nky$.

$\{a_n\}$, the Fourier coefficients of Figure 3, are given by

(18) $$a_n = \frac{4J_0}{n\pi} \sin n\pi\, a/\ell \quad , \quad n = 1, 3, 5, \ldots \text{ (odd)}$$
$$= 0 \quad , \quad n = 2, 4, 6, \cdots \text{ (even)}.$$

Hence, our final expressions for $\bar{B}$ are given by

(19)

(a) $$B_{x_I}(x,y) = -4J_0\left(\frac{\mu\mu_0}{\mu+\mu_0}\right)\sum_{n=odd}\frac{\sin n\pi(a/\ell)}{n\pi} e^{nkx}\sin nky$$

(b) $$B_{y_I}(x,y) = -4J_0\left(\frac{\mu\mu_0}{\mu+\mu_0}\right)\sum_{n=odd}\frac{\sin n\pi(a/\ell)}{n\pi} e^{nkx}\cos nky$$

(c) $$B_{x_{II}}(x,y) = -4J_0\left(\frac{\mu\mu_0}{\mu+\mu_0}\right)\sum_{n=odd}\frac{\sin n\pi(a/\ell)}{n\pi} e^{-nkx}\sin nky$$

(d) $$B_{y_{II}}(x,y) = +4J_0\left(\frac{\mu\mu_0}{\mu+\mu_0}\right)\sum_{n=odd}\frac{\sin n\pi(a/\ell)}{n\pi} e^{-nkx}\cos nky.$$

We can give a graphical representation to a vector field by plotting "field lines". In our case we are interested in plotting lines of $\bar{B}$. By definition a field line is given as the curve in x-y space which is, at each point of space, tangent to the vector field at that point. This means that the field lines satisfy the differential equation

(20) $$\frac{dy}{dx} = \frac{B_y(x,y)}{B_x(x,y)}$$

The left-hand side of (20) is the tangent to the field line curve and the right-hand side is the slope of the vector field at the point (x,y). By substituting (19) into (20), we can plot the field lines in Regions I and II. The process is broken up into two parts. First we must sum the Fourier series representation for B_x, B_y at each point (x,y), and then solve the resulting differential equation, (20). Both steps usually require machine computation for accuracy.

Now that we have calculated the $\bar{B}$ field in Regions I and II, we can proceed to derive an expression for the leakage inductance by applying the formula for magnetic stored energy, W_m:

(21) $$W_m = \frac{1}{2}L\,i^2 = \frac{1}{2}\iiint\frac{B^2}{\mu}\,dv,$$

where the integral is taken over the volume occupied by the leakage flux. This volume is precisely Regions I and II in Figure 2. Thus, we have for the energy stored in Region I to a depth, Δ, in the z-direction (normal to the page)

$$W_{m_I} = \frac{\Delta}{2\mu_0}\int_{-\infty}^{0} dx \int_{-\ell/2}^{\ell/2} dy \left(B_{x_I}^2 + B_{y_I}^2 \right)$$

(22)

$$= \frac{\Delta}{2\mu_0}\int_{-\infty}^{0} dx \int_{-\ell/2}^{\ell/2} dy \cdot 16 J_0^2 \left(\frac{\mu\mu_0}{\mu+\mu_0}\right)^2 \left\{ \left(\sum_{n=odd}' \frac{\sin n\pi(a/\ell)}{n\pi} e^{nkx} \sin nky \right)^2 + \left(\sum_{n=odd}' \frac{\sin n\pi(a/\ell)}{n\pi} e^{nkx} \cos nky \right)^2 \right\}.$$

$$= 8 J_0^2 \frac{\mu^2\mu_0}{(\mu+\mu_0)^2} \Delta \int_{-\infty}^{0} dx \int_{-\ell/2}^{\ell/2} dy \cdot \sum_{m,n=odd}' \frac{\sin m\pi(a/\ell)}{m\pi} \cdot \frac{\sin n\pi(a/\ell)}{n\pi} e^{mkx} \cdot e^{nkx} \cdot (\sin mky \cdot \sin nky + \cos mky \cos nky).$$

Note that we have written the square of a series as a series involving double summations and double subscripts. For example

$$(a_1+a_2+a_3)^2 = (a_1a_1 + a_1a_2 + a_1a_3 + a_2a_1 + a_2a_2 + a_2a_3 + a_3a_1 + a_3a_2 + a_3a_3)$$

$$= \sum_{m,n=1}^{3}{}' (a_m a_n).$$

The integrals over the trigometric functions can be reduced first

(23)

(a) $$\int_{-\ell/2}^{\ell/2} \cos m\cdot\frac{2\pi}{\ell}\cdot y \cos n\cdot\frac{2\pi}{\ell} y \, dy = 0, \quad m \neq n$$
$$= \frac{\ell}{2}, \quad m = n.$$

(b) $$\int_{-\ell/2}^{\ell/2} \sin m\cdot\frac{2\pi}{\ell}\cdot y \sin n\cdot\frac{2\pi}{\ell} y \, dy = 0, \quad m \neq n$$
$$= \frac{\ell}{2}, \quad m = n.$$

Hence, (22) vanishes when $m \neq n$ (because of (23)), and the double ummation reduces to a single sum, with the result

(24) $$W_{m_I} = 4 J_0^2 \frac{\mu_0 \mu^2 \Delta}{(\mu+\mu_0)^2} \ell \int_{-\infty}^{0} \sum_{n=odd} \left(\frac{\sin n\pi(a/\ell)}{n\pi} \right)^2 e^{2nkx} \, dx$$

$$= \frac{J_0^2 \mu_0 \mu^2 \ell^2 \Delta}{\pi(\mu+\mu_0)^2} \sum_{n=odd} \left(\frac{1}{n}\right) \left(\frac{\sin n\pi(a/\ell)}{n\pi} \right)^2.$$

The stored energy in Region II is obtained by interchanging μ and μ_0 :

(25) $$W_{m_{II}} = \frac{J_0^2 \mu_0^2 \mu \ell^2 \Delta}{\pi(\mu+\mu_0)^2} \sum_{n=odd} \left(\frac{1}{n}\right) \left(\frac{\sin n\pi(a/\ell)}{n\pi} \right)^2.$$

Hence, the total stored energy due to the leakage flux is given by the sum of (24) and (25)

(26) $$W_m = \frac{J_0^2 \ell^2 \Delta}{\pi} \left(\frac{\mu \mu_0}{\mu+\mu_0} \right) \sum_{n=odd} \left(\frac{1}{n}\right) \left(\frac{\sin n\pi(a/\ell)}{n\pi} \right)^2.$$

Now J_o is the number of primary ampere-turns per meter. Hence, (See Fig. 3) $J_o a = N_p i_p = N_s i_s$, where $N_{p,s}$ are, respectively, the total number of primary, secondary turns, and $i_{p,s}$, are, respectively, the primary and secondary currents. The equality of the total ampere turns on the primary and secondary c@ils follows from the assumption of highly permeable iron.

If we refer W_m to the primary current, we have

(27) $$W_m = N_p^2 i_p^2 \frac{\Delta}{\pi} \left(\frac{\mu \mu_0}{\mu+\mu_0} \right) \left(\frac{\ell}{a} \right)^2 \sum_{n=odd} \left(\frac{1}{n}\right) \left(\frac{\sin n\pi(a/\ell)}{n\pi} \right)^2$$

$$= \frac{1}{2} L_p i_p^2.$$

where

(28) $$L_p = 2 N_p^2 \left(\frac{\Delta}{\pi} \right) \left(\frac{\mu \mu_0}{\mu+\mu_0} \right) \left(\frac{\ell}{a} \right)^2 \sum_{n=odd} \left(\frac{1}{n}\right) \left(\frac{\sin n\pi(a/\ell)}{n\pi} \right)^2$$

is the total <u>leakage inductance</u> (primary and secondary) <u>referred to the primary side</u>.

Similarly, the total leakage inductance referred to the secondary side is

(29) $$L_s = 2 N_s^2 \left(\frac{\Delta}{\pi} \right) \left(\frac{\mu \mu_0}{\mu+\mu_0} \right) \left(\frac{\ell}{a} \right)^2 \sum_{n=odd} \left(\frac{1}{n}\right) \left(\frac{\sin n\pi(a/\ell)}{n\pi} \right)^2.$$

We note from (28), (29) that

$$\frac{L_p}{L_s} = \left(\frac{N_p}{N_s}\right)^2, \tag{30}$$

in agreement with the results of the theory of transformer equivalent circuits. Note that L_p and L_s are not the primary and secondary leakage inductances referred to either the primary or secondary sides. Rather,

$$\begin{aligned} L_p &= L_1 + L_2\left(\frac{N_p}{N_s}\right)^2 \\ L_s &= L_2 + L_1\left(\frac{N_s}{N_p}\right)^2, \end{aligned} \tag{31}$$

where L_1 and L_2 are, respectively, the individual leakage inductances of the primary and secondary coils, separately.

... 3C. CALCULATION OF ARMATURE-FIELD MUTUAL INDUCTANCE

... the rotor-stator configuration of section 2-8, imagine two ... windings essentially concentrated at the stator surface R_f. ... "coil sides" of each winding are located at $\theta = -\alpha$, $\pi - \alpha$ ... separated from each other by a thin slit. When the coils are ...ed in series and carry currents in the directions shown, they ... stator "north" and "south" pole. These windings are called *field windings*. The distributed winding located on the surface ... of course, the *armature* winding.

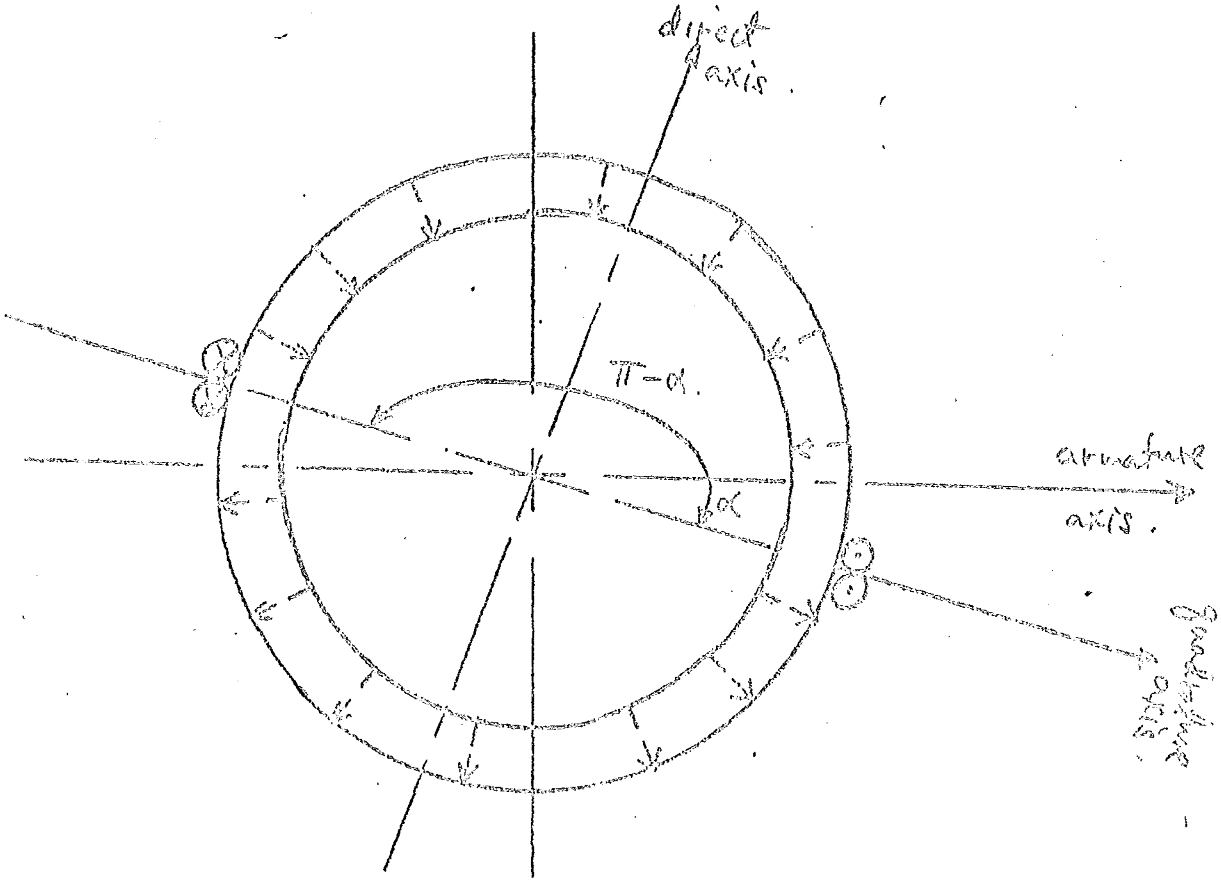


Fig. 3C-1. Idealized Stator Windings.

... flux density $\bar{B}$ established by the armature current sheet, ... radial component $B_r(R_f, \theta)$, at $r = R_f$ links the field windings. ... armature flux linking the *upper field* winding is (per unit length)

$$\lambda_{fa} = N_f \int_{-\alpha}^{\pi-\alpha} B_r(R_f,\theta) R_f \, d\theta = \frac{2 N_f \mu_0 R_a J_s}{\pi R_f} \int_{-\alpha}^{\pi-\alpha} \left(\sum_{n=odd} \frac{1}{n} \frac{\cos n\theta}{\left(\frac{R_a}{R_f}\right)^n - \left(\frac{R_f}{R_a}\right)^n} \right)$$

$$= \frac{16 \mu_0}{\pi} N_f J_s R_a \sum_{n=odd} \frac{1}{n^2} \frac{\sin n\alpha}{\left(\frac{R_a}{R_f}\right)^n - \left(\frac{R_f}{R_a}\right)^n}$$

here N_f is the number of concentrated turns on the upper (and lower) field winding

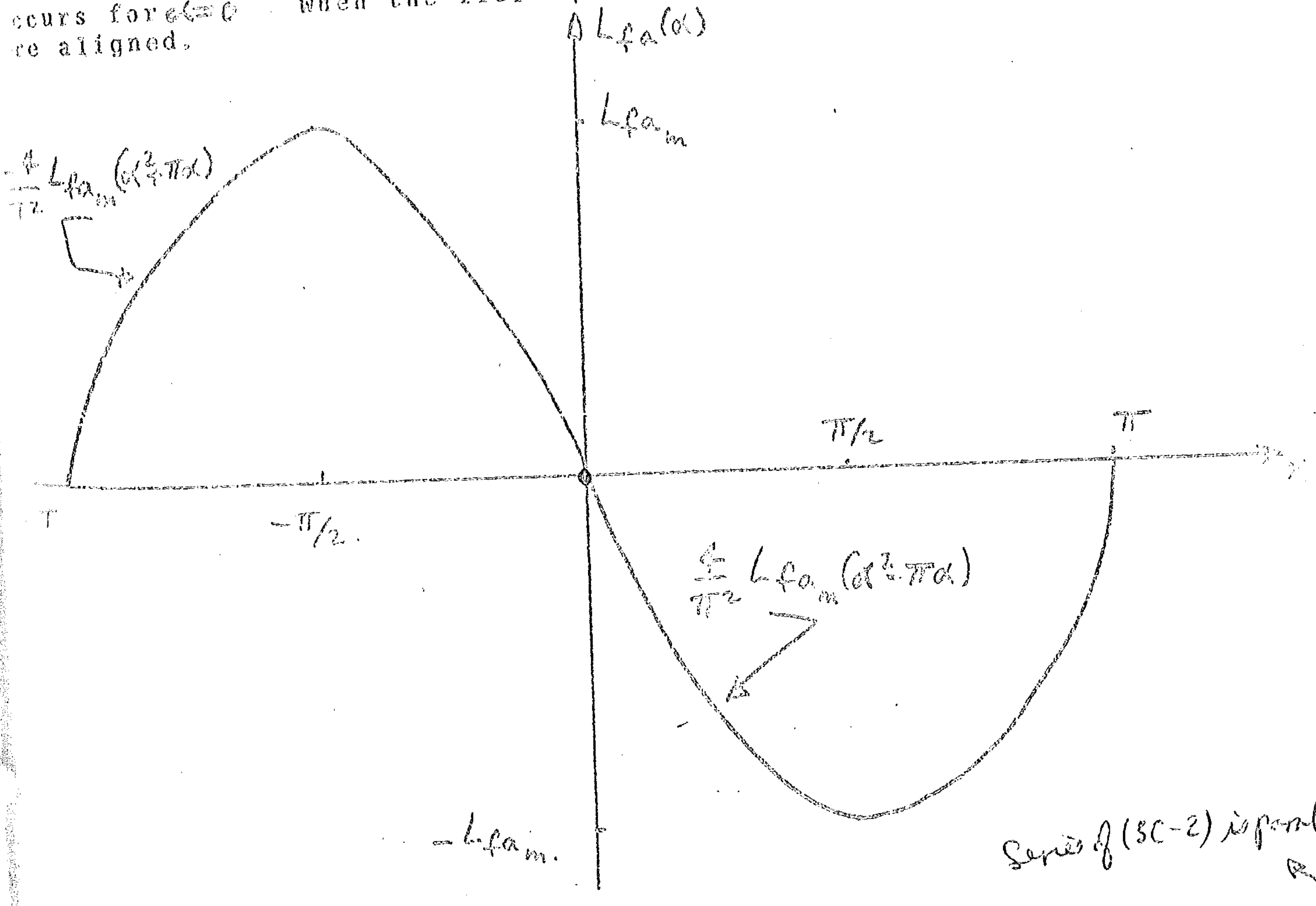
As in Appendix 3B, we replace J_g by $N_a' \, i_a$ and get for the armature-field mutual inductance

$$L_{fa} = \frac{\lambda_{fa}}{i_a} = \frac{\frac{16}{8}\mu_0 N_f (N_a' R_a)}{\pi} \sum_{n=odd} \frac{1}{n^2} \cdot \frac{\sin n\alpha}{\left(\frac{R_a}{R_f}\right)^n - \left(\frac{R_f}{R_a}\right)^n} \qquad \text{(3C-2)}$$

In this case, we note that if $R_f \to \infty$, then $L_{fa} \to 0$ i.e., there is zero coupling between armature and field winding (because the field winding has been removed to infinity).

The important idea to gain from (3C-2) is that the mutual coupling between armature and field depends on the relative orientation of the field and armature axes. We define the armature axis to be along $\Theta = 0$ and the field axis (also called the <u>direct axis</u>) to be along $\Theta = \frac{\pi}{2} - \alpha$ (see Fig. 3C-1). <u>The quadrature axis</u> lies along $\Theta = -\alpha$

Note that maximum coupling occurs for $\alpha = \frac{\pi}{2}$, i.e., when the armature axis is aligned with the field direct axis. Minimum coupling occurs for $\alpha = 0$ when the field quadrature axis and armature axis are aligned.



Series of (3C-2) is parabolic

Fig. 3C-2. $L_{fa}(\alpha)$, armature-field mutual inductance vs α, i.e., the Fourier ... arcs of parabolas

Appendix 3D. CALCULATION OF ARMATURE-FIELD INTERACTION FORCE.

In this appendix, we shall calculate the torque on the rotor (or armature) of the system in Appendix 3C, assuming that the armature and field coils are excited. We shall perform the calculations two ways, using the results of Section 3-11 (equation 3-147, in particular) and using the stress-tensor.

1. Calculation of torque using mutual inductance.

In order to apply (3-147) we must determine the manner of variation of the self and mutual inductances with angle. From (3B-10) it follows that the armature self inductance is independent of the position (angle) of the rotor, while a study of Figure 3C-1 indicates that the field self inductance is independent of α because the symmetry involved precludes any variation of reluctance of the field flux path with α. Thus, this leaves only the mutual inductance term for torque, τ :

$$\tau = i_a i_f \frac{dL_{fa}}{d\alpha} \qquad \text{(3D-1)}$$

$$= \frac{i_a i_f\, 16 \mu_0 N_a N_f}{\pi^2} \cdot \sum_{n=odd} \frac{1}{n} \cdot \frac{\cos n\alpha}{\left(\frac{R_a}{R_f}\right)^n - \left(\frac{R_f}{R_a}\right)^n},$$

where we have written $N_a = \pi R_a N_a'$ for the total number of ~~terms~~ turns per armature pole.

The fourier series in (3D-1) is identical to that plotted in Chapter 2, for $B_r(R_f, \theta)$ (see equation (2-50)), with the result that $\tau(\alpha)$ is given in Figure (3D-1):

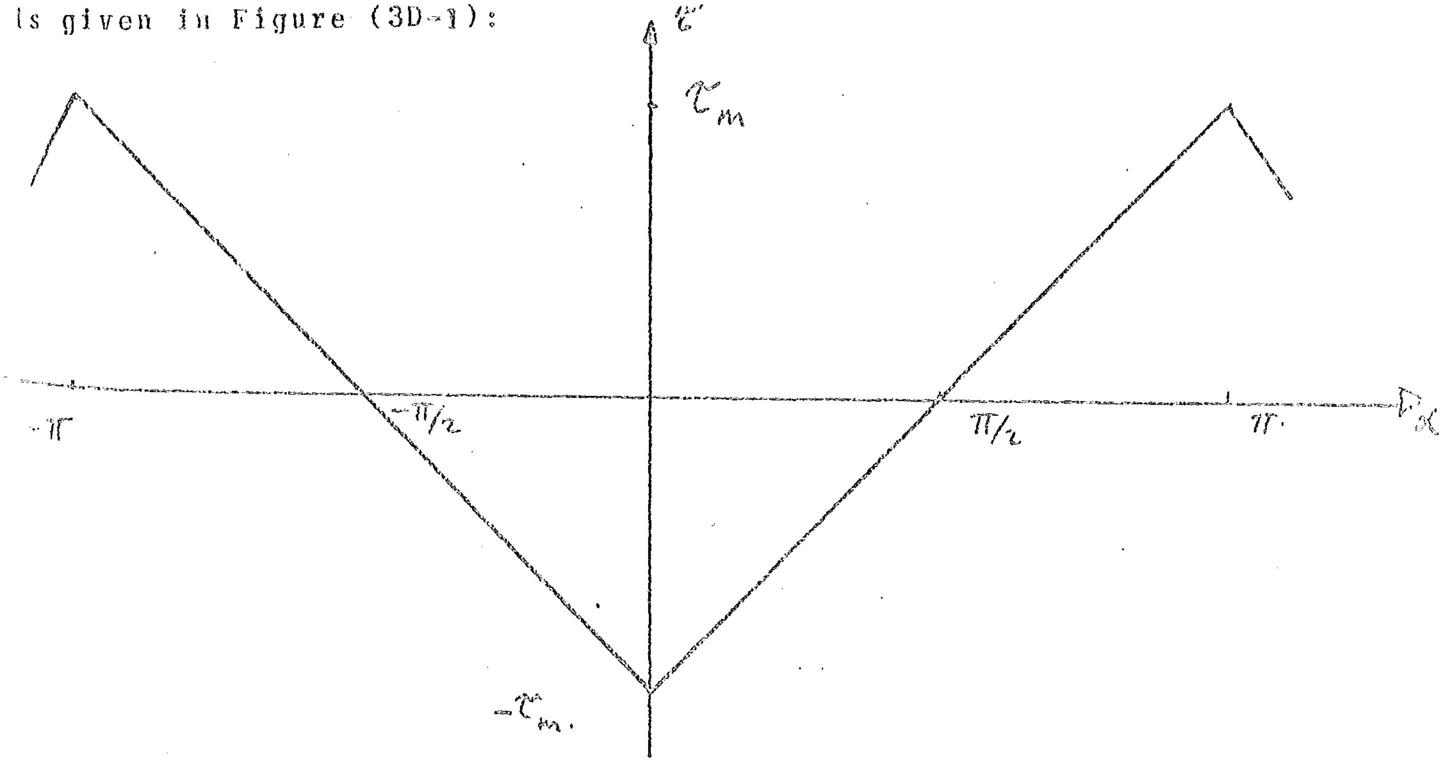


Figure (3D-1). Torque on armature versus α, the angle between armature axis and field quadrature axis. The maximum torque, proportional to the product $i_f i_a N_a N_f$

From this figure we see that when $\alpha = 0$, i.e., when the quadrature of the field is aligned with the armature axis, maximum torque results, when the direct axis of the field is aligned with the armature axis torque vanishes.

Calculation of torque using the stress tensor.

Because there is no air gap flux density in the z-direction (out the page) in Fig. (3C-1), it follows that the magnetic stress tensor $r = R_a$ is given by

$$\bar{\bar{\mathcal{T}}}_M(R_a,\theta) = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix} \tag{3D-2}$$

The net traction (or stress) acting on the armature (whose outward normal is $\bar{a}_R$) is given by

$$\bar{P} = \bar{\bar{\mathcal{T}}}_M(R_a,\theta)\cdot\bar{a}_R = \begin{bmatrix} \frac{B_r^2 - B_\theta^2}{2\mu_0} & \frac{B_r B_\theta}{\mu_0} & 0 \\ \frac{B_r B_\theta}{\mu_0} & \frac{B_\theta^2 - B_r^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B_r^2 + B_\theta^2}{2\mu_0} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3D-3}$$

$$= \left(\frac{B_r^2 - B_\theta^2}{2\mu_0}\right)\bar{a}_R + \frac{B_r B_\theta}{\mu_0}\,\bar{a}_\theta .$$

Upon taking the cross-product of (3D-3) with $R_a\bar{a}_R$, the radius vector from the origin to any point on the armature, we get the torque-per-unit area established by the air gap flux

$$\bar{\tau}' = R_a \bar{a}_R \times \bar{P}$$

$$= \frac{R_a B_r B_\theta}{\mu_0}\,\bar{a}_R \times \bar{a}_\theta \tag{3D-4}$$

$$= \frac{R_a B_r B_\theta}{\mu_0}\,\bar{a}_z .$$

The torque-per-unit length of rotor (armature) is then given by (dropping the unit vector)

$$\tau = \frac{R_a}{\mu_0}\int_0^{2\pi} B_r B_\theta R_a\, d\theta .$$

In order to carry out the integration in (3D-5), we must know the total (armature plus field) contributions to B_r and B_θ.

A common approximation is to say that the field winding (Fig. 3C-1) establishes a purely radial flux density which is uniform at the rotor, and

(3D-6) $$\bar{B}_f(R_a, \theta) = -B_{rf}\,\bar{a}_R \quad , \quad -\alpha < \theta < \pi - \alpha$$
$$= B_{rf}\,\bar{a}_R \quad , \quad \pi - \alpha < \theta < 2\pi - \alpha .$$

The constant value of B_{r_f} is easily determined to be $\dfrac{\mu_0 N_f i_f}{g}$

where g is the gap length, $R_f - R_a$.

The integrand of (3D-5) is written

(3D-7) $$B_r(R_a,\theta)\,B_\theta(R_a,\theta) = (B_{ra} + B_{rf})\,B_{\theta_a}$$

where B_{r_a} and B_{θ_a} are the armature components given by (2-50) and B_{r_f} is given by (3D-6).

It follows immediately from the expressions for B_{r_a} and B_{θ_a} that the integral of $B_{r_a} \cdot B_{\theta_a}$ over $0 \leq \theta \leq 2\pi$ vanishes because the integral of $\sin m\theta \cdot \cos n\theta$ vanishes over that range. Thus, we are left with the integral of $B_{r_f} \cdot B_{\theta_a}$.

From the analysis of section 2-8, we found that

(3D-8) $$B_{\theta_a}(R_a, \theta) = -\mu_0 J_s(\theta)$$

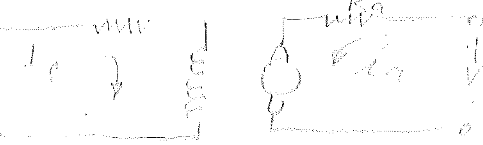
with $J_s(\theta)$ shown in Figure 2-10. Hence, utilizing Figure 2-10 and (3D-6) we sketch $B_{r_f} \cdot B_{\theta_a}$ in Figure 3D-2.

OPTIMUM CONTROL OF MACHINES*

Let us consider the task of optimizing the performance of a d.c. motor which turns a platform of an excavator. The duty of the electric motor is to turn the platform a definite amount, starting at the excavation site and ending at the site where the dirt is to be dumped. By "optimum" we mean "best", in the sense that some aspect of the motor's performance is largest or smallest during the duty cycle.

Let us be more precise and consider the problem of determining the armature current, $i_a(t)$, and angular velocity, $\omega(t)$, which will simultaneously produce the given angular rotation of the platform with least heating of the armature windings. This problem, called the minimum loss problem, will be stated mathematically.

If T is the time of operation of the platform, then the angle turned is given by



(1) $$\alpha = \int_0^T \omega(t)\,dt.$$

The energy dissipated in the armature during this time is

(2) $$Q = \int_0^T i_a^2(t)\,R_a\,dt$$

where $i_a(t)$ is the armature current and Ra the armature resistance.

There exists, of course, a coupling equation between $\omega(t)$ and $i_a(t)$, namely the electromechanical equation

(3) $$K_t\, i_f(t)\, i_a(t) = J\frac{d\omega_m}{dt} + T_L,$$

where K_t is the torque constant, $i_f(t)$ the field current, which will be assumed to be constant ($=I_f$) for the separately excited machine considered here, J the rotational moment of inertia of the rotor plus load and T_L is the load torque. We will assume T_L to be the constant T_0, for simplicity.

We require, of course, the boundary conditions

(4) $$\omega(0) = \omega(T) = 0.$$

Our problem, then, is to minimize (2), given a fixed value of (1) and subject to the coupling equation (3) and boundary conditions (4).

* I. P. Petrov, Variational Methods in Optimum Control Theory, Academic Press, 1968.

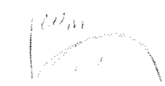
The determination of the optimum $i_a(t)$ and $\omega(t)$ is important for many other electric motors performing cyclic load displacements, as for example, in reversing rolling mills.

Problems of the type we are here posing belong to the subject matter treated in the calculus of variations. We will sketch enough of the theory to solve our problem. First consider the result of substituting (3) into (2)

(5) $$Q = \frac{J^2 R_a}{K_t^2 T_f^2} \int_0^T \left[\frac{d\omega}{dt} + \frac{T_L}{J} \right]^2 dt .$$

Equation (5) is a special form of the more general functional

(6) $$J = \int_a^b F[t, y(t), \frac{dy(t)}{dt}] dt.$$



The object of the calculus of variations is to determine $y(t)$ such that J (not to be confused with moment of inertia) is a maximum or minimum. Whenever one seeks an extremum, he is led to compare the values of J obtained for different functions, y(t). Thus, instead of the function y(t), which is supposed to maximize or minimize (6), we are led to consider the function $y(t) + \varepsilon\eta(t)$, where ε is a real number and $\eta(t)$ is an arbitrary "smooth" function subject only to the conditions $\eta(a) = \eta(b) = 0$. That is $y(t)$ and $y(t) + \varepsilon\eta(t)$ pass through the same end points when t = a or b.

The difference in the values of the functional

(7) $$\Delta J = \int_a^b F(t, y+\varepsilon\eta, y'+\varepsilon\eta')dt - \int_a^b F(t, y, y')dt$$

where we are calling $y' \equiv \frac{dy}{dt}$ J will be a function of ε. Hence, we can expand ΔJ in a power series in ε about $\varepsilon = 0$!

(8) $$\Delta J = J(\varepsilon) - J(0) = J(0) + \varepsilon \frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} + \frac{\varepsilon^2}{2!} \frac{d^2J}{d\varepsilon^2}\Big|_{\varepsilon=0} + \cdots - J(0)$$

$$= \varepsilon \frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} + \frac{\varepsilon^2}{2!} \frac{d^2J}{d\varepsilon^2}\Big|_{\varepsilon=0} + \cdots .$$

The first term, $\varepsilon \frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0}$ is denoted by δJ and is called the first variation of the functional J. Similarly, the second variation, $\delta^2 J$ denotes the second term, $\frac{\varepsilon^2}{2!} \frac{d^2J}{d\varepsilon^2}\Big|_{\varepsilon=0}$.

As $\varepsilon \to 0$ the first term dominates

(9) $$\Delta J \approx \varepsilon \frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} = \delta J.$$

If y(t) maximizes J, then the second term on the right-hand side of (7) is not smaller than the first, <u>no matter how</u> $\varepsilon \to 0$, i.e., <u>through positive or negative values</u>. Thus, we must have

$$(10) \qquad \Delta J \approx \delta J = \varepsilon \frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} \leq 0 \qquad (\varepsilon \to 0 \text{ through positive or negative values.})$$

If, in (10), however, $\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} > 0$, then $\delta J > 0$ if $\varepsilon \to 0$ through positive values. Likewise, if $\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} < 0$, then $\delta J > 0$ if $\varepsilon \to 0$ through negative values. We conclude, therefore, that

$$(11) \qquad \boxed{\delta J = \frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} = 0},$$

in order for J to be maximized. Precisely the same reasoning applies (with a reversal of the inequality in (10) for J to be minimized.

<u>Hence, the necessary condition for an extremum is the vanishing of the first variation</u>. Assuming that an extremum of J exists at y(t), we may check to see whether it is a maximum or a minimum by returning to (8), which now reads

$$(12) \qquad \Delta J = \frac{\varepsilon^2}{2!} \cdot \frac{d^2 J}{d\varepsilon^2}\Big|_{\varepsilon=0} + \cdots\cdots = \delta^2 J + \cdots\cdots.$$

Clearly, for small ε, $\Delta J \approx \delta^2 J$, so that if J is maximized, then, according to (7)

$$(13) \qquad \delta^2 J < 0$$

and if minimized

$$(14) \qquad \delta^2 J > 0.$$

Conditions (11), (13) and (14) are entirely analogous to the corresponding conditions for extrema of functions of a single variable in elementary calculus.

Let us now see what all of this implies. We have

$$(15) \qquad \begin{aligned} \frac{dJ(\varepsilon)}{d\varepsilon} &= \frac{d}{d\varepsilon}\int_a^b F(t, y(t)+\varepsilon\eta(t), y'(t)+\varepsilon\eta'(t))\,dt \\ &= \int_a^b \frac{d}{d\varepsilon} F(t, y(t)+\varepsilon\eta(t), y'(t)+\varepsilon\eta'(t))\,dt \\ &= \int_a^b \left[\frac{\partial F}{\partial y}\cdot\frac{d}{d\varepsilon}(y+\varepsilon\eta) + \frac{\partial F}{\partial y'}\cdot\frac{d(y'+\varepsilon\eta')}{d\varepsilon}\right]dt \\ &= \int_a^b \left(\eta(t)\frac{\partial F}{\partial y} + \eta'(t)\frac{\partial F}{\partial y'}\right)dt. \end{aligned}$$

We integrate by pqrts the second term in (15)

$$\int_a^b \eta'(t)\frac{\partial F}{\partial y'}\,dt = \eta(t)\frac{\partial F}{\partial y'}\Big|_a^b - \int_a^b \eta(t)\frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right)dt$$

$$= -\int_a^b \eta(t)\frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right)dt.$$

In dropping the first term on the right-hand side, we made use of the condition $\eta(a) = \eta(b) = 0$. Upon substituting the above result back into (15), we obtain

(16) $$\frac{dJ(\varepsilon)}{d\varepsilon} = \int_a^b \eta(t)\left[\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right)\right]dt,$$

and in order for this to vanish, as required by (11), for arbitrary $\eta(t)$, it is necessary that the factor of the integrand within braces vanish

(17) $$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right) = 0}.$$

This equation, called the Euler equation, is a necessary condition for the occurrence of an extremum of J. In general, it yields a differential equation for the unknown function, y(t), which extremizes J.

Let us return to a consideration of (13) and (14). Because ε^2 is positive in (12), it follows that the inequalities in (13) and (14) imply identical inequalities on $d^2J/d\varepsilon^2|_{\varepsilon=0}$. But

(18) $$\frac{d^2J}{d\varepsilon^2} = \int_a^b \frac{d^2}{d\varepsilon^2}F(t, y(t)+\varepsilon\eta(t), y'(t)+\varepsilon\eta'(t))\,dt$$

$$= \int_a^b \frac{d}{d\varepsilon}\left[\eta(t)\frac{\partial F}{\partial y} + \eta'(t)\frac{\partial F}{\partial y'}\right]dt$$

$$= \int_a^b \left[\eta(t)\left(\frac{\partial^2 F}{\partial y^2}\eta + \frac{\partial^2 F}{\partial y\partial y'}\eta'\right) + \eta'(t)\left(\frac{\partial^2 F}{\partial y'\partial y}\eta + \frac{\partial^2 F}{\partial y'^2}\eta'\right)\right]dt$$

$$= \int_a^b \left(\frac{\partial^2 F}{\partial y^2}\eta^2(t) + 2\frac{\partial^2 F}{\partial y\partial y'}\eta(t)\eta'(t) + \frac{\partial^2 F}{\partial y'^2}\eta'^2(t)\right)dt.$$

The second member of the integrand can be transformed by integrating by parts, using $\eta(a) = \eta(b) = 0$:

$$2\int_a^b \frac{\partial^2 F}{\partial y \partial y'} \eta \eta' \, dt = \int_a^b \frac{\partial^2 F}{\partial y \partial y'} \, d(\eta^2) = \frac{\partial^2 F}{\partial y \partial y'} \eta^2 \Big|_a^b - \int_a^b \eta^2 \frac{d}{dt}\left(\frac{\partial^2 F}{\partial y \partial y'}\right) dt$$

$$= -\int_a^b \eta^2 \frac{d}{dt}\left(\frac{\partial^2 F}{\partial y \partial y'}\right) dt.$$

Upon substituting this result into (18), there obtains

$$(19) \qquad \frac{d^2 J}{d\varepsilon^2} = \int_a^b \left[\eta^2(t)\left(\frac{\partial^2 F}{\partial y^2} - \frac{d}{dt}\left(\frac{\partial^2 F}{\partial y \partial y'}\right)\right) + \eta'^2(t) \frac{\partial^2 F}{\partial y'^2}\right] dt.$$

Suppose, now, that we choose for our arbitrary function, $\eta(t)$, one possessing small values but rapidly varying (hence large derivatives) within $[a,b]$. Hence, the second term in (19) dominates the first, and we conclude that if $d^2J/d\varepsilon^2 \big|_{\varepsilon=0}$ is to be greater than (less than) zero, $\partial^2 F/\partial y'^2$ must satisfy the same inequalities. Thus

(20)

(a) $$\boxed{\frac{\partial^2 F}{\partial y'^2} > 0} \qquad \text{for a minimum}$$

(b) $$\boxed{\frac{\partial^2 F}{\partial y'^2} < 0} \qquad \text{for a maximum.}$$

If $\frac{\partial^2 F}{\partial y'^2} \equiv 0$ is the interval, the functional is said to be degenerate. Conditions (20) are called Legendre conditions.

The problem with which we are contending is one of determining a conditional extremum, because the minimum of the functional (2) is sought conditioned on a fixed value of of ( ). This problem belongs to the class of isoperimetric problems wherein one seeks that curve bounding the greatest area among all curves possessing the same length.

The treatment of such problems requires the use of Lagrange multipliers. Let us state the rule for solving such problems and then apply it to our problem.

If a function y(t) yields an extremum of (6) subject to the condition that $\int_a^b G(t, y(t), y'(t))\, dt$ is given then y(t) satisfies the Euler equation.

(21) (a) $$\boxed{\frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial y'}\right) = 0}$$

where

(21) (b) $H(t,y,y') = F + \lambda G(t,y,y')$.

λ <u>is a constant called the Lagrange multiplier.</u>

Basically, then, all we have done is augment the kernel, $F(t,y,y')$, of the original functional, J, by the kernel, $G(t,y,y')$, of the conditioning functional multiplied by the Langrange multiplier. We shall not prove (21) but refer you to Petrov's book.

In applying the preceding theory to our problem, we identify Q of (5) with J of (6) and (1) with the conditioning functional. Hence, upon replacing y by ω, y' by ω' we get

$$F(t,y,y') = \frac{J^2 R_a}{K_t^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_0}{J}\right)^2, \quad G(t,y,y') = \omega,$$

$$\therefore \quad H(t,y,y') = \frac{J^2 R_a}{K_t^2 I_f^2}\left(\frac{d\omega}{dt} + T_0/J\right)^2 + \lambda\omega.$$

Thus, the Euler equation, (21)(a), becomes

$$\frac{\partial}{\partial\omega}\left[\frac{J^2 R_a}{K_t^2 I_f^2}\left(\omega' + \frac{T_0}{J}\right)^2 + \lambda\omega\right] - \frac{d}{dt}\left\{\frac{\partial}{\partial\omega'}\left[\frac{J^2 R_a}{K_t^2 I_f^2}\left(\omega' + T_0/J\right)^2 + \lambda\omega\right]\right\}$$

(22)

$$= \lambda - \frac{d}{dt}\left[\frac{2J^2 R_a}{K_t^2 I_f^2}\left(\omega' + T_0/J\right)\right]$$

$$= \lambda - \frac{2J^2 R_a}{K_t^2 I_f^2} \cdot \frac{d^2\omega}{dt^2}$$

$$= 0.$$

The solution of (22) which yields the optimum angular velocity characteristic is

(23) $$\omega(t) = \frac{\lambda K_t^2 I_f^2}{4J^2 R_a} t^2 + C_1 t + C_2,$$

where C_1 and C_2 are arbitrary constants.

In order to satisfy the boundary conditions, (4), C_1 and C_2 must satisfy $C_1 = \frac{-\lambda K_t^2 I_f^2}{4J^2 R_a} T$, $C_2 = 0$. Hence,

$$\omega(t) = \frac{\lambda K_t^2 I_f^2 T}{4J^2 R_a}\left(\frac{t^2}{T} - t\right).$$

λ may be evaluated in terms of the known angle, α, through which the motor turns by substituting this expression for $\omega(t)$ into (1), with the result $\lambda = -\frac{2\alpha T^2 R_a}{K_t^2 I_f^2 T^3}\alpha$.

Thus, the final result for $\omega(t)$ is

(24) $$\omega(t) = \frac{6\alpha}{T^2}\left(t - t^2/T\right).$$

The armature current which minimizes heat losses is obtained from (3)

(25) $$i_a(t) = \left(\frac{6\alpha J/T^2 + T_o}{K_t I_f}\right) - \frac{12\alpha J}{K_t I_f T^3}\, t$$

We conclude, therefore, that the current vs. time relationship which minimizes heating while rotating the rotor through an angle α in T seconds is linear while the angular velocity is parabolic. These results are sketched in Figure 1.

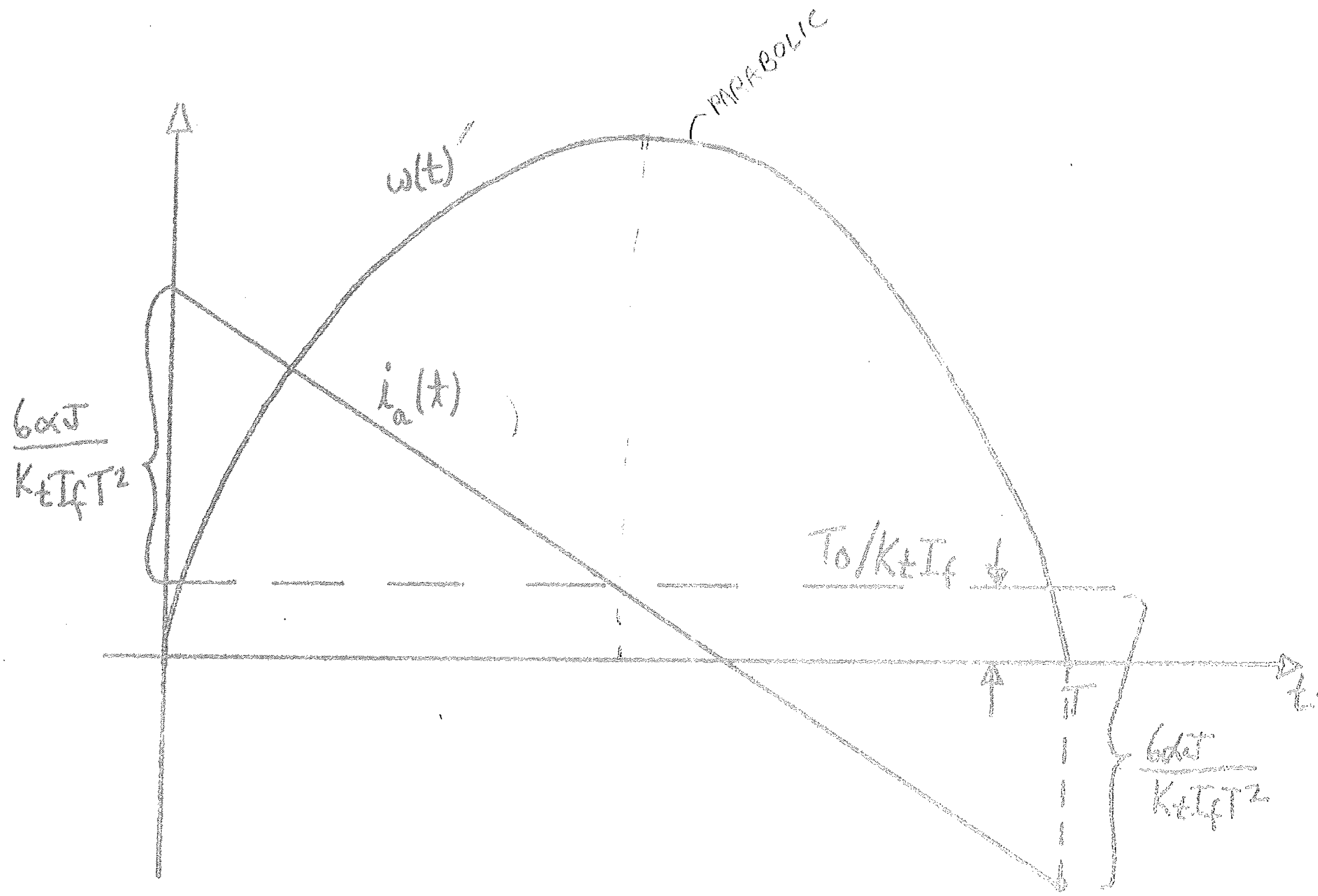


Figure 1. Optimum i(t) and ω(t).

In order to check to see if we, indeed, have minimized the heat dissipated (rather than maximized) we apply the Legendre conditions (20) to the functional $H = \frac{J^2 R_a}{K_f^2 I_f^2}\left(\frac{d\omega}{dt} + \frac{T_0}{J}\right)^2 + \lambda \frac{d\omega}{dt}$

(26) $$\frac{\partial^2 H}{\partial \omega'^2} = \frac{2J^2 R_a}{K_f^2 I_f^2} > 0.$$

Hence, we are actually minimizing the functional as desired. Next, let us calculate the heat dissipated using (25) and then compare the result with another intuitively useful current waveform:

(27) $$Q = R_a \int_0^T \left[\frac{6\alpha J/T^2 + T_0}{K_f I_f} - \frac{12\alpha J}{K_f I_f T^3} t \right]^2 dt$$

$$= \frac{12\alpha^2 J^2 R_a}{K_f^2 I_f^2 T^3} + \frac{R_a T_0^2 T}{K_f^2 I_f^2}.$$

Let us compare this result with the heat dissipated by using a current waveform having the piecewise constant characteristic

(28) $$i_a(t) = \frac{T_0}{K_f I_f} + \frac{4\alpha J}{K_f I_f T^2}, \quad 0 \le t \le T/2$$

$$= \frac{T_0}{K_f I_f} - \frac{4\alpha J}{K_f I_f T^2}, \quad T/2 \le t \le T.$$

This waveform, shown in Figure 2, produces an angular velocity, in accordance with the differential equation (3), that satisfies (1) and (4). Hence, it is a legitimate "comparison function" for our problem (we leave the proof to the reader. It is not difficult). The power dissipated over the interval $[0,T]$ due to (28) is

(29) $$Q = \frac{16\alpha^2 J^2 R_a}{K_f^2 I_f^2 T^3} + \frac{R_a T_0^2 T}{K_f^2 I_f^2}$$

and is greater than (27). In fact, for $T_0=0$, it is greater by 33%. This could be significant if the motor is to be operated as described over a long period of time.

We should point out that the $\omega(t)$ versus t characteristic corresponding to (28) is piecewise linear, rather than parabolic.

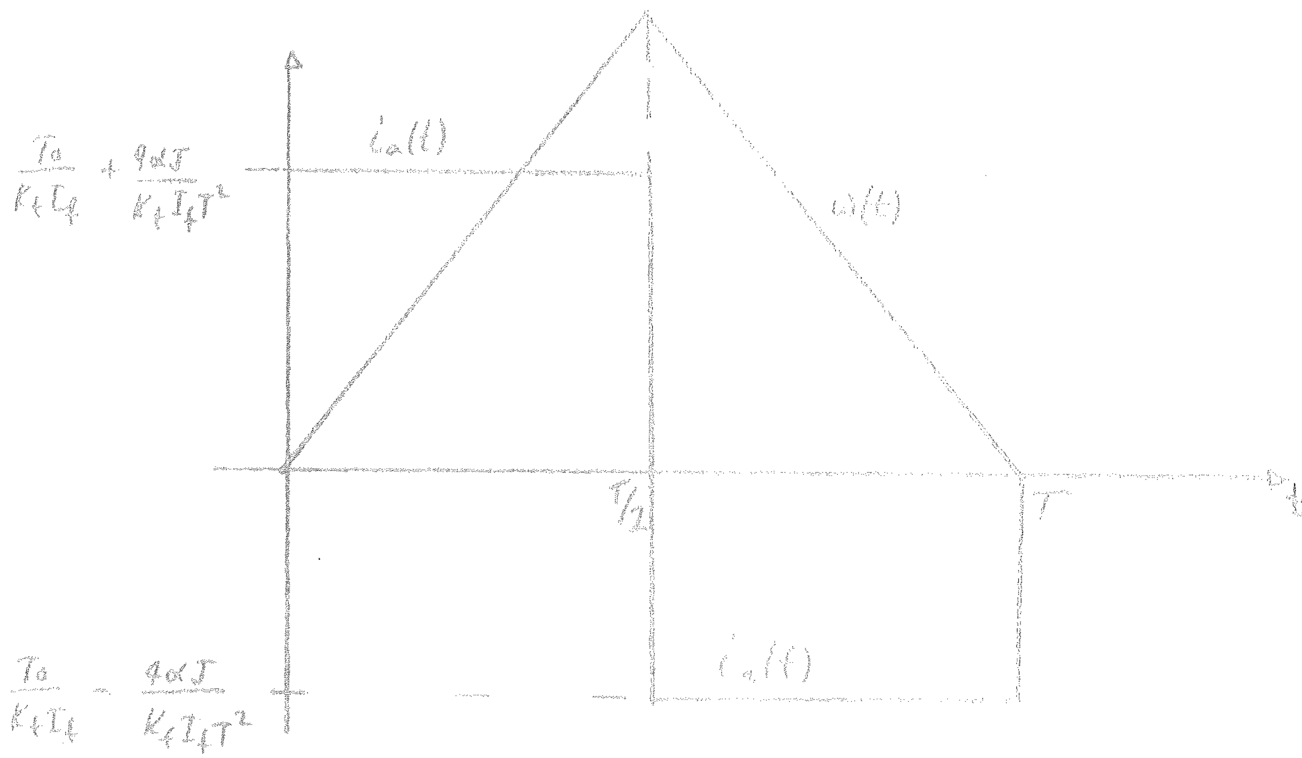


Figure 2. An alternative current waveform that produces the prescribed mechanical motion but dissipates more heat than the optimum waveform of Figure 1

OPTIMUM CONTROL OF MACHINES (II)

In these notes we shall extend the discussion of the first set of notes, on optimum control of d.c. machines, to that of the [illegible] of the linear induction machine (or, in general, of [illegible] machines). We number the equations consecutively, starting from the last set of notes.

We start with the expression for thrust force, F_x, which [illegible] (1-27?):

$$(30?)\qquad F_x = \frac{\mu_0 (N' I_m)^2 l a}{2} \cdot \frac{s R_m'}{(\sinh kg)^2 + (s R_m' \coth kg)^2}$$

$$= \frac{\mu_0 (N' I_m)^2 l a}{2} \cdot \frac{s \mu_0 \sigma' v_{sync}}{(\sinh kg)^2 + (s \mu_0 \sigma' v_{sync} \coth kg)^2}$$

$$= \frac{\mu_0^2 \sigma' (N' I_m)^2 l a}{2} \cdot \frac{s\omega/k}{(\sinh kg)^2 + \left(\frac{s\omega}{k}\right)^2 \mu_0^2 \sigma'^2 \coth^2 kg}$$

$$= \frac{\mu_0^2 \sigma' (N' I_m)^2 l a}{2 (\sinh kg)^2} \cdot \frac{s\omega/k}{1 + \left(\frac{s\omega}{k}\right)^2 \mu_0^2 \sigma'^2 \coth^2 kg}$$

In deriving these expressions we have used $v_{sync} = \omega/k$ and [illegible].

Now, if we define a normalized (dimensionless) angular [illegible] Ω:

$$(31?)\qquad \Omega = s\omega \left(\frac{\mu_0 \sigma' \coth kg}{k}\right),$$

then simple algebra permits expressing F_x as

$$(32?)\qquad F_x = \frac{\mu_0 l a (N' I_m)^2}{2 \sinh kg \cosh kg} \cdot \frac{\Omega}{1+\Omega^2} = 2F_m \frac{\Omega}{1+\Omega^2}$$

where $F_m = [illegible]\, \mu_0 l a (N' I_m)^2 / \sinh(2kg)$ [illegible] (with respect to Ω).

In optimum control problems we seek to either minimize heat losses in the rotor of the induction machine, or, given a value of these losses, maximize displacement, acceleration, etc. Thus, we need an expression for rotor losses. We can derive a useful expression for such losses by referring to Figure 1-20 (b). If we call the current in the right-hand loop, I_R, then, clearly the power dissipated as heat in the rotor is $|I_a|^2 R_a$, whereas the electrical power converted into mechanical energy is $|I_a|^2 R_a \left(\frac{1-s}{s}\right)$. Thus, we conclude that the power dissipated in the rotor equals $\frac{s}{1-s}$ times the mechanical power developed.

The mechanical power developed, however, is $F_x V_m = F_x (1-s) V_{syn}$, which means that

$$\text{(33)} \quad \text{Power dissipated in rotor} = \frac{s}{1-s} F_x (1-s) V_{syn}$$

$$= F_x s V_{syn}$$

$$= \frac{F_x}{k} s\omega$$

$$= \frac{F_x}{k} \cdot \frac{k\Omega}{\mu_0 \sigma' \coth kg}$$

$$= \frac{F_x \Omega}{\mu_0 \sigma' \coth kg}$$

$$= \frac{\ell a (N' I_a)^2}{2\sigma' \coth^2 kg} \cdot \frac{\Omega^2}{1+\Omega^2}$$

$$= B \frac{\Omega^2}{1+\Omega^2}.$$

Let us now pose the following optimum control problem: For a fixed current, I_a, determine the excitation frequency, ω, such that for a given (fixed) change of velocity ΔV_m of the rotor, the heat dissipated in the rotor is a minimum.

Before stating the problem mathematically, let us point out that there are two fundamentally different schemes that could be used for control of the induction machine: (a) amplitude (voltage or current magnitude) control, or (b) frequency control. The former is analogous to "amplitude modulation" whereas the latter is analogous to "frequency modulation". By holding Ω_s fixed, we are fixing F_m and B in (32) and (33), [illegible] the control of Ω to be determined. The efficiency of frequency control is [illegible] generally better than for amplitude control, although the [illegible] it requires a variable frequency source, is more involved. [illegible] Physically, the idea behind frequency control is that because [illegible] and k is fixed by the winding "pole" pitch, then V_{syn} varies directly as ω. For a low slip machine V_m is practically equal to V_{syn}, thus allowing the control of the mechanical velocity.

Mathematically, the problem is stated as

$$(34)\quad \text{(a) minimize:}\quad \Phi = \int_0^T B\,\frac{\Omega^2}{1+\Omega^2}\,dt$$

$$\text{(b)}\quad \Delta V_m = \int_0^T \frac{dV_m}{dt}\,dt \qquad \text{given as a constant.}$$

$$\text{(c)}\quad F_e = 2F_m\,\frac{\Omega}{1+\Omega^2} = M\,\frac{dV_m}{dt} + F_L .$$

Equation (34) (c) is the usual electromechanical equation of motion. We will take the load force, F_L, to be constant. We divide (34) (c) by F_m, and introduce a new time parameter $\tau = (F_m/M)t$, which means that the electromechanical equation becomes

$$(35)\quad \frac{2\Omega}{1+\Omega^2} = \frac{dV_m}{d\tau} + F_0,$$

where $F_0 = F_L/F_m$. In terms of τ, (34) (a) becomes $\Phi = $ [illegible] and (34)(b), $\Delta V_m = \int_0^{(F_m/M)T} (dV_m/d\tau)\,d\tau$.

In order to solve for Ω in terms of V_m', we start with (35) in the form

$$1+\Omega^2 - \frac{2\Omega}{(V_m'+F_0)} = 0,$$

and using the quadratic formula, obtain

$$(36)\quad \Omega = \frac{1}{V_m'+F_0} - \left(\frac{1}{(V_m'+F_0)^2} - 1\right)^{1/2}.$$

Therefore, the integrand of the functional (34)(a), [illegible] is

$$\frac{2\Omega^2}{(1+\Omega^2)} = \left(\frac{2\Omega}{1+\Omega^2}\right)\Omega = (V_m' + F_0)\left[\frac{1}{(V_m'+F_0)} - \left(\frac{1}{(V_m'+F_0)^2} - 1\right)^{1/2}\right]$$

$$= 1 - \left[1 - (V_m' + F_0)^2\right]^{1/2} .$$

The integrand of the constraint functional, (34) (b), is V_m'. Hence, the function $H(V_m, V_m', \tau)$ which enters the Euler equation, (31)(a) is

(38) $$H = 1 - \left(1 - (V_m' + F_0)^2\right)^{1/2} + \lambda_0 V_m' .$$

Using (38), we have

(39) $$\frac{\partial H}{\partial V_m} = 0 , \quad \frac{\partial H}{\partial V_m'} = \lambda_0 + \frac{V_m' + F_0}{\left(1 - (V_m' + F_0)^2\right)^{1/2}}$$

$$\frac{d}{d\tau}\left(\frac{\partial H}{\partial V_m'}\right) = \frac{V_m''}{\left[1 - (V_m' + F_0)^2\right]^{3/2}} .$$

Thus, the Euler equation satisfied by the optimal control is

(40) $$\frac{V_m''}{\left[1 - (V_m' + F_0)^2\right]^{3/2}} = 0 ,$$

which is, of course, equivalent to $\dfrac{d^2 V_m}{d\tau^2} = 0$.

The solution for the optimum V_m is, therefore,

(41) $$V_m = a\tau + b ,$$

where a and b are arbitrary constants. If we assume that the motor starts from rest, then, clearly b = 0. In order to determine the remaining arbitrary constant, we use the constraint (34) (b)

(42) $$\Delta V_m = \int_0^{(F_m/\mu)T} (dV_m/d\tau)\, d\tau = \int_0^{(F_m/\mu)T} a\, d\tau = a\left(\frac{F_m}{\mu}\right)T ,$$

from which we obtain $\tau = \left(\frac{[illegible]}{F_m}\right) \frac{[illegible]}{T}$.

Hence, the optimum velocity profile between 0 and T is a linear ramp

(43) $$V_m(t) = \left(\frac{A}{F_m}\right) \frac{\Delta V_m}{T} \tau = \frac{\Delta V_m}{T} t.$$

The optimum Ω profile is derived upon substituting (43) into the electromechanical equation (34) (c)

(44) $$2 F_m \frac{\Omega}{1+\Omega^2} = \frac{A \, \Delta V_m}{T} + F_L = C_1 \quad (\text{constant})$$

The quadratic equation, $\Omega^2 - \frac{2F_m}{C_1}\Omega + 1 = 0$, that (44) yields, has a positive solution

(45) $$\Omega = \frac{F_m}{C_1} + \sqrt{\left(\frac{F_m}{C_1}\right)^2 - 1}.$$

Thus, the optimum Ω profile is a constant, because F_m and C_1 are constants. We should note that by virtue of (44), it follows that

$$F_m / C_1 \geq 1.$$

Our interest is not so much in Ω, but rather in the time variation of the applied angular frequency, ω. This is easily obtained by returning to (31) and solving for s(t):

(46) $$S(t) = \frac{\Omega}{\omega(t)} \cdot \frac{k}{\mu_0 \sigma' \coth kg} = \frac{V_{sync} - V_m}{V_{sync}} = 1 - \frac{V_m}{V_{sync}} = 1 - [illegible]$$

Thus, upon solving for $\omega(t)$, we get our final result

(47) $$\omega(t) = k\left[\frac{\Omega}{\mu_0 \sigma' \coth kg} + V_m(t)\right] = k\left[\frac{\Omega}{\mu_0 \sigma' \coth kg} + [illegible]\right]$$

where Ω is the constant given in (45). <u>The optimum frequency [illegible] variation is a linear ramp commencing with the frequency</u> $\frac{k\Omega}{\mu_0 \sigma' \coth kg}$ [illegible]

If $F_m \gg C_1$, then (45) reduces approximately to $\Omega \approx [illegible]$ and the optimum $\omega(t)$ equation becomes

(48) $$\boxed{\omega(t) = \frac{\mu_0 k^2 a (N I_m)^2}{2 C_1 \coth^2 kg} + \frac{k \, \Delta V_m}{T} t}$$

The optimum control laws for $\omega(t)$ and $V_m(t)$ are shown in Figure 3.

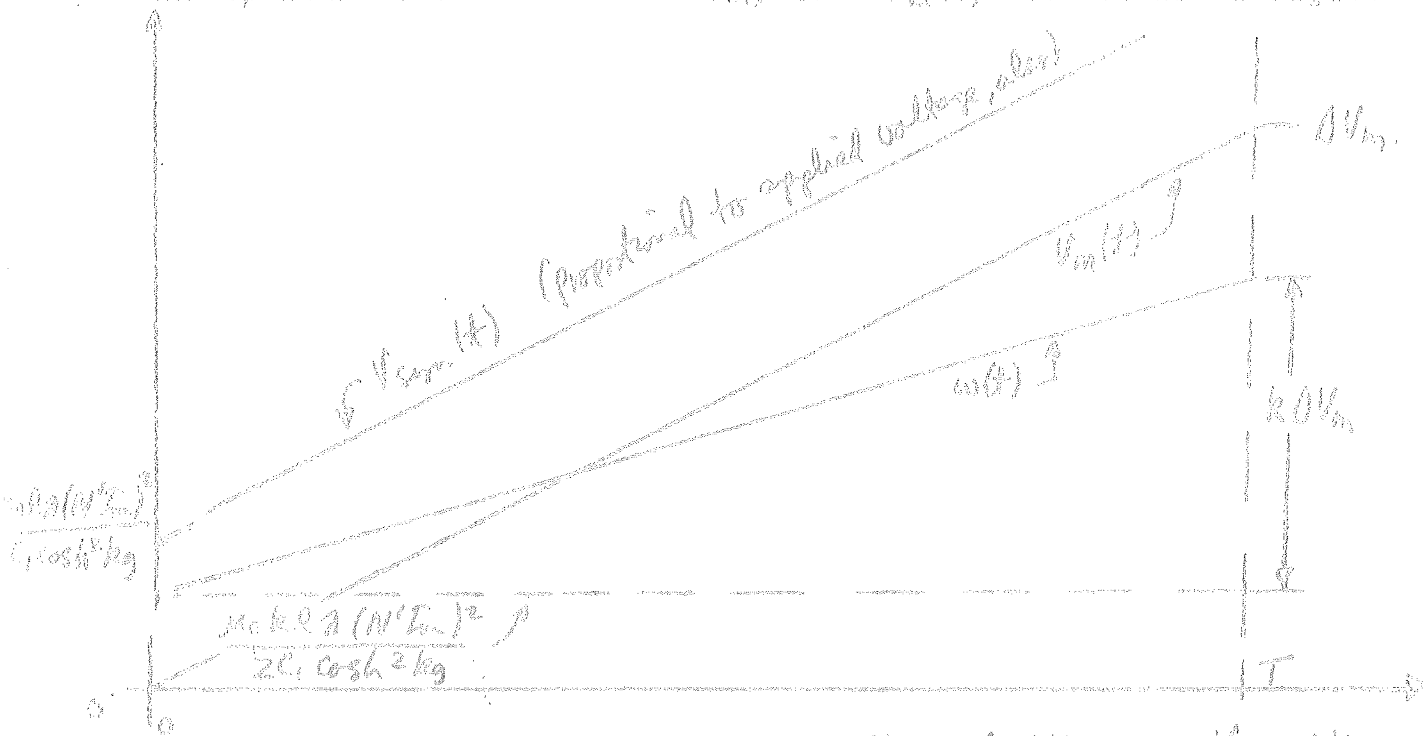


Figure 3. The optimum control laws $\omega(t)$, $V_m(t)$ and $V_{syn}(t)$, $|V(t)|$ for $F_m/C_1 >> 1$.

The typical excitation of a linear induction machine (indeed, of any machine) is from a voltage source, not a current source. Because we maintained $|I_m|$ fixed throughout this development, let us return to (1-225), and determine the optimum law for the voltage variation. Because $sR_m' = s\mu_0\sigma' V_{syn} = \left(\frac{\mu_0\sigma'}{k}\right) s\omega = \frac{a}{\coth kg}$ = constant, it follows from (1-225) that $|V| \propto V_{syn}|I_m| \propto V_{syn}$. Hence, <u>the magnitude of the applied terminal voltage must vary with time directly as does the synchronous velocity in order to maintain the line current constant, as demanded by our optimum solution.</u> From (46), however,

$$(49)\qquad V_{syn}(t) = \frac{\omega(t)}{k} = \frac{\mu_0 l a (N'I_m)^2}{2C_1 \cosh^2 kg} + \frac{\theta V_m}{T} t,$$

and this is proportional to the law of optimum applied voltage.

CHAPTER 5. SYNCHRONOUS MACHINES

The torque in a reluctance machine is generated by the rate of change of the self inductance of the exciting system with respect to the angle between its axis and the axis (center) of saliency of the rotor. In other words, the self inductance of the exciting system should be a function of the above angle. The machine is excited with a.c. voltage and the speed of its rotor is constant and is equal, in electrical radians, to the angular velocity (ω) of the excitation voltage. The instantaneous torque of a singly excited (single phase) reluctance motor consists of a constant torque and a number of harmonics. To cancel the harmonic torques a polyphase winding with balanced polyphase excitation is used. When this is done it is found that the torque is constant and has a higher value than the average torque under single phase excitations.

In this chapter we shall study synchronous machines wherein the stator and rotor are excited, one by a.c. and the other by d.c. Electric energy is carried to the rotor by means of brushes and slip rings. The brushes are stationary while the slip rings, being fastened to the rotor, rotate. This machine, when used as a generator, is also called an alternator and is the backbone of the electrical generating industry.

As was the case with the reluctance motor, we cancel torque harmonics by using balanced polyphase a.c. excitation, in particular, balanced three-phase excitation. Thus, we have three a.c. windings and one d.c. winding. Each winding is sinusoidally distributed in space around the periphery of the air gap.

The sinusoidal distribution can be accomplished in one of the following ways:

a) The slots on the field structure are equally spaced, the number of conductors per slot is constant, but the current per conductor varies sinusoidally in space along the gap circumference.

b) The slots are equally spaced, the current per conductor is constant but the number of conductors per slot varies in a sinusoidal manner in space.

c) The current per conductor is constant in all conductors, the number of conductors per slot is the same for all slots, but the number of slots is sinusoidally distributed along the circumference of the gap.

The three phase (a.c.) coils have the same number of sinusoidally distributed turns and are displaced from each other by $\frac{2\pi}{3}$ electrical radians, or 120 electrical degrees.

(180 electrical degrees span consecutive north and south poles)

In large machines the d.c. winding is placed on the rotor and the three phase winding (a.c.) is placed on the stator, thereby allowing more space for a higher number of larger slots. These slots are insulated for high voltage and can accommodate a larger number of conductors for high current. In small size machines the three phase winding is placed on the rotor and the d.c. winding on the stator. The analysis of the system is the same regardless of which part of the machine supports the a.c. and d.c. windings. We shall take the d.c. field to be on the stator. The schematic representation of the three phase machine is shown in Figure 5-1.

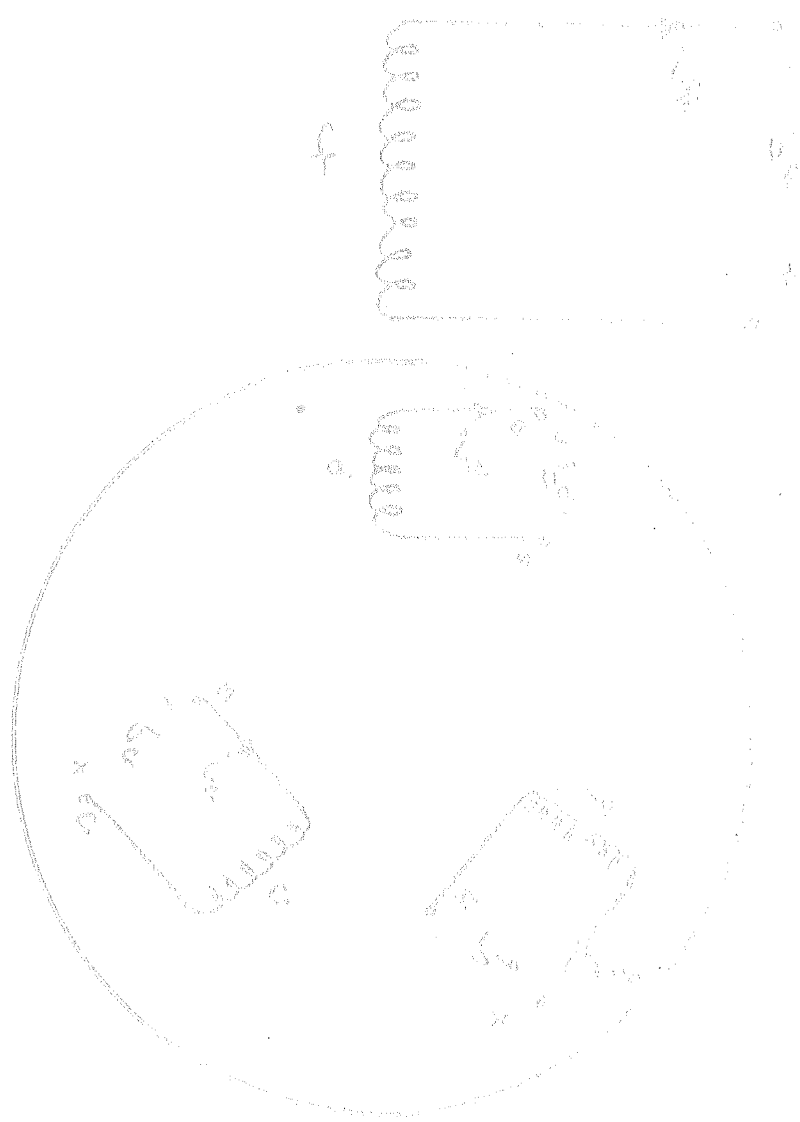


Figure 5-1. Schematic diagram of a three phase machine showing the d-c. excited stator field and the three phase (a.c.) windings on the rotor

5-1. Analysis of the Constant Gap Machine

The stator windings [illegible] has a constant [illegible] self inductance L_{ff}. Each winding [illegible] a constant [illegible] and a constant (i.e., independent of [illegible]) [illegible] inductance. The fact that the self inductances of [illegible] independent of [illegible] rotation angle follows from the assumed [illegible]. From this latter assumption follows, also, the [illegible] between the three rotor windings [illegible] it follows that these self and mutual [illegible] are equal.

Finally, however, the mutual inductance [illegible] rotor windings is a function of the angle [illegible] for the electromechanical energy conversion [illegible] that in order for electromechanical energy [illegible] mutual inductances must be variable). [illegible] mutual inductance between the stator and [illegible] Figure 5-2 for phase a.

The idealized sinusoidal nature of [illegible] to the ideal sinusoidal distribution of the [illegible]

Figure 5-2. Mutual inductance between phase [illegible] vs. α in electrical degrees

The mutual inductances are given by

$$L_{fa} = L_{af} = M_{af} \cos\alpha = M_{af} \cos p\theta$$

(5-1) $$L_{fb} = L_{bf} = M_{af} \cos(\alpha - 120°) = M_{af} \cos(p\theta - 120°)$$

$$L_{fc} = L_{cf} = M_{af} \cos(\alpha + 120°) = M_{af} \cos(p\theta + 120°)$$

where P is the number of pairs of poles on the machine and θ is the angle between the axes of phase a and [illegible] in mechanical degrees (or radians). Note that α electrical degrees [illegible] mechanical degrees.

Our equations may be written compactly [illegible] (Appendix 3A). Thus, Kirchhoff's voltage equation for [illegible] becomes

(5-2) $$[v] = [R][i] + \frac{d}{dt}[\lambda]$$

where

(5-3) $$[v] = \begin{bmatrix} v_f \\ v_a \\ v_b \\ v_c \end{bmatrix}, \quad [i] = \begin{bmatrix} i_f \\ i_a \\ i_b \\ i_c \end{bmatrix}, \quad [\lambda] = \begin{bmatrix} \lambda_f \\ \lambda_a \\ \lambda_b \\ \lambda_c \end{bmatrix}, \quad [R] = \begin{bmatrix} r_f & 0 & 0 & 0 \\ 0 & r_a & 0 & 0 \\ 0 & 0 & r_b & 0 \\ 0 & 0 & 0 & r_c \end{bmatrix}$$

In addition we have

(5-4) $$[\lambda] = [L][i],$$

where

(5-5) $$[L] = \begin{bmatrix} L_{ff} & L_{fa} & L_{fb} & L_{fc} \\ L_{af} & L_{aa} & L_{ab} & L_{ac} \\ L_{bf} & L_{ba} & L_{bb} & L_{bc} \\ L_{cf} & L_{ca} & L_{cb} & L_{cc} \end{bmatrix}$$

The diagonal elements of the inductance matrix [illegible] are, of course, the self inductances of the respective coils. The derivative of the column matrix $[\lambda]$ is given by [illegible]

(5-6) $$\frac{d}{dt}[\lambda] = \frac{d}{dt}\{[L][i]\} = \left(\frac{d}{dt}[L]\right)[i] + [L]\frac{d}{dt}[i]$$

$$= \left\{\frac{\partial[L]}{\partial\theta_m}\cdot\frac{d\theta_m}{dt}\right\}[i] + [L]\frac{d}{dt}[i].$$

Thus, equation (5-2) becomes

(5-7) $$[v] = [R][i] + [illegible]\frac{d}{dt}[illegible]$$

The last term in (5-7), depending as it does on the rotor angular velocity $\omega_m = d\theta_m/dt$, is called the <u>generated</u>, or rotary (and sometimes <u>speed</u>) <u>voltage</u>, or <u>emf</u>.

Utilizing (5-1) and the fact that the self inductances are all constant, we obtain

$$\frac{d[L]}{d\theta_m}\cdot\frac{d\theta_m}{dt} = -P\omega_m M_{afm}\begin{bmatrix} 0 & \sin P\theta_m & \sin(P\theta_m-120^\circ) & \sin(P\theta_m-240^\circ) \\ \sin P\theta_m & 0 & 0 & 0 \\ \sin(P\theta_m-120^\circ) & 0 & 0 & 0 \\ \sin(P\theta_m-240^\circ) & 0 & 0 & 0 \end{bmatrix} \quad (5\text{-}8)$$

Now we see from the form of $[L]$, as well as the <u>mutual inductance matrix</u>, (5-6), that the coefficients of the terms in (5-7) are time-varying, which makes the general analysis quite complicated. We shall undertake to analyze two special, albeit important cases; <u>no load</u> and <u>balanced load</u>.

Case I. <u>No load</u> ($i_a = i_b = i_c = 0$) <u>and</u> v_f is a d.c. voltage. Under these conditions (5-7) becomes

$$v_f = i_f R_{ff} + L_{ff}\frac{di_f}{dt}$$

$$v_a = M_{afm}\cos P\theta_m \frac{di_f}{dt} - P\omega_m M_{afm} i_f \sin P\theta_m$$

$$(5\text{-}9) \qquad v_b = M_{afm}\cos(P\theta_m-120^\circ)\frac{di_f}{dt} - P\omega_m M_{afm} i_f \sin(P\theta_m-120^\circ)$$

$$v_c = M_{afm}\cos(P\theta_m-240^\circ)\frac{di_f}{dt} - P\omega_m M_{afm} i_f \sin(P\theta_m-240^\circ)$$

After steady state has been reached, $di_f/dt = 0$ and $I_f = v_f/R_{ff}$. Then

$$(5\text{-}10) \qquad \boxed{\begin{aligned} v_a &= P\omega_m M_{afm} I_f \sin P\theta_m \\ v_b &= P\omega_m M_{afm} I_f \sin(P\theta_m-120^\circ) \\ v_c &= P\omega_m M_{afm} I_f \sin(P\theta_m-240^\circ) \end{aligned}}$$

[illegible] v_b [illegible] negative when the [illegible]

The [illegible] a, b and c are balanced sine wave [illegible] displaced from each other by 120°. With no current in the [illegible] the generated and terminal voltages are equal.

Case 1. Balanced three phase load and i_f is d.c.

As we have just seen, with d.c. field excitation, in the steady state the generated voltages across the three phases are balanced sine waves. If we load the three phases with equal impedances (a balanced load) we expect the current in the three phases to be balanced sine waves given by the expressions

$$i_a = I_m \sin(p\theta_m - \theta)$$

(5-11) $$i_b = I_m \sin(p\theta_m - \theta - 120^\circ)$$

$$i_c = I_m \sin(p\theta_m - \theta - 240^\circ),$$

where θ is the phase angle (positive for a lagging load, as written) of the impedance. The results (5-11) will hold provided that these currents (the load currents i_a, i_b, i_c) do not change the field current. Therefore, let us first find the effect of the load currents on the field.

When the three phase currents are given by (5-11), then the field flux linkages, by virtue of (5-1), (5-4) and (5-5), are given by

$$\lambda_f = L_{ff} i_f + M_{af} I_m [\sin(p\theta_m - \theta) \cos p\theta_m$$

(5-12) $$+ \sin(p\theta_m - \theta - 120^\circ) \cos(p\theta_m - 120^\circ) + \sin(p\theta_m - \theta - 240^\circ) \cos(p\theta_m - 240^\circ)]$$

$$= L_{ff} i_f + \frac{M_{af} I_m}{2} [\sin(2p\theta_m - \theta) + \sin(-\theta) + \sin(2p\theta_m - \theta - 240^\circ)$$

$$+ \sin(-\theta) + \sin(2p\theta_m - \theta - 120^\circ) + \sin(-\theta)]$$

$$= L_{ff} i_f - \frac{3}{2} M_{af} I_m \sin\theta .$$

Note that the balanced second harmonic contributions cancel, leaving only a d.c. term $-\frac{3}{2} M_{af} I_m \sin\theta$, which represents a decrease in the field flux linkages if θ > 0 (lagging load) and an increase if θ < 0 (leading load). The term $\frac{3}{2} M_{af} I_m \sin\theta$, is the total effect of the armature current on the field and is called armature reaction.

The field current, however is still d.c. and is given, again, by $i_f = v_f/R_{ff}$. This follows because the armature reaction, being d.c., does not [illegible] $v_f = i_f R_{ff} + \frac{d\lambda_f}{dt}$ [illegible]. However, the [illegible] field will have [illegible] and all even harmonic [illegible] have the fundamental and all odd harmonics [illegible] the armature is still a sine wave, [illegible] remains d.c. and the same [illegible].

The flux linkages of phases a, b, c [illegible] given by

(5-13) $$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \end{bmatrix} = \begin{bmatrix} M_{af_m} \cos \theta_{de} & L_{aa} & [illegible] & [illegible] \\ M_{af_m} \cos(\theta_{de} - 120°) & L_{ba} & [illegible] & [illegible] \\ M_{af_m} \cos(\theta_{de} - 240°) & L_{ca} & [illegible] & [illegible] \end{bmatrix} \begin{bmatrix} [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \end{bmatrix}$$

Consider λ_a

(5-14) $$\lambda_a = I_f M_{af_m} \cos \theta_{de} + L_{aa} [illegible]$$
$$+ L_{ab} \left[I_m \sin(\theta_{de} - \theta - 120°) + I_m \sin(\theta_{de} - \theta - 240°) \right]$$

(note that we have used $L_{ab} = L_{ac}$ which follows from the [illegible] nature of the windings)

But

(5-15) $$I_m \left[\sin(\theta_{de} - \theta - 120°) + \sin(\theta_{de} - \theta - 240°) \right] = - I_m \sin(\theta_{de} - \theta)$$

so that (5-14) becomes

(5-16) $$\lambda_a = M_{af_m} I_f \cos \theta_{de} + (L_{aa} - L_{ab}) I_m \sin(\theta_{de} - \theta)$$
$$= M_{af_m} I_f \cos \theta_{de} + L_s I_m \sin(\theta_{de} - \theta)$$

where $L_s = L_{aa} - L_{ab}$ is called the synchronous [illegible] inductance has a meaning only when the [illegible] balanced three phase. L_{ab} may be negative, depending on the manner in which coil a is wound relative to b

When the machine is operating as a generator, the [illegible] phase a is given by

(5-17) $$v_a = -\left(i_a R_{a} + \frac{d\lambda_a}{dt} \right) = [illegible]$$
$$+ [illegible]$$

Similarly

$$V_b = -R_{bb} I_m \sin(P\theta_m - \theta - 120°) - P\omega_m L_s I_m \cos(P\theta_m - \theta - 120°) + P\omega_m M_{afm} I_f \sin(P\theta_m - 120°) \quad (5\text{-}18)$$

$$V_c = -R_{cc} I_m \sin(P\theta_m - \theta - 240°) - P\omega_m L_s I_m \cos(P\theta_m - \theta - 240°) + P\omega_m M_{afm} I_f \sin(P\theta_m - 240°) \quad (5\text{-}19)$$

Notice that the last term on the right hand side of (5-17), (5-18), and (5-19) is the generated (no load) voltage in phases a, b, and c, respectively. The structure of each of the above equations is identical, so that we may consider (5-17) only.

It will be useful to represent (5-17) by a phasor diagram. To that end let

$$E_g = P\omega_m M_{afm} I_f \quad (5\text{-}20)$$

and

$$X_s = P\omega_m L_s = \omega L_s. \quad (5\text{-}21)$$

X_s is called the synchronous reactance and $Z_s = R_a + jX_s$ is called the synchronous impedance.

The current $I_a = I_m \sin(P\theta_m - \theta)$ lags the generated voltage by θ degrees, which means that the first term leads the generated voltage by (180-θ°) because of the negative sign. The second term lags the first by 90°, with the result that Figure 5-3 is the phasor diagram with E_g as the reference.

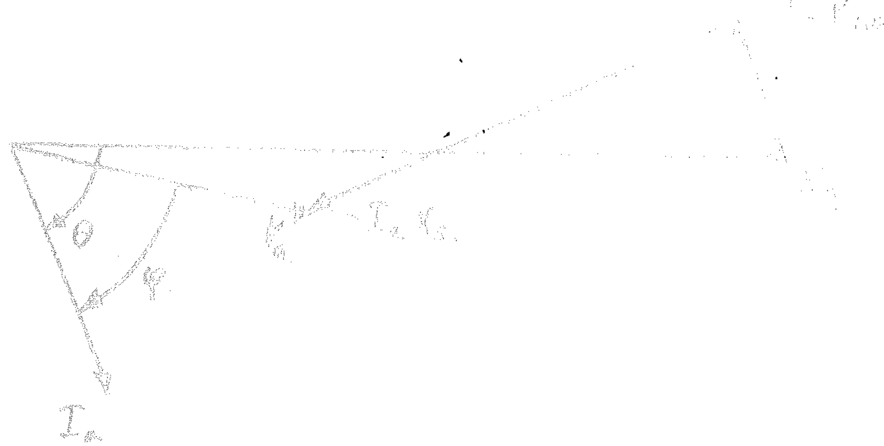


Figure 5-3. Phasor diagram for phase a of an alternator with the generated voltage, E_g, as the reference.

The generated power per phase is $1/2\, E_g I_a \cos\theta$, so that the total generated power of the machine is

$$P = \frac{3}{2} E_g I_a \cos\theta \qquad (5\text{-}22)$$

Of course, we are using peak values for E_g and I_a in the preceding equations. If E_g and I_a stand for rms values the 1/2 in (5-22) is omitted.

Usually the terminal voltage V_a, the load current I_a and the load impedance phase angle, φ, between V_a and I_a are known. Equation (5-[illegible]7) may be written, therefore,

$$E_g = V_a + I_a (R_{aa} + jX_s) \qquad (5\text{-}23)$$

$$= V_a + I_a Z_s \,.$$

Figure 5-4 is the phasor diagram that results. Note that it simply is figure 5-3 rotated counterclockwise $\theta - \varphi$ degrees to put V_a as the reference.

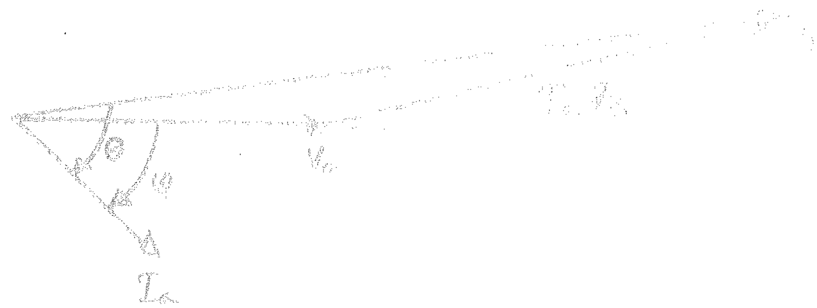


Figure 5-4. Phasor diagram with V_a as reference.

5-2. Torque Analysis for a Constant Gap Machine.

In the preceding analysis we have assumed that the machine was driven by an outside mechanical device at constant speed. Let us find the mechanical torque exerted on the shaft of this machine.

From equation (3-47) we have

$$\tau = -\frac{\partial W_m'}{\partial \alpha_m} \qquad (5\text{-}24)$$

ROSE-HULMAN INSTITUTE OF TECHNOLOGY

Non-Sinusoidal Periodic Waves

__Preliminary__

The current waveform being examined may be represented by a Fourier Series of the form

$$i(t) = \sum_{n=0}^{\infty} C_n \cos(n\omega t + \theta_n)$$

The audio oscillator output may be represented by

$$v(t) = V_m \cos(h\omega t + \phi)$$

the instantaneous torque on the wattmeter movement is given by:

$$T(t) = k\, i(t)\, v(t)$$

$$= k \sum_{n=0}^{\infty} C_n \cos(n\omega t + \theta_n)\, V_m \cos(h\omega t + \phi)$$

Since $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

We have $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

$$\therefore \quad T(t) = k \sum_{n=0}^{\infty} \frac{C_n V_m}{2} \Big[\cos(n\omega t + h\omega t + \theta_n + \phi)$$

$$+ \cos(n\omega t - h\omega t + \theta_n - \phi)\Big]$$

$$T(t) = \frac{K V_m}{2} \sum_{n=0}^{\infty} C_n \Big\{ \cos\left[(n+h)\omega t + \theta_n + \phi\right]$$

$$+ \cos\left[(n-h)\omega t + \theta_n - \phi\right] \Big\}$$

The first term $\cos\left[(n+h)\omega t + \theta_n + \phi\right]$ represents a torque varying too rapidly for the meter movement to follow; hence it contributes zero average value to the meter indication. In the second term, $\cos\left[(n-h)\omega t + \theta_n - \phi\right]$ as h approaches a particular value of n for the current wave the radian frequency (n-h)w will become quite small. The watt-meter will respond, reading first up scale and then down scale according to the sign and magnitude of the torque. If the oscillator is adjusted very carefully, the period of oscillation can be made very long, minimizing error due to inertia and damping. The maximum deflection of the wattmeter will be

$$W_{max} = \frac{V_m C_n}{2} = \frac{V_m}{\sqrt{2}} \frac{C_n}{\sqrt{2}}$$

$$\therefore \quad W_{max} = V_{RMS} \, I_{n\,RMS}$$

Since the RMS value of the oscillator voltage can be measured readily I_{RMS} can be immediately calculated.

SKETCH OF THE THEORY OF THE WATTMETER

The instantaneous torque of electromagnetic origin developed within a wattmeter is proportional to the product of the voltage, $e(t)$, applied to the voltage coil, with the current, $i(t)$, supplied to the current coil. The wattmeter movement has a moment of inertia, J, a damping constant, D, and is connected to a torsional spring whose spring constant is K_s.

Thus, the equation of motion is

(1) $$T = A\,e(t)\,i(t) = J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K_s\theta,$$

where A is the torque constant of the meter and θ is the angular deflection of the meter movement. Because the damping is large in conventional meters, thereby reducing the acceleration of the movement and the associated inertial reaction torque, we simplify (1) by omitting the first term

$$A\,e(t)\,i(t) = D\,d\theta/dt + K_s\theta$$

or

(3) $$\frac{A}{D}e(t)\,i(t) = \frac{d\theta}{dt} + \frac{K_s}{D}\theta = \frac{d\theta}{dt} + \frac{\theta}{T},$$

where $T = D/K_s$ is the movement time constant.

The impulse response of the simple first-order system on the right is

(4) $$h(t) = e^{-t/T}, \quad t \geq 0.$$

Hence, applying the convolution integral to (3) and (4) gives

(5) $$\theta(t) = \frac{A}{D}\int_0^t e(\tau)\,i(\tau)\,e^{-(t-\tau)/T}\,d\tau.$$

We see from (5) that if the time constant, T, is large, than for $0 \leq \tau \leq t$, the exponential $e^{-(t-\tau)/T}$ will be practically constant with value unity. This indicates that the response, (5), is essentially the integral of the product of $e(t)$ and $i(t)$. Put another way, the response at time t is the average value of the power over the interval 0 to t. Hence, the wattmeter measures average power.

/ Of course, if T ~~is~~ were truly infinite, the system could never settle into a steady-state condition. This can be understood from equation (3), with the second term on the right-hand side omitted. Then for a non-zero input on the left-hand side, the movement would have to continually spin in order to achieve a balance.

Now, in order to be specific, let

(6) $e(t) = \mathcal{E} \cos(\omega_1 t + \varphi_1)$, $i(t) = I \cos(\omega_2 t + \varphi_2)$

Then

(7) $$e(t)\cdot i(t) = \frac{\mathcal{E} I}{2}\left\{\cos[(\omega_1-\omega_2)t + (\varphi_1-\varphi_2)] + \cos[(\omega_1+\omega_2)t + (\varphi_1+\varphi_2)]\right\}$$
$$= \frac{\mathcal{E} I}{4}\left\{ e^{j[(\omega_1-\omega_2)t + (\varphi_1-\varphi_2)]} + e^{-j[(\omega_1-\omega_2)t+(\varphi_1-\varphi_2)]} + e^{j[(\omega_1+\omega_2)t+(\varphi_1+\varphi_2)]} + e^{-j[(\omega_1+\omega_2)t+(\varphi_1+\varphi_2)]}\right\}.$$

The integral in (5) is easily evaluated to be

(8) $$\theta(t) = \frac{A\mathcal{E}I}{4D}\left\{ e^{j\varphi_a}\,\frac{e^{j\omega_a t} - e^{-t/T}}{\frac{1}{T} + j\omega_a} + e^{-j\varphi_a}\,\frac{e^{-j\omega_a t} - e^{-t/T}}{\frac{1}{T} - j\omega_a} + e^{j\varphi_b}\,\frac{e^{j\omega_b t} - e^{-t/T}}{\frac{1}{T} + j\omega_b} + e^{-j\varphi_b}\,\frac{e^{-j\omega_b t} - e^{-t/T}}{\frac{1}{T} - j\omega_b}\right\},$$

where $\varphi_a = \varphi_1 - \varphi_2$, $\omega_a = \omega_1 - \omega_2$, $\varphi_b = \varphi_1 + \varphi_2$, $\omega_b = \omega_1 + \omega_2$.

After the transient terms involving $e^{-t/T}$ die out, (8) becomes

(9) $$\theta(t) = \frac{A\mathcal{E}I}{2D}\left(\frac{\frac{1}{T}\cos(\omega_a t + \varphi_a) + \omega_a \sin(\omega_a t + \varphi_a)}{(\frac{1}{T})^2 + \omega_a^2} + \frac{\frac{1}{T}\cos(\omega_b t + \varphi_b) + \omega_b \sin(\omega_b t + \varphi_b)}{(\frac{1}{T})^2 + \omega_b^2}\right)$$
$$= \frac{A\mathcal{E}I}{2D}\left(\frac{\cos(\omega_a t + \varphi_a + \psi_a)}{[(\frac{1}{T})^2 + \omega_a^2]^{1/2}} + \frac{\cos(\omega_b t + \varphi_b + \psi_b)}{[(\frac{1}{T})^2 + \omega_b^2]^{1/2}}\right),$$

where $\psi_a = -\tan^{-1}\omega_a T$, $\psi_b = -\tan^{-1}\omega_b T$.

The response, therefore, consists of a high frequency term, the ω_b term, and a low frequency term, the ω_a term. That the wattmeter movement acts as a low pass filter is clear when we observe that the high frequency term is smaller than the low frequency term ~~is smaller than the low frequency term~~ due to the larger denominator at higher frequencies in (9).

Usually, e(t) and i(t) have the same frequency, ω. In this case, $\omega_a = 0$, $\varphi_a = 0$, $\omega_b = 2\omega$ and (9) consists of a d.c. term and a much smaller term at 2ω

$$\theta(t) = \frac{AT}{D}\left[\frac{EI}{2}\cos\varphi_a + \frac{EI}{2}\cdot\frac{\cos(2\omega t + \varphi_b + \psi_b)}{[1 + 4\omega^2\tau^2]^{1/2}}\right] \tag{10}$$

$$= \frac{A}{K_s}\left[\frac{EI}{2}\cos\varphi_a + \frac{EI}{2}\cdot\frac{\cos(2\omega t + \varphi_b + \psi_b)}{[1 + 4\omega^2\tau^2]^{1/2}}\right].$$

Clearly, the first term within the brackets is the <u>average power</u> delivered by the electrical system, and it is also the <u>average reading</u> of the wattmeter. The second term, being so much smaller than the first, does not contribute significantly to the wattmeter fluctuation. <u>It could, if ω is very small, i.e., if not only do e(t) and i(t) have the same frequency, ω, but this frequency is small.</u> A wattmeter actually is designed to operate over a limited frequency range to preclude such anomolous behavior.

Note, from (10), that as far as sinusoidal steady-state operation is concerned, we want $\tau \to \infty$. Clearly, from (8), however, this implies that the transient term, $e^{-t/\tau}$, never dies out, but remains fixed at unity.

A. Waves in Fluid Media

A fluid field is defined if at each point of space and at each instant of time we specify the density, pressure, temperature and velocity of the fluid. The Eulerian method (K. R. Symon, Mechanics, Addison-Wesley, 1953, ch. 8), thus, focuses attention on what is happening at point x at time t. The Lagrangian method, which we will not develop, focusses attention on a particular fluid particle (say the one at x_0 when $t = t_0$).

Suppose one of the state functions of the fluid, e.g., pressure, is given by $p(x,t)$. If we move with a fluid particle, the total time derivative $\frac{dp}{dt}$ is given by

$$(1) \qquad \frac{dp}{dt} = \frac{\partial p}{\partial t} + \underbrace{\frac{\partial p}{\partial x} \cdot \frac{dx}{dt}}_{\text{CHAIN RULE}} = \underset{1}{\frac{\partial p}{\partial t}} + \underset{2}{\mathbf{v} \frac{\partial p}{\partial x}}$$

where v is the velocity of motion. Thus, the total time derivative consists of two parts: 1) a part due to a time change at a fixed point and 2) a part due to the motion of the particle.

Consider a small volume of cross sectional area A and extending from x to x + Δx at time t. At time t + Δt the volume changes as shown due to the motion of it with the fluid.

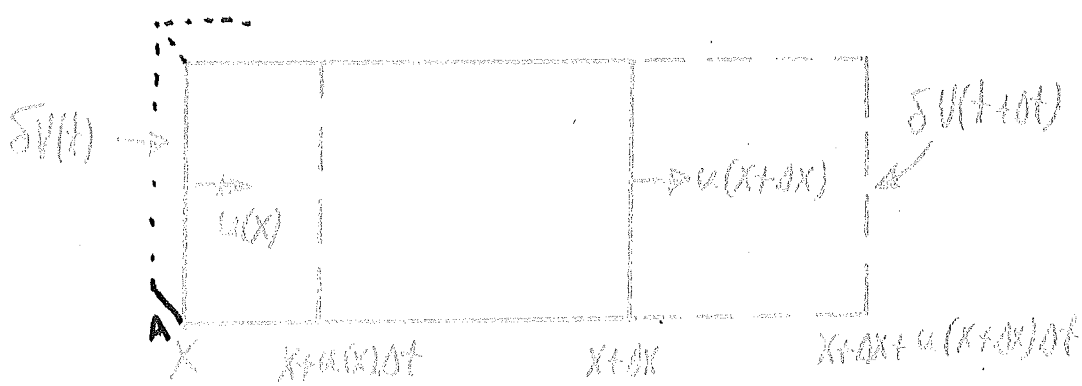


Figure A-1. Volume in a fluid field.

We calculate the total time rate of change of

$$\delta V(t+\Delta t) = A\left[X+\Delta X + u(X+\Delta X)\Delta t - X - u(X)\Delta t\right]$$

$$= A\left[\Delta X + \left[u(X+\Delta X) - u(X)\right]\Delta t\right]$$

$$\delta V(t) = A\left[X + \Delta X - X\right] = A\Delta X$$

$$\therefore \delta V(t+\Delta t) - \delta V(t) = A\left[u(X+\Delta X) - u(X)\right]\Delta t$$

$$= A\,\frac{\partial u(X,t)}{\partial X}\,\Delta X\,\Delta t.$$

Hence

(2) $$\frac{d\,\delta V(t)}{dt} = \lim_{\Delta t \to 0} \frac{\delta V(t+\Delta t) - \delta V(t)}{\Delta t} = A\frac{\partial u}{\partial x}\Delta x = \frac{\partial u}{\partial x}\,\delta V(t).$$

Let $\rho(x,t)$ be the volume density of fluid contained in a small volume δV. The net mass enclosed within the small volume at time t is $\delta m = \rho\,\delta V$. As we travel with this "fluid parcel", we ask how must ρ and δV change in order for δm to remain constant. The reason for asking the question is because Newton's law applies to a closed system only, i.e., a system which neither gains nor loses mass.

Thus,

(3) $$\frac{d}{dt}(\delta m) = \frac{d}{dt}(\rho\,\delta V) = 0 = \frac{d\rho}{dt}\delta V + \rho\frac{d}{dt}(\delta V) = \left(\frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x}\right)\delta V,$$

(where we have used (2)).

Hence,

(4) $$\frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x} = 0.$$

This result is called the "equation of continuity". Using (1), we rewrite (4) as

(5) $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,$$

Our next job is to apply Newton's law and derive the equation of motion of an ideal fluid (no viscosity).

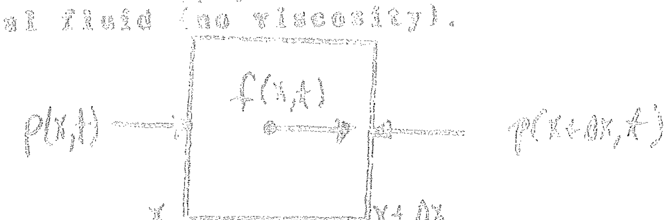


Figure A-2. Force density acting on small volume.

Let $f(x,t)$ be the body force density applied to a small region $(\delta x\,\delta y\,\delta z)$ of fluid. $p(x,t)$ and $p(x+\Delta x,t)$ are, respectively, the pressure at x at time t and at $x+\Delta x$, also at time t. The region (or fluid parcel) contains a constant mass $\delta m = \rho\,\delta V$. Newton's law yields

(6) $$p(x,t)\,\delta y\,\delta z - p(x+\Delta x,t)\,\delta y\,\delta z + f(x,t)\,\delta x\,\delta y\,\delta z = \rho\,\delta x\,\delta y\,\delta z\,\frac{du}{dt}.$$

Dividing by $\delta x \delta y \delta z$:

(7) $$\frac{p(x,t) - p(x+\delta x,t)}{\delta x} + f(x,t) = \rho \frac{du}{dt}.$$

Or, passing to the limit:

(8) $$-\frac{\partial p(x,t)}{\partial x} + f(x,t) = \rho(x,t)\frac{du}{dt} = \rho(x,t)\left(\frac{\partial u(x,t)}{\partial t} + u\frac{\partial u}{\partial x}\right).$$

Hence, the system of equations is:

(9) $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$
$$-\frac{\partial p}{\partial x} + f = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right).$$

It may be interesting to digress and note that (9) applies to meteorological forecasting when f includes gravity and the Coriolis force. This latter force occurs in rotating systems (including the Earth) and accounts for wind directions in extratropical latitudes.

Note that (9) involves two equations but three variables, p, ρ, u. Thus, we need an additional relation. Such a relation, called a "constitutive" relation involves p and ρ;

(10) $$\frac{\delta \rho}{\rho} = \frac{\delta p}{B},$$

where B is the bulk modulus of the fluid. For a gas varying adiabatically $B = \gamma p_0$ where γ is the ratio of specific heat at constant pressure to that at constant volume.

Equation (9) is non-linear because the unknowns p, ρ, u and their derivatives are multiplied together. To linearize, we use the method of small perturbations. Write $p = p_0 + p'$, $\rho = \rho_0 + \rho'$, where the primed variables are small compared to the unprimed ones. We substitute these into (9) and (10) and neglect products of primed variables as being of second order smallness. Thus, (10) becomes

(11) $$\frac{\rho'}{\rho_0} = \frac{p'}{B},$$

and (9) becomes

(12) $$\frac{\partial \rho'}{\partial t} + u_0\frac{\partial \rho'}{\partial x} + \rho_0\frac{\partial u'}{\partial x} = 0$$
$$-\frac{\partial p'}{\partial x} + f = \rho_0\left(\frac{\partial u'}{\partial t} + u_0\frac{\partial u'}{\partial x}\right)$$

(assume p_0, ρ_0 and u_0 are constant in space and time).

Let's drop primes, now, and set f and u_0 to zero. Then

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0$$

(13)

$$-\frac{\partial p}{\partial x} - \rho_0 \frac{\partial u}{\partial t} = 0$$

$$\rho/\rho_0 = p/B,$$

or

(14)

$$\frac{\rho_0}{B}\frac{\partial p}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial p}{\partial x} + \rho_0 \frac{\partial u}{\partial t} = 0.$$

B: BULK MODULUS
ρ_0: CONSTANT DENSITY
$u \Rightarrow i$
$p \Rightarrow V$
$\rho_0 \Rightarrow L$
$1/B \Rightarrow C$
} ANALOGOUS

Differentiating the top equation with respect to x and the bottom with respect to t:

(15)

$$\frac{\rho_0}{B}\frac{\partial^2 p}{\partial x \partial t} + \rho_0 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 p}{\partial x \partial t} + \rho_0 \frac{\partial^2 u}{\partial t^2} = 0.$$

$$\therefore \quad \frac{\rho_0}{B}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

This is a wave equation with velocity $c = \left(\frac{B}{\rho_0}\right)^{1/2}$ for a fluid and $c = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}$ for a gas.

$$\frac{\delta^2 p}{dx^2} = -\rho_0 \frac{d^2 u}{\delta x \delta t};\ \frac{\delta^2 u}{\delta x \delta t} = -\frac{1}{B}\frac{\delta^2 p}{\delta t^2} \Rightarrow \frac{\delta^2 u}{\delta x \delta t} = -\frac{1}{B}\frac{\delta^2 p}{\delta t^2} \Rightarrow$$ SOUND IS VARIATION OF PRESSURE

$$V_{P_{SOUND}} = \left(\frac{B}{\rho_0}\right)^{\frac{1}{2}}$$

MARK

Consider the thin, straight rod shown in Figure B-1.

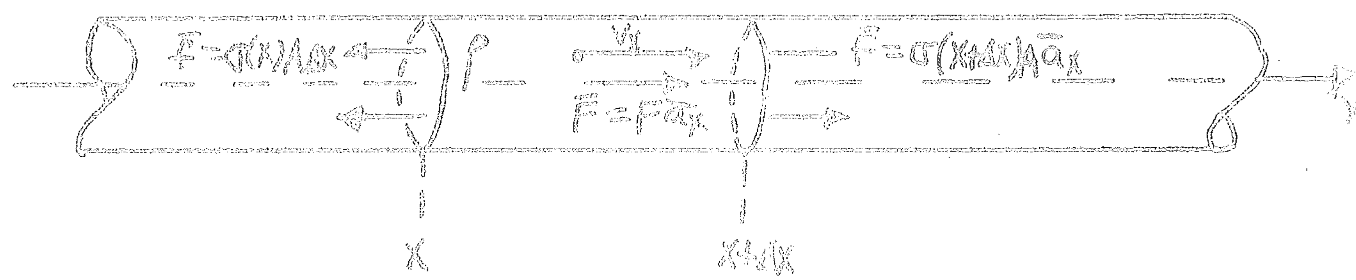


Figure B-1. Straight, thin elastic rod of cross-sectional area A. F is the body force density $\frac{\text{Newtons}}{(\text{Meter})^3}$, σ = stress $\left(\frac{\text{Newtons}}{(\text{Meter})^2}\right)$. ρ is the mass density $\left(\frac{\text{KG}}{(\text{Meter})^3}\right)$.

The stress, σ, is the resultant traction applied by the remainder of the rod to the end faces of the small cylindrical section between x and x + Δ x. Following the reasoning of (6) of the "Fluid Media" notes, we write Newton's law for Figure B-1 as

$$(1) \qquad \left[\sigma(x+\Delta x)-\sigma(x)\right]A + FA\cdot\Delta x = \rho A\Delta x\frac{du}{dt} = \rho A\Delta x\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}\right),$$

where $V_x(x,t)$ is the x-directed velocity of the center of mass of the small cylindrical section of Figure B-1.

In elasticity we can neglect the non-linear term $v_x\frac{\partial v_x}{\partial x}$ and rewrite (1), upon division by $A\Delta x$ and passage to the limit $\Delta x \to 0$ as

$$(2) \qquad \frac{\partial\sigma}{\partial x}+F=\rho\frac{\partial v_x}{\partial t}.$$

In terms of the x-directed displacement, u(x,t), of the center of mass of the incremental section, we have

$$(3) \qquad v_x=\frac{\partial u}{\partial t}+v_x\frac{\partial u}{\partial x}.$$

We again neglect the term $v_x\frac{\partial u}{\partial x}$ as being the product of two small quantities, and arrive at our first important result

$$(4) \qquad \frac{\partial\sigma}{\partial x}+F=\rho\frac{\partial^2 u}{\partial t^2}$$

This is a statement about the dynamics of an elastic system and followed from Newton's law. It is now necessary to derive a result based purely on the elastic nature of the rod. Such a result is the stress-strain relation of linear elasticity theory.

Consider the motion of the cross-section at x in Figure B-1. When the rod vibrates, this section moves to the x-coordinate $x + u(x,t)$, because u(x,t) is, by definition, the x-directed displacement of the cross-section which was initially at x. Similarly, the cross section at $x + \Delta x$ moves to the coordinate $x + \Delta x + u(x+\Delta x, t)$. The new length of the deformed incremental section becomes, therefore,

$$(5) \qquad u(x+\Delta x, t) + x + \Delta x - u(x,t) - x = u(x+\Delta x, t) - u(x,t) + \Delta x.$$

The original length, however, was Δx, so that the change in length of the incremental section is $u(x+\Delta x) - u(x,t)$. The longitudinal strain, e, is, by definition, the change in length divided by the original length

$$(6) \qquad e = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x} = \frac{\partial u}{\partial x}.$$

Hook's law states that the stress, σ, is linearly related to the strain, e, with the constant of proportionality being Young's modulus, Y:

$$(7) \qquad \sigma = Ye = Y\frac{\partial u}{\partial x}.$$

Thus, our coupled system (4) and (7) reduce to the single equation for u:

$$(8) \qquad Y\frac{\partial^2 u}{\partial x^2} + F = \rho\frac{\partial^2 u}{\partial t^2}.$$

Equations (4) and (7) can be put into a form analogous to the lossless transmission-line equations by dropping F, differentiating (7) with respect to t and writing $v_x \equiv \partial u/\partial t$. The resulus are

$$(9)\ (a) \qquad \frac{\partial \sigma}{\partial x} = \rho\frac{\partial v_x}{\partial t}$$

$$(b) \qquad \frac{\partial v_x}{\partial x} = \frac{1}{Y}\frac{\partial \sigma}{\partial t}.$$

Similar reasoning can be applied to determine the equations for the torsional vibrations of a rod. Figure B-2 shows an incremental section of a rod undergoing torsional displacement.

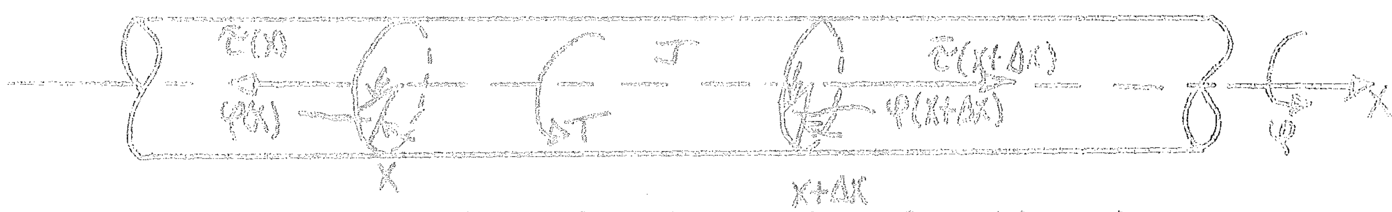


Figure B-2. Relevant to discussing the torsion of a thin rod. T is the applied torque per unit length and J is the moment of inertia about the x-axis per unit length, of the rod.

Shearing stresses acting upon the cross-sections at x and x + Δx produce torques, respectively, $\vec{\tau}(x) = -\tau(x)\,\vec{a}_x$, $\vec{\tau}(x+\Delta x) = \tau(x+\Delta x)\,\vec{a}_x$. Newton's law for rotational motion of the incremental section between x and x + Δx states

(10) $$\tau(x+\Delta x) - \tau(x) + T\Delta x = J\Delta x\,\frac{\partial^2\varphi}{\partial t^2},$$

where T is the applied body torque (due to external, or non-elastic, agents) per-unit-length and J is the moment of inertia about the center line (x-axis) per-unit-length of rod. Upon dividing by Δ x and passage to the limit $\Delta x \to 0$, (10) becomes

(11) $$\frac{\partial\tau}{\partial x} + T = J\,\frac{\partial^2\varphi}{\partial t^2}.$$

There is a relation between torque, τ, and <u>twist</u>, $\frac{\partial\varphi}{\partial x}$, analogous to (7)

(12) $$\tau = C\,\frac{\partial\varphi}{\partial x}$$

where C is a constant called the <u>coefficient of torsional rigidity</u>. Substitution of (12) into (11) yields the wave equation for torsional motion

(13) $$C\,\frac{\partial^2\varphi}{\partial x^2} + T = J\,\frac{\partial^2\varphi}{\partial t^2}.$$

Equations (11) and (12) can be brought into analogy with the electrical transmission-line equations by differentiating (12) with respect to time and defining the angular velocity $\omega(x,t) = \partial\varphi/\partial t$. Thus,

(14) (a) $$\frac{\partial\tau}{\partial x} = J\,\frac{\partial\omega}{\partial t}$$

(b) $$\frac{\partial\omega}{\partial x} = \frac{1}{C}\,\frac{\partial\tau}{\partial t},$$

where we have set the applied torque-per-unit length, T, equal to zero.

Our treatment of the electrical transmission line and other distributed systems, up to this point, has dealt with the <u>sinusoidal steady-state</u>. We have been concerned only with finding the response, or wave propagation, due to a sinusoidal input that has been "switched on" for an infinite length of time.

Clearly, there exist many classes of non-sinusoidal steady-state systems that must be investigated, because it is important to know how transmission lines respond to pulses, initial conditions or "recently switched on" signals. It is the purpose of this chapter to develop mathematical tools for the analysis of transient distributed systems and then study the variety of phenomena that result when the tools are applied to the investigation of these systems.

1. <u>Elementary solutions of the initially charged line.</u>

Consider the equations for the lossless transmission line

$$(1\text{-}1)\qquad \frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}\,, \qquad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}\,,$$

which together yield the second order wave equation for v(x,t) or i(x,t).

$$(1\text{-}2)\qquad \frac{\partial^2 v}{\partial x^2} = \frac{1}{v_p^2}\frac{\partial^2 v}{\partial t^2}\,, \qquad \frac{\partial^2 i}{\partial x^2} = \frac{1}{v_p^2}\frac{\partial^2 i}{\partial t^2}\,, \qquad v_p = \frac{1}{\sqrt{LC}}\,.$$

A general solution of the wave equation is

$$(1\text{-}3)\qquad v(x,t) = f(x+v_pt) + g(x-v_pt)\,,$$

where $f(\xi)$ and $g(\xi)$ are arbitrary (twice differentiable) functions. Upon substitution of (1-3) into (1-1) we can solve for i(x,t):

$$-L\frac{\partial i}{\partial t} = f'(x+v_pt) + g'(x-v_pt)\,,$$

where the prime denotes differentiation with respect to the argument $\xi = x \pm v_pt$ of the functions $f(\xi)$ and $g(\xi)$. It follows that

$$(1\text{-}4)\qquad i(x,t) = \frac{f(x+v_pt)}{-Lv_p} + \frac{g(x-v_pt)}{Lv_p}$$

$$= \sqrt{\frac{C}{L}}\left[g(x-v_pt) - f(x+v_pt)\right]$$

$$= 1/Z_0\left[g(x-v_pt) - f(x+v_pt)\right],$$

where $Z_0 = \sqrt{\frac{L}{C}}$ is, of course, the <u>characteristic impedance</u> of the transmission line.

Before proceeding, let us prove the statement that (1-3) satisfies the wave equation. The proof is an elementary application of the chain rule of calculus and proceeds as follows

$$\frac{\partial v}{\partial x} = f'(x+v_p t) + g'(x-v_p t)$$

$$\frac{\partial^2 v}{\partial x^2} = f''(x+v_p t) + g''(x-v_p t)$$

$$\frac{\partial v}{\partial t} = \frac{\partial f(x+v_p t)}{\partial (x+v_p t)} \cdot \frac{\partial (x+v_p t)}{\partial t} + \frac{\partial g(x-v_p t)}{\partial (x-v_p t)} \cdot \frac{\partial (x-v_p t)}{\partial t}$$

$$= v_p f'(x+v_p t) - v_p g'(x-v_p t)$$

$$\frac{\partial^2 v}{\partial t^2} = v_p^2 f''(x+v_p t) + v_p^2 g''(x-v_p t)$$

$$= v_p^2 \left[f''(x+v_p t) + g''(x-v_p t) \right]$$

$$= v_p^2 \frac{\partial^2 v}{\partial x^2},$$

which concludes the proof.

Suppose, now, that the following <u>initial conditions</u> are prescribed on the infinitely long line at $t = 0$:

$$(1\text{-}5) \qquad v(x,t) = v_0(x), \quad \frac{\partial v(x,t)}{\partial t} = -\frac{1}{C}\frac{\partial i(x,t)}{\partial x} = u_0(x), \quad t = 0,$$

i.e., we specify the <u>initial value of voltage</u> and the <u>initial value of the rate-of-change</u> of voltage (which is proportional to the space rate-of-change of the initial value of <u>current</u>) at $t = 0$.

Upon substituting (1-5) into (1-3) we obtain the relations

$$(1\text{-}6) \qquad v_0(x) = f(x) + g(x), \quad v_p f'(x) - v_p g'(x) = u_0(x).$$

The latter equation could have been obtained by applying the second of (1-5) to (1-4). Upon integrating the second of (1-6) we obtain

$$(1\text{-}7) \qquad f(x) - g(x) = \frac{1}{v_p}\int_a^x u_0(\xi)\, d\xi$$

where ξ is a "dummy variable" of integration (having the dimensions of x) and a is an arbitrary lower limit of integration.

[illegible] and [illegible] furnish us with linear equations in the two unknowns f(x), g(x), which unknowns will be determined in terms of the given data $v_0(x)$, $u_0(x)$:

$$f(x) + g(x) = v_0([illegible])$$

$$f(x) - g(x) = \frac{1}{v_p}\int_a^x u_0(\xi)\,d\xi \ .$$

These equations can be solved by elimination to yield

$$(1\text{-}8) \qquad f(x) = \frac{1}{2} v_0(x) + \frac{1}{2v_p}\int_a^x u_0(\xi)\,d\xi$$

$$g(x) = \frac{1}{2} v_0(x) - \frac{1}{2v_p}\int_a^x u_0(\xi)\,d\xi \ .$$

Finally, upon substituting these results back into (1-3) we obtain

$$(1\text{-}9) \qquad v(x,t) = \frac{1}{2}\left[v_0(x+v_pt) + v_0(x-v_pt) + \frac{1}{v_p}\int_{x-v_pt}^{x+v_pt} u_0(\xi)\,d\xi\right] .$$

This equation may be reduced by substituting for $u_0(\xi)$, $-\frac{1}{c}\frac{\partial i}{\partial \xi}$, where $i_0(x)$ is the initial value of current on the line (recall the second of (1-5)), and then carrying out the integral of the derivative

$$(1\text{-}10) \quad v(x,t) = \frac{1}{2}\left[v_0(x-v_pt) + z_0 i_0(x-v_pt) + v_0(x+v_pt) - z_0 i_0(x+v_pt)\right]$$

Example 1. Let $v_0(x)$ be the spatial pulse $v_0(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$, and assume the initial current, $i_0(x)$, vanishes. Then, from (1-10)

$$v(x,t) = \frac{1}{2}\left[v_0(x-v_pt) + v_0(x+v_pt)\right] .$$

The resulting voltage waveforms on an infinite transmission line are shown in Figure 1-1. The dotted lines of slope ± 1 are called characteristic lines. They indicate geometrically how a disturbance propagates in the two dimensional (x, vpt) "space-time continuum".

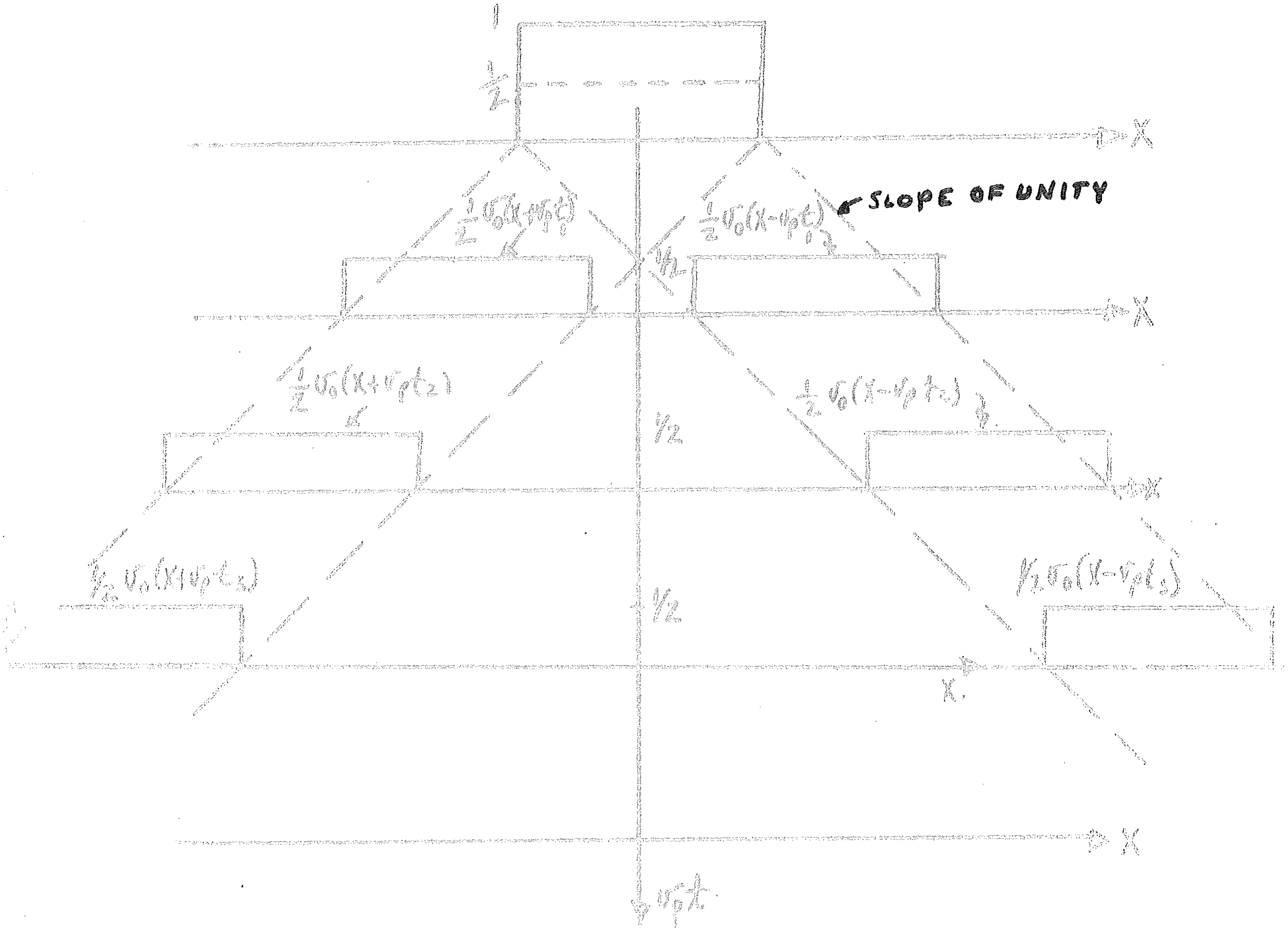


Figure 1-1. Illustrating propagation of an initial voltage along the characteristic curves (dotted lines). Note the "space-time continuum" axes (x, vpt).

[illegible] contributes a wave traveling to the left with increasing t, while $\frac{1}{2} V_0(x-v_p t)$ contributes a wave traveling to the right.

If, now, the transmission line is terminated at $x = 0$ in a short circuit, and remains infinitely extended for $x>0$, then it is called semi-infinite. The initial conditions, (1-5), must be supplemented by a boundary condition at $x = 0$. Hence, we have the initial-boundary value problem data

$$v(x,t) = V_0(x), \quad \frac{\partial v(x,t)}{\partial t} = -\frac{1}{c} \frac{\partial i(x,t)}{\partial x} = U_0(x), \quad x \geq 0, \; t = 0,$$

$$\text{(1-11)} \qquad v(x,t) = 0, \quad \frac{\partial v(x,t)}{\partial t} = 0, \quad t \geq 0, \; x = 0.$$

Because $V_0(x)$ is defined only for positive values of its argument, it follows that the solutions (1-9) or (1-10) are no longer valid because $V_0(x-v_p t)$ has no meaning when $t > x/v_p$.

The troublesome situation is due to the finite boundary $x = 0$. Let us, therefore, consider an infinite line subject to the initial conditions (no boundary conditions because there are no boundaries)

$$v(x,t) = V_0(x), \quad \frac{\partial v(x,t)}{\partial t} = -\frac{1}{c} \frac{\partial i(x,t)}{\partial x} = U_0(x), \quad t = 0$$

where

$$V_0(x) = \begin{cases} V_0(x), & x \geq 0 \\ -V_0(-x), & x < 0 \end{cases}$$

$$U_0(x) = \begin{cases} U_0(x), & x \geq 0 \\ -U_0(-x), & x < 0 \end{cases}$$

Hence, $V_0(x)$ and $U_0(x)$ are odd functions of the argument x, which now ranges over the fully infinite line $-\infty < x < \infty$. According to (1-10), the solution of this modified (infinitely long) problem is

$$\text{(1-12)} \qquad v(x,t) = \frac{1}{2} \left[V_0(x-v_p t) + Z_0 I_0(x-v_p t) + V_0(x+v_p t) - Z_0 I_0(x+v_p t) \right],$$

where $I_0(x)$, the initial current on the line, is proportional to the integral of $U_0(x)$. Because $U_0(x)$ is odd, its integral, $I_0(x)$, is an even function of its argument.

When $x = 0$,

$$v(0,t) = \frac{1}{2} \left[V_0(-v_p t) + Z_0 I_0(-v_p t) + V_0(v_p t) - Z_0 I_0(v_p t) \right] = 0$$

$$\frac{\partial v(0,t)}{\partial t} = \frac{1}{2} \left[-v_p V_0'(-v_p t) - v_p Z_0 I_0'(-v_p t) + v_p V_0'(v_p t) - v_p Z_0 I_0'(v_p t) \right] = 0$$

The first identity results when we recall that V_0 is an odd function of its argument, so that $V_0(\xi) = -V_0(-\xi)$, and I_0 is an even function, so that $I_0(\xi) = I_0(-\xi)$. The second identity is due to the fact that the derivative, V_0', of the odd function, V_0, is even whereas, the derivative, I_0', of the even function, I_0, is odd. Hence, (1-12) satisfies the boundary condition of (1-11) at $x = 0$.

It is a straightforward matter to verify that the initial conditions of (1-11) are satisfied by (1-12), when use is made of the definitions of $V_0(x)$, $\mathcal{J}_0(x)$. This will be left as an exercise.

<u>Example 2.</u> Let the semi-infinite line carry no initial current, so that $i_0(x) = 0$. Then, from (1-12)

$$v(x,t) = \frac{1}{2}\left[V_0(x - v_p t) + V_0(x + v_p t) \right]$$

$$= \frac{1}{2}\left[v_0(x - v_p t) + v_0(x + v_p t) \right], \quad x \geq v_p t.$$

$$= \frac{1}{2}\left[v_0(x + v_p t) - v_0(v_p t - x) \right], \quad x \leq v_p t.$$

where we have used the definition of $V_0(x)$ in terms of $v_0(x)$.

The graphical representation of the wave motion is obtained in Figure 1-2 similarly to the representation in Figure 1-1, except now we introduce an <u>image initial voltage distribution</u> for $x < 0$, which is just the negative of the <u>real initial voltage distribution</u> for $x > 0$. The resulting wave disturbances of the real and image distributions superposed upon each other ensure the vanishing voltage at $x = 0$. The method of images finds wide-spread use in boundary value problems, especially when the boundary condition is that of the vanishing of a field disturbance.

Note, also, the usual role played by the characteristic lines.

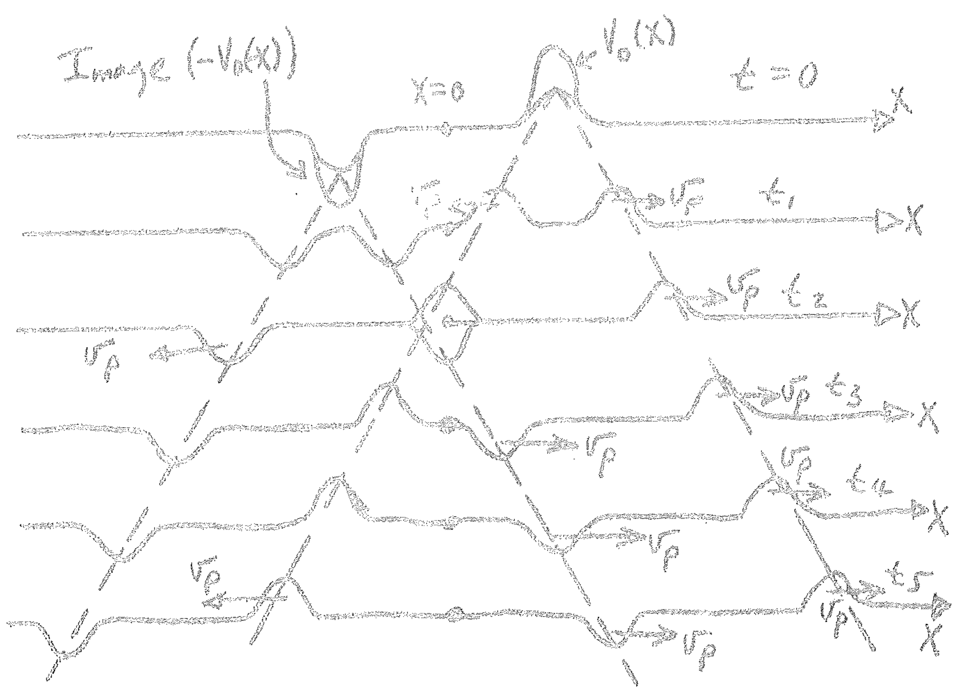


Figure 1-2. Illustrating the superposition of disturbances due to the real ($x>0$) and image ($x<0$) initial conditions. The resulting wave motion satisfies the boundary condition $v(x,t) \equiv 0, \quad x \geq 0$.

2. The Laplace Transform Method of Solution.

The solutions (1-9), (1-10) and (1-12) of the preceding section were originally given by d'Alembert in his analysis of wave motions on strings. They stress the role played by the initial conditions $v(x,0)=v_0(x)$, $\frac{\partial v(x,0)}{\partial t} = a_0(x)$ or $i(x,0)=i_0(x)$.

It is not so easy to apply d'Alembert's approach to lossy or dispersive transmi on lines, so we must look for other methods of analysis. Because we are familiar with the method of the Laplace transform in lumped parameter, linear systems we ask if it is applicable for transmission lines. It is, as we shall now see.

The Laplace transform of a function of two variables, v(x,t), is defined in precisely the same manner as for a function of the single time variable t, except now the transformation function is not only dependent on s, the complex frequency variable, but on x, as well. Thus,

(1-13) $$V(x,s) = \mathcal{L}\left[v(x,t)\right] = \int_0^\infty v(x,t)\, e^{-st}\,dt,$$

As far as derivatives of v(x,t) and the corresponding Laplace transforms are concerned, we have the following results

(1-14)

$$\text{(a)}\quad \mathcal{L}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] = \int_0^\infty \frac{\partial^2 v(x,t)}{\partial x^2} e^{-st}dt = \frac{d^2}{dx^2}\int_0^\infty v(x,t)e^{-st}dt = \frac{d^2V(x,s)}{dx^2}.$$

$$\text{(b)}\quad \mathcal{L}\left[\frac{\partial v(x,t)}{\partial t}\right] = \int_0^\infty \frac{\partial v(x,t)}{\partial t} e^{-st}dt = sV(x,s) - v(x,0) = sV(x,s) - v_0(x)$$

$$\text{(c)}\quad \mathcal{L}\left[\frac{\partial^2 v(x,t)}{\partial t^2}\right] = \int_0^\infty \frac{\partial^2 v(x,t)}{\partial t^2} e^{-st}dt = s^2V(x,s) - s\,v(x,0) - \frac{\partial v(x,0)}{\partial t}$$

$$= s^2V(x,s) - s\,v_0(x) - a_0(x).$$

The last two relations follow from the usual properties of the Laplace transform of time derivatives*, while the first simply states that transforming the time variable does nothing to the spatial variable, so that spatial derivatives are not affected by the transformation.

The beauty in applying the Laplace transform to initial value problems is that the initial data, $v_0(x)$, $u_0(x)$, are automatically involved in the solution via (1-14) (b)(c) when transformations are taken of time derivatives in a differential equation.

We may work with the transformations of either the system of first order partial differential equations, (1-1), or the second order wave equation (1-2). Taking the Laplace transform of the former yields

$$\text{(1-15)} \quad \begin{aligned} &(a) \quad \frac{dV(x,s)}{dx} = -sL\,I(x,s) + L\,i_0(x) \\ &(b) \quad \frac{dI(x,s)}{dx} = -sCV(x,s) + C v_0(x), \end{aligned}$$

and of the latter,

$$\text{(1-16)} \quad \frac{d^2V(x,s)}{dx^2} - \frac{s^2}{v_p^2}V(x,s) = -\frac{s}{v_p^2}v_0(x) - \frac{u_0(x)}{v_p^2}.$$

In (1-15), $I(x,s)$ is the transform of $i(x,t)$ and $i_0(x)$ is the initial current distribution along the line. An important distinction between the Laplace transform of an unknown variable in lumped systems and that in distributed systems is that in the former the transformed equations are algebraic, whereas in the latter (e.g., (1-15) or (1-16)) they are differential equations in the spatial variable x.

Let us now apply the above results to some examples.

<u>Example 3.</u> Redo Example 2 using the Laplace transform.

We can start with either (1-15) or (1-16). Let us start with (1-16). If $i_0(x)$ is identically zero, then by the second equation (1-5) it follows that $u_0(x)$ vanishes identically. Hence, (1-16) becomes

$$\text{(1-17)} \quad \frac{d^2V}{dx^2} - \frac{s^2}{v_p^2}V = -\frac{s}{v_p^2}v_0(x).$$

---

* G. R. Cooper and C. D. McGillem, *Methods of Signal and System Analysis*, chapter 6, Holt, Rinehart and Winston, 1967.

This equation is inhomogeneous, the initial data $v_0(x)$ acting as a spatial forcing function. In order to solve this equation we will use some physical intuition in order to establish the form of the solution.

In Figure 1-3 we illustrate the partitioning of $v_0(x)$ into strips Δx_i units wide and $v_0(x_i)$ units high. We then associate with the <u>ith</u> strip a delta function source of magnitude (remember the "magnitude" of a delta function is really the area contained within it) $v_0(x_i)\Delta x_i$. Then, we argue, if we know the response at point x of (1-17) to a unit delta function placed at point x_i, it follows that the response at point x due to a delta function of magnitude $v_0(x_i)\Delta x_i$ placed at x_i will be $v_0(x_i)\Delta x_i$ times the unit delta function response. Finally, when we sum up all of the delta functions and their associated responses we get the solution of (1-17).

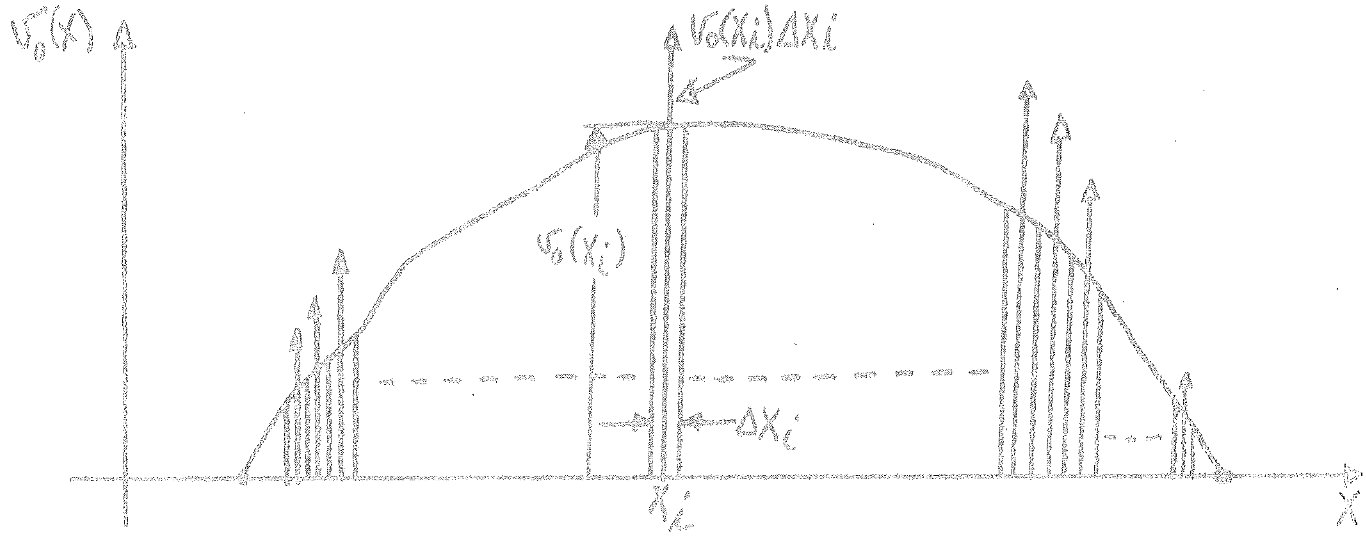


Figure 1-3. Showing the decomposition of $v_0(x)$ into equivalent delta function sources.

Thus, let the solution of (1-17) at x, when the righthand side is a unit delta function located at x_i, be $G(x|x_i)$. This function is also called the <u>Green's function</u> for (1-17). Then by superposition, the solution of (1-17) is given by

$$v(x) = -\frac{s}{v_p^2}\sum_i G(x|x_i)\,v_0(x_i)\,\Delta x_i \longrightarrow -\frac{s}{v_p^2}\int_0^\infty G(x|x')\,v_0(x')\,dx'. \tag{1-18}$$

The variable point x' is called the <u>source point</u> while x is called the <u>field</u> point.

Our job now is to determine the Green's function $G(x|x')$, which by our reasoning above, must satisfy

$$\frac{d^2G(x|x')}{dx^2} - \frac{s^2}{v_p^2}G(x|x') = \delta(x-x') \tag{1-19}$$

as well as the boundary condition ($G(0|x') = 0$ for all x'. This condition follows from the assumed short circuit placed at $x = 0$.

Another important condition that must be satisfied by G can be determined from the following consideration. Integrate (1-19) with respect to x from $x'-\varepsilon$ to $x'+\varepsilon$

$$\text{(1-20)} \quad \int_{x'-\varepsilon}^{x'+\varepsilon} \frac{d^2 G(x|x')}{dx^2} dx - \frac{s^2}{v_p^2}\int_{x'-\varepsilon}^{x'+\varepsilon} G(x|x') dx = \int_{x'-\varepsilon}^{x'+\varepsilon} \delta(x-x') dx$$

The first term of (1-20) approaches $\frac{dG(x'_+|x')}{dx} - \frac{dG(x'_-|x')}{dx}$ as $\varepsilon \to 0$, where x'_+ denotes the approach to x' from the right, while x'_- denotes the approach from the left. Because $G(x|x')$ is to be continuous in x, the second term of (1-20) vanishes as $\varepsilon \to 0$, while the righthand side equals $+1$. Hence, (1-20) yields the <u>discontinuity</u> condition on the derivative of $G(x|x')$ at $x = x'$.

$$\text{(1-21)} \quad \frac{dG(x'_+|x')}{dx} - \frac{dG(x'_-|x')}{dx} = +1.$$

In summary, then, $G(x|x')$ satisfies (1-19), which can also be written

$$\text{(1-22)} \quad \text{(a)} \quad \frac{d^2 G}{dx^2} - \frac{s^2}{v_p^2} G = 0, \qquad \text{for } x \neq x'$$

as well as the boundary conditions

$$\text{(1-22)} \quad \text{(b)} \quad \frac{dG(x'_+|x')}{dx} - \frac{dG(x'_-|x')}{dx} = +1$$

(discontinuity in slope at delta function location)

(c) $G(0|x') = 0$, for all x'. (short circuit at $x=0$)

(d) $G(x'_+|x') = G(x'_-|x')$ (continuity of G at $x=x'$)

A solution which satisfies (1-22) (a) and (c) is

$$G(x|x') = 2A \sinh(sx/v_p), \qquad x < x'$$

where we consider this only for the region $x < x'$ because this region includes the left-most x point, the origin, $x=0$. A is an arbitrary constant.

A solution that represents a right-going wave for $x > x'$, i.e., propagating to the right of the source point x' is

$$G(x|x') = B e^{-sx/v_p}, \qquad x > x'.$$

Note that if we set $s = j\omega$, this expression would correspond to a sinusoidal wave propagating to the right. If we used the solution e^{sx/v_p} and let $s = j\omega$ we would obtain a sinusoidal wave traveling to the left.

The remaining conditions (1-22) (b) and (d) are easily applied. Eqn. (1-22)(d) implies that

$$2A \sinh(sx'/v_p) = Be^{-sx'/v_p},$$

while (1-22)(b) yields the relation

$$-s/v_p\, B e^{-sx'/v_p} - 2A\, s/v_p \cosh(sx'/v_p) = +1.$$

Observe the functions used in the appropriate regions $x > x'$ or $x < x'$. These two equations can be easily solved for A and B

$$A = -\frac{v_p}{s} \cdot \frac{e^{-sx'/v_p}}{2}, \qquad B = -\frac{v_p}{s} \sinh(sx'/v_p),$$

with the resulting expressions for $G(x|x')$.

$$G(x|x') = -\frac{v_p}{s} e^{-sx/v_p} \sinh\left(\frac{sx'}{v_p}\right), \qquad x > x' \tag{1-23}$$

$$= -\frac{v_p}{s} e^{-sx'/v_p} \sinh\left(\frac{sx}{v_p}\right), \qquad x < x'.$$

Note that if we interchange x and x' in (1-23), we get the same expressions, i.e., $G(x|x') = G(x'|x)$. This is interpreted as a <u>reciprocity theorem</u>: if a unit delta function at point x' yields a certain response at x, then the unit delta function at x will yield the same response at x'.

Upon substituting (1-23) into (1-18), after breaking up the range of integration in (1-18) into the ranges $0 < x' < x$ and $x < x' < \infty$, we obtain the solution for the Laplace transformed voltage

$$V(x,s) = \frac{1}{v_p}\left\{\int_0^x e^{-sx/v_p} \sinh\left(\frac{sx'}{v_p}\right) v_0(x')\,dx' + \int_x^\infty \sinh\left(\frac{sx}{v_p}\right) e^{-sx'/v_p} v_0(x')\,dx'\right\}$$

$$= \frac{1}{2v_p}\left\{\int_0^x \left[e^{-s(-x'+x)/v_p} - e^{-s(x'+x)/v_p}\right] v_0(x')\,dx' \right. \tag{1-24}$$

$$\left. + \int_x^\infty \left[e^{-s(-x+x')/v_p} - e^{-s(x'+x)/v_p}\right] v_0(x')\,dx'\right\}.$$

Each of the terms involving the variable s is of the form $e^{-s\tau}$ whose inverse Laplace transform is $\delta(t-\tau)$. Hence, when we take the inverse Laplace transform of (1-24) we get

$$v(x,t) = \mathcal{L}^{-1}\{V(x,s)\} = \frac{1}{2}\left\{\int_0^x \left[\delta\left(t-\frac{x}{v_p}+\frac{x'}{v_p}\right)-\delta\left(t-\frac{x}{v_p}-\frac{x'}{v_p}\right)\right]v_0(x')\frac{dx'}{v_p}\right.$$

(1-25)

$$\left. + \int_x^\infty \left[\delta\left(t+\frac{x}{v_p}-\frac{x'}{v_p}\right)-\delta\left(t-\frac{x}{v_p}-\frac{x'}{v_p}\right)\right]v_0(x')\frac{dx'}{v_p}\right\}.$$

This expression may be easily reduced further by consideration of the location of the delta functions. Consider, for example, $\delta\left(t-\frac{x}{v_p}+\frac{x'}{v_p}\right)$ which is located at the point $x' = -v_p t + x$. When $0 < v_p t < x$, this point lies between 0 and x, so that the first integral, because it involves a delta function lying within its range of integration, contributes to v(x,t), the expression $\frac{1}{2}\cdot v_0(x-v_pt)$. Similarly $\delta\left(t+x/v_p-x'/v_p\right)$, because it lies at $x'=v_pt+x$, contributes to the second integral an amount $\frac{1}{2}\cdot v_0(x+v_pt)$. Neither of the other delta functions contributes when $v_pt < x$. Hence, for $v_pt < x$, we obtain the result

$$v(x,t) = \frac{1}{2}\left[v_0(x-v_pt)+v_0(x+v_pt)\right].$$

The same type of thinking applied to the remaining delta functions leads to the conclusion that $v(x,t) = \frac{1}{2}\left[v_0(x+v_pt)-v_0(v_pt-x)\right]$ for $v_pt > x$. Upon comparing these results with those for Example 2, we see that the answers are the same, although the approaches are entirely different.

Although it appears that we have spent a lot of time getting the same answer by using the Laplace transform, we will find that in many cases this method is virtually the only way to get answers systematically to initial value problems. In addition we introduced some important ideas, such as the Green's function, and that always takes time.

Example 4. Find the voltage at any point x and time t on a finite transmission line short-circuited at x=0 and x=L when $v(x,0) = v_0(x)$ and $i(x,0) = 0$.

The solution of this problem proceeds in exactly the same way as that for Example 3 except now the Green's function must vanish at x = L as well as at x = 0. The remaining conditions in (1-22) remain unchanged. [illegible] Hence, two functions which satisfy (1-22)(a) and (c), as well as the condition that $G(L|x') = 0$ are

$$G(x|x') = A\sinh(sx/v_p), \qquad x < x'$$

$$= B\sinh\left(\frac{s(L-x)}{v_p}\right), \qquad x > x'.$$

Note that $L > x'$, because L is the righthand boundary. Hence, that solution for $G(x|x')$ which vanishes at $x=L$ must correspond to the range $x > x'$.

Continuity of $G(x|x')$ at $x \to x'_{\pm}$ $\big((1\text{-}22)(d)\big)$ requires that

$$A \sinh(sx'/v_p) = B \sinh\left(s(L-x')/v_p\right).$$

Finally, the discontinuity in slope condition (1-22)(b) implies that

$$-B\frac{s}{v_p}\cosh\left[\frac{s}{v_p}(L-x')\right] - A\frac{s}{v_p}\cosh\left(\frac{sx'}{v_p}\right) = 1.$$

Solution of these two equations for A and B yields

$$A = \frac{-\sinh\left[\frac{s}{v_p}(L-x')\right]}{\frac{s}{v_p}\sinh\left(\frac{sL}{v_p}\right)}, \qquad B = \frac{-\sinh\left(\frac{sx'}{v_p}\right)}{\frac{s}{v_p}\sinh\left(\frac{sL}{v_p}\right)},$$

so that $G(x|x')$ becomes

(1-26)
$$G(x|x') = \frac{-\sinh\left[\frac{s}{v_p}(L-x')\right]\sinh\left(\frac{sx}{v_p}\right)}{\frac{s}{v_p}\sinh\left(\frac{sL}{v_p}\right)}, \qquad x < x'$$

$$= \frac{-\sinh\left(\frac{sx'}{v_p}\right)\sinh\left[\frac{s}{v_p}(L-x)\right]}{\frac{s}{v_p}\sinh\left(\frac{sL}{v_p}\right)}, \qquad x > x'$$

Once again we deduce the reciprocity relation alluded to earlier; $G(x|x') = G(x'|x)$, which means that the <u>response</u> at point <u>x</u> due to a unit delta <u>source</u> at <u>x'</u> is equal to the <u>response</u> at <u>x'</u> due to a unit delta <u>source</u> at <u>x</u>.

Upon substituting (1-26) into (1-18), after breaking up the range of integration into the ranges $0 < x' < x$ and $x < x' < L$, where we now note that the range of integration of x' extends over the finite interval $[0, L]$, we obtain the solution for $V(x,s)$

(1-27)
$$V(x,s) = \frac{1}{v_p}\left\{\int_0^x \frac{\sinh\left[\frac{s}{v_p}(L-x)\right]\sinh\left(\frac{sx'}{v_p}\right)}{\sinh(sL/v_p)}\, v_0(x')\,dx' + \int_x^L \frac{\sinh\left(\frac{sx}{v_p}\right)\sinh\left[\frac{s}{v_p}(L-x')\right]}{\sinh(sL/v_p)}\, v_0(x')\,dx'\right\}.$$

The development from this point on is quite similar to that of Example 3, (1-24), when we recall the exponential form of $\sinh\theta$:

$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}.$$

We easily expand the numerators of the integrands in (1-27) into the sum of products of exponentials. The denominator is written

$$\frac{e^{sL/v_p}}{2}\left(1-e^{-2sL/v_p}\right),$$

which, when taken into the numerator becomes $\frac{e^{-sL/v_p}}{2}\left(1+e^{-2sL/v_p}+e^{-4sL/v_p}+e^{-6sL/v_p}+\dots\right)$.

In arriving at this we have used the result $\frac{1}{1-X} = 1+X+X^2+X^3+\dots$, and put $X = e^{-2sL/v_p}$.

The result of all the algebra is that we rewrite (1-27) as

$$V(x,s) = \frac{1}{2v_p}\Big\{\int_0^x\Big[e^{-s/v_p(x-x')}-e^{-s/v_p(x+x')}+e^{-s/v_p(2L+x-x')}-e^{-s/v_p(2L+x+x')}$$

$$+e^{-s/v_p(2L-x+x')}-e^{-s/v_p(2L-x-x')}+\dots\Big]v_0(x')\,dx'$$

(1-28)

$$+\int_x^L\Big[e^{-s/v_p(x'-x)}-e^{-s/v_p(x'+x)}+e^{-s/v_p(2L+x'-x)}-e^{-s/v_p(2L+x'+x)}$$

$$+e^{-s/v_p(2L-x'+x)}-e^{-s/v_p(2L-x'-x)}+\dots\Big]v_0(x')\,dx'\Big\}.$$

Now taking inverse Laplace transforms gives us a series of delta functions with which to contend (as before in (1-25)).

(1-29) $$v(x,t) = \mathcal{L}^{-1}\{V(x,s)\} = \frac{1}{2v_p}\Big\{\int_0^x\Big[\delta\Big(t-\frac{x}{v_p}+\frac{x'}{v_p}\Big)-\delta\Big(t-\frac{x}{v_p}-\frac{x'}{v_p}\Big)+\delta\Big(t-\frac{2L}{v_p}-\frac{x}{v_p}+\frac{x'}{v_p}\Big)$$

$$-\delta\Big(t-\frac{2L}{v_p}-\frac{x}{v_p}-\frac{x'}{v_p}\Big)+\delta\Big(t-\frac{2L}{v_p}+\frac{x}{v_p}-\frac{x'}{v_p}\Big)-\delta\Big(t-\frac{2L}{v_p}+\frac{x}{v_p}+\frac{x'}{v_p}\Big)+\dots\Big]v_0(x')\,dx'$$

$$+\int_x^L\Big[\delta\Big(t-\frac{x'}{v_p}+\frac{x}{v_p}\Big)-\delta\Big(t-\frac{x'}{v_p}-\frac{x}{v_p}\Big)+\delta\Big(t-\frac{2L}{v_p}-\frac{x'}{v_p}+\frac{x}{v_p}\Big)-\delta\Big(t-\frac{2L}{v_p}-\frac{x'}{v_p}-\frac{x}{v_p}\Big)$$

$$+\delta\Big(t-\frac{2L}{v_p}+\frac{x'}{v_p}-\frac{x}{v_p}\Big)-\delta\Big(t-\frac{2L}{v_p}+\frac{x'}{v_p}+\frac{x}{v_p}\Big)+\dots\Big]v_0(x')\,dx'\Big\}.$$

In order to better see what is going on, let us focus our attention on the voltage at the point x = L/2 as a function of time. Thus, in (1-29) we put x = L/2 and then argue, as we did below equation (1-25), that only certain of the delta functions will contribute to the two integrals as time increases.

For example, $\delta(t - L/2v_p + x'/v_p)$ will contribute to the first integral only when $0 < v_p t < L/2$, whereas $\delta(t - x'/v_p + L/2v_p)$ contributes to the second integral during this same time interval.

$-\delta(t - \ldots - x'/v_p)$ [illegible] contributes to the first integral only for $L < v_p t < \frac{3L}{2}$ which is the same time interval during which $-\delta(t - L/2v_p - x'/v_p)$ contributes to the second integral.

In fact the reader may check the result that

$$v\left(\tfrac{L}{2}, t\right) = \frac{1}{2}\left\{v_0\left(\tfrac{L}{2} - v_p t\right) + v_0\left(\tfrac{L}{2} + v_p t\right)\right\}, \quad 0 < v_p t < \tfrac{L}{2}$$

$$= -\frac{1}{2}\left\{v_0\left(v_p t - \tfrac{L}{2}\right) + v_0\left(\tfrac{3L}{2} - v_p t\right)\right\}, \quad \tfrac{L}{2} < v_p t < L$$

(1-30)

$$= -\frac{1}{2}\left\{v_0\left(\tfrac{3L}{2} - v_p t\right) + v_0\left(v_p t - \tfrac{L}{2}\right)\right\}; \quad L < v_p t < \tfrac{3L}{2}$$

$$= \frac{1}{2}\left\{v_0\left(v_p t - \tfrac{3L}{2}\right) + v_0\left(\tfrac{5L}{2} - v_p t\right)\right\}, \quad \tfrac{3L}{2} < v_p t < 2L.$$

$$= \frac{1}{2}\left\{v_0\left(\tfrac{5L}{2} - v_p t\right) + v_0\left(v_p t - \tfrac{3L}{2}\right)\right\}, \quad 2L < v_p t < \tfrac{5L}{2}$$

An interpretation of these results can be made along the lines of Fig. 1-2 wherein we have "bouncings" or "reflections" of an incident pulse from each of the boundaries x = 0 and L. Every time the voltage pulse is reflected it is inverted, thus giving rise to the change in signs shown in (1-30). Figure 1-4 displays the infinite bouncing phenomenon for a given initial waveform

One way of visualizing the generation of the waveform in Fig. 1-4 (b) is to start at L/2 in Fig. 1-4(a) and send out two messengers, one going to the right and the other to the left. As they travel they "collect" voltages, summing their values at time t to produce the appropriate waveform. When they come to a boundary, they "bounce off" and invert their meters to read the negative of what they were able to read before reaching a boundary. Their meters are calibrated to read 1/2 the original waveform values, $v_0(\xi)$.

Fig. 1-4. Multiple boundings or reflections of an initial disturbance when the boundaries x = 0 and L are short circuits. x = L/2 is the fixed point of observation. (a) Initial waveform (b)

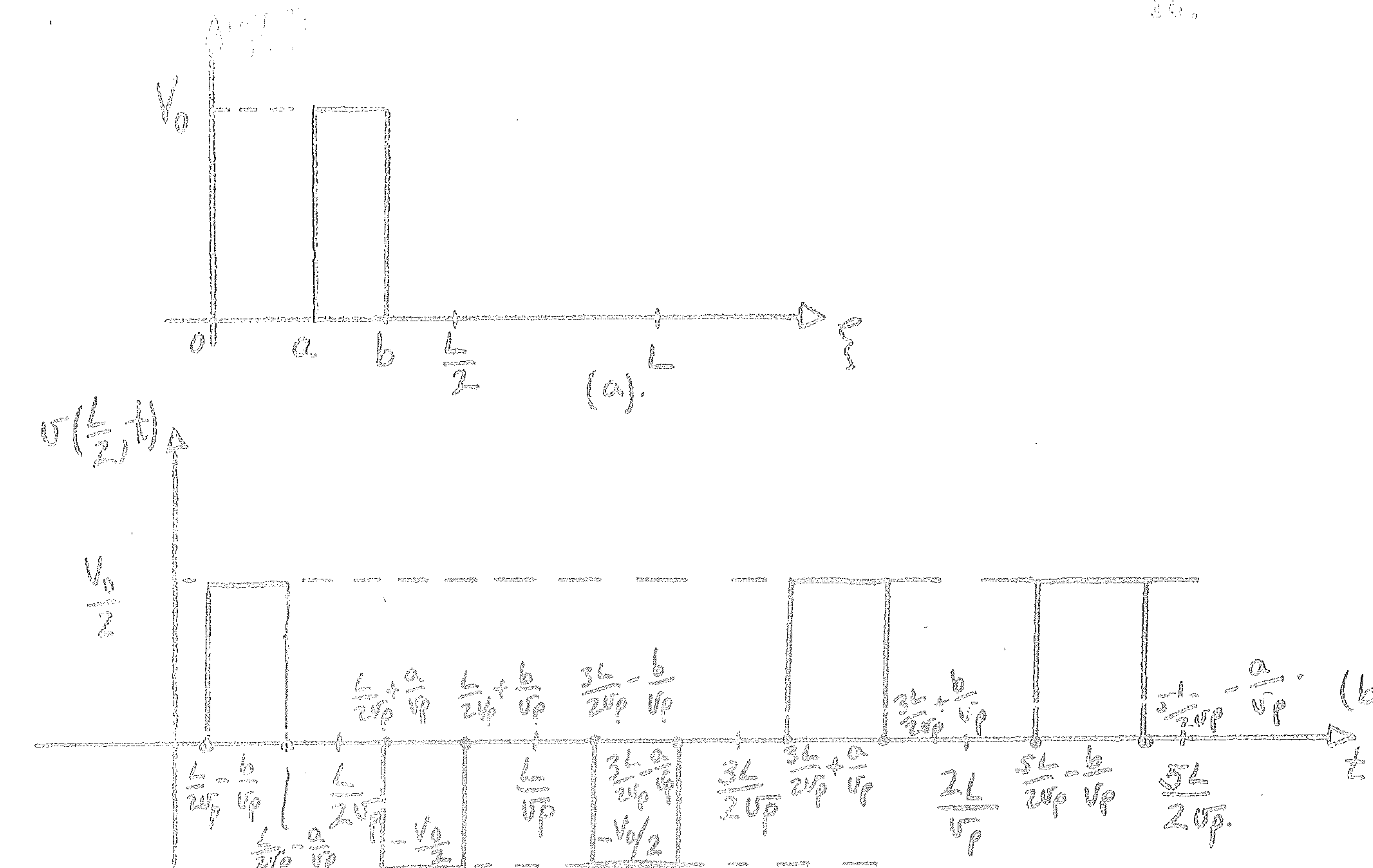


Figure 1-4. Multiple bouncings or reflections of an initial disturbance when the boundaries $x = 0$ and L are short circuits. $x = L/2$ is the fixed point of observation. (a) Initial waveform $v_0(\xi)$. (b) $v(\frac{L}{2},t)$.

The arrows in (1-30) illustrate this "bounce diagram" by giving the values of voltage read by each messenger at time t. Of course, an alternative interpretation can be made in terms of image sources to the left of $x = 0$ and to the right of $x = L$, but the picture now will be more complicated than in Fig. 1-2 because the images will be reflected from their "image boundaries" as well.

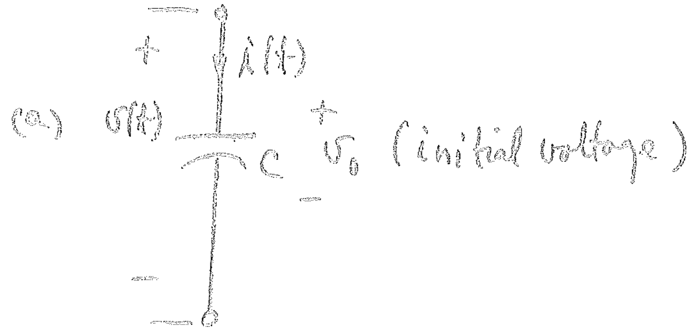
The significance of the preceding examples is that the initial conditions on the transmission line (i.e., $i_0(x)$, $v_0(x)$) can be modeled as independent sources of either voltage or current distributed according to the function $i_0(x)$ or $v_0(x)$. That such modeling can be done should not surprise us because of our familiarity with the corresponding replacement of initial conditions by sources in lumped circuits.

For example, Fig. 1-5(a) shows a capacitor initially charged to v_0 volts, but with no external sources present.

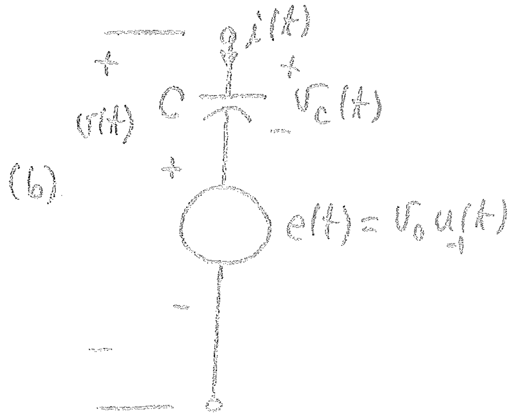
In the time domain, the appropriate equation is $i(t) = C\,dv/dt$, where v is the capacitor voltage. Upon Laplace transformation this equation becomes $I(s) = sCV(s) - Cv_0$.

In part (b) of the same figure we remove the initial charge (or voltage) on the capacitor and introduce the series source $e(t) = v_0 u_{-1}(t)$ with the polarity shown. Here $u(t)$ is the unit step function. The time domain equation now becomes $i(t) = C\,dv_c/dt = C\frac{d}{dt}[v(t) - v_0 u_{-1}(t)] = C\frac{dv}{dt} - Cv_0\delta(t)$.

If we assume that $v(0+)=0$, then the Laplace transformed version of this last equation is $I(s) = sCV(s) - Cv_0$, in agreement with the result of part (a).



$$i(t) = C\frac{dv}{dt};$$
$$I(s) = sCV(s) - Cv_0.$$



$$i(t) = C\frac{dv_c}{dt} = C\frac{d}{dt}\left[v(t) - v_0 u_{-1}(t)\right]$$
$$= C\frac{dv}{dt} - Cv_0\,\delta(t);$$
$$I(s) = sCV(s) - Cv_0.$$

Figure 1-5. Showing the modeling of a capacitor (as viewed from the terminals) possessing an initial charge (a) by an uncharged capacitor, (b) in series with a unit step voltage source. The Laplace transformed terminal variables $(v(t), i(t))$ satisfy the same equation in either case.

Let us see what this modeling does for our distributed system (lossless transmission line). In Fig. 1-6 we show an increment of transmission line in which (a) there is an initial voltage on the line (which is equivalent to charging the distributed capacitors).

Part (b) of the figure shows the replacement of the initial voltage by the source $v_0 u_{-1}(t)$, corresponding to Fig. 1-5(b).

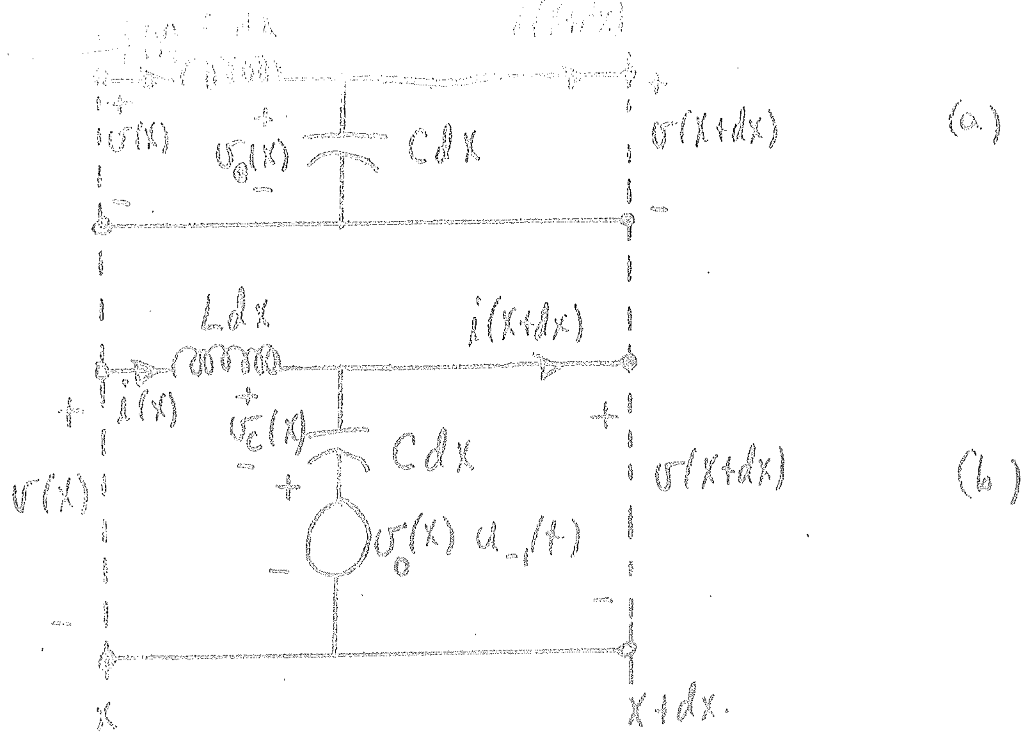


Figure 1-6. Replacing an initially charged (to voltage $v_0(x)$) increment of transmission line (a) by an increment of line containing a voltage source $v_0(x)\,u_{-1}(t)$, (b). In part (b) $v_0(x)$ is initially zero.

The transmission-line equations for Fig. 1-6(b) are easily obtained by using Kirchoff's voltage and current laws. Thus Kirchoff's voltage law yields $v(x) = L\,dx \cdot \frac{\partial}{\partial t} i(x+dx, t) + v(x+dx)$,

which by rearrangement, division by dx, and passage to the limit $dx \to 0$, yields the usual result $\partial v/\partial x = -L\,\partial i/\partial t$. Thus, the presence of the hypothetical voltage source does not affect this particular one of the telegrapher's equations.

Kirchoff's current law, on the other hand, yields

$$i(x) = C\,dx \frac{\partial}{\partial t} v_c(x,t) + i(x+dx) = i(x+dx) + C\,dx \frac{\partial}{\partial t}\left[v(x,t) - v_0(x)\,u_{-1}(t)\right]$$

$$= i(x+dx) + C\,dx \frac{\partial v}{\partial t} - C\,dx\, v_0(x)\,\delta(t).$$

This equation can be rearranged, as before, to yield the modified telegrapher's equation

$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} + C v_0(x)\,\delta(t).$$

Hence, we see quite clearly that the system of equations which results when we model initial conditions by source voltages is no longer homogeneous but includes the inhomogeneous term $C v_0(x)\,\delta(t)$:

(1-31) $$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}, \qquad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} + C v_0(x)\,\delta(t).$$

Upon taking the Laplace transform of (1-31), assuming that $i(x,0+) = 0$, $v(x,0+) = 0$, we get exactly the same results that we had in (1-15) when $i_0(x) \equiv 0$, thus demonstrating the equivalence of an <u>initial value problem</u> (such as (1-15)) to a transmission line with <u>spatially distributed impulsive</u> (in time) <u>sources</u>. Problems involving sources are usually treated by means of Green's functions techniques, as we have already demonstrated.

3. <u>Boundary Excitation</u>.

The usual means of exciting transmission lines is not via initial conditions and the concomitant distributed impulsive sources, although in many problems the initial conditions are present and have to be dealt with, but rather by attaching a voltage or current source at one end (as, for example, a radio transmitter or a receiving antenna). Such excitation we will call <u>boundary excitation</u> to distinguish it from the aforementioned distributed (or volume) excitation.

A. <u>Finite lossless line</u>. One of the simplest problems that we can analyze, and yet one that is very instructive in that it illustrates pulse reflection and transmission phenomena quite well, is shown in Fig. 1-7. This arrangement can be easily implemented and demonstrated in the laboratory.

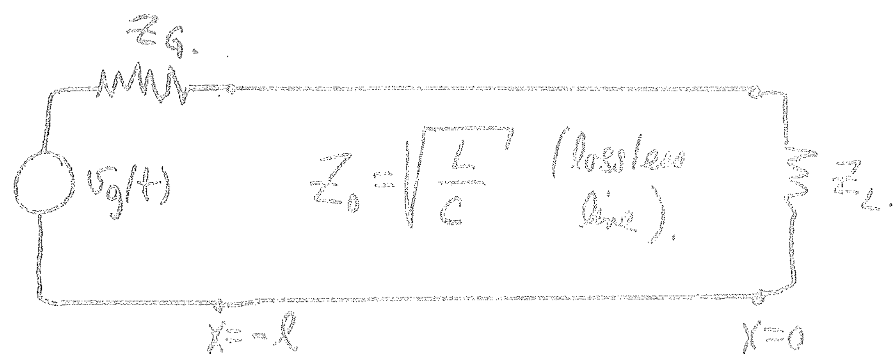


Figure 1-7. A generator with internal impedance Z_G exciting a lossless transmission line loaded at $x = 0$ by Z_L.

We have a source $v_g(t)$, with internal impedance Z_G, exciting (at $x = -\ell$) a lossless transmission line which is loaded at $x = 0$ by Z_L. Our interest is in determining the voltage waveform that would be observed on an oscilloscope when connected across the generator terminals at $x = -\ell$. Of course, we could inquire about the voltage anywhere along the line, as well. The line will be taken to be initially relaxed, i.e., all initial voltages and currents on the line are zero.

Our analysis starts with the first order system (1-1) or its second order wave equation equivalents (1-2) $\partial^2 v/\partial x^2 = (1/v_p^2)\,\partial^2 v/\partial t^2$.

Upon taking the Laplace transform, as in (1-16), and setting $v_0(x) = u_0(x) \equiv 0$, because of our assumption of initial relaxation, we obtain

$$\frac{d^2 V(x,s)}{dx^2} = \left(\frac{s}{v_p}\right)^2 V(x,s).$$

The solution of this equation, of course, is immediately known

(1-32) $$V(x,s) = V_+ e^{-(s/v_p)x} + V_- e^{(s/v_p)x},$$

where V+ and V- are arbitrary constants (i.e., they are independent of x but will depend on the transform variable, s). We have already pointed out in our discussion of the Green's function in Example 3 that if $s = j\omega$, then the first term on the righthand side of (1-32) corresponds to a positive going wave (traveling to the right in Fig. 1-7), while the second term corresponds to a negatively traveling wave. Hence, V+ and V- stand for, respectively, the magnitude of the net positively and negatively traveling waves.

At the load, x = 0, we have from (1-32)

(1-33) $$V(0,s) = V_+ + V_- = V_+(1+\Gamma_L) = V_+\left(1 + \frac{V_-(s)}{V_+(s)}\right)$$ ← REFLECTED

where $\Gamma_L = \frac{V_-}{V_+}$ is the load-end reflection coefficient. In order to determine it we must solve for the current, $I(x,s)$, by substituting (1-32) into (1-15)(a), with $i_o(x) = 0$

$$I(x,s) = -\frac{1}{s\ell}\frac{\partial V(x,s)}{\partial x} = \frac{1}{s\ell}\left(\frac{s}{v_p}\right)\left[V_+ e^{-sx/v_p} - V_- e^{sx/v_p}\right]$$

$$= \frac{1}{\sqrt{\frac{\ell}{c}}}\left[V_+ e^{-sx/v_p} - V_- e^{sx/v_p}\right]$$

$$= \frac{1}{Z_0}\left[V_+ e^{-sx/v_p} - V_- e^{sx/v_p}\right].$$

Hence, the load current is given by $I(0,s) = \frac{1}{Z_0}[V_+ - V_-] = \frac{V_+}{Z_0}[1-\Gamma_L]$ which must equal $\frac{V(0,s)}{Z_L(s)} = \frac{V_+}{Z_L}(1+\Gamma_L)$. Upon equating these last two expressions we get $\frac{V_+}{Z_L}(1+\Gamma_L) = \frac{V_+}{Z_0}(1-\Gamma_L)$ or

(1-34) $$\Gamma_L = \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0},$$ IF PROPERLY TERMINATED, $\Gamma_L = 0$, $V_-(s) = 0$

which is exactly what we had derived for the sinusoidal steady-state reflection coefficient with $s = j\omega$.

Thus, (1-32) becomes

(1-35) $$V(x,s) = V_+\left[e^{-sx/v_p} + \Gamma_L e^{sx/v_p}\right], \quad \Gamma_L = \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0},$$

and I(x,s) can be written

(1-36) $$I(x,s) = \frac{V_+}{Z_0}\left[e^{-sx/v_p} - \Gamma_L e^{sx/v_p}\right]$$

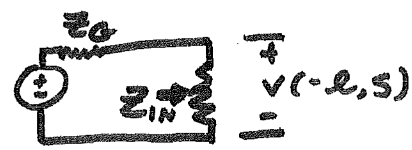
Using these two results we can define the impedance at point x (looking in the (+) x-direction) as

$$Z(x, s) = \frac{V(x,s)}{I(x,s)} = Z_0\left[\frac{e^{-sx/u_p} + \Gamma_L e^{sx/u_p}}{e^{-sx/u_p} - \Gamma_L e^{sx/u_p}}\right]. \tag{1-37}$$

In particular, $Z(0,s) = Z_0\left[\frac{1+\Gamma_L}{1-\Gamma_L}\right] = Z_L(s)$, and the impedance seen at the terminals of the generator becomes

$$Z_{in}(s) = Z(-\ell,s) = Z_0\left[\frac{e^{s\ell/u_p} + \Gamma_L e^{-s\ell/u_p}}{e^{s\ell/u_p} - \Gamma_L e^{-s\ell/u_p}}\right] \tag{1-38}$$

$$= Z_0\left[\frac{1+\Gamma_L e^{-2s\ell/u_p}}{1-\Gamma_L e^{-2s\ell/u_p}}\right].$$



Referring to Fig. 1-7, and according to the "voltage divider" equation

$$V(-\ell,s) = \frac{Z_{in}}{Z_G + Z_{in}} V_g(s) = \left(\frac{Z_0}{Z_0 + Z_G}\right)\frac{1+\Gamma_L e^{-2s\ell/u_p}}{1-\Gamma_G\Gamma_L e^{-2s\ell/u_p}} V_g(s), \tag{1-39}$$

where $V_g(s) = \mathcal{L}[v_g(t)]$, and $\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0}$ is the <u>generator-end reflection coefficient.</u>

Upon taking the inverse transform of (1-39), we will have the time response of the voltage at the transmission line input. To that end, let us expand the function $\frac{1}{1-\Gamma_G\Gamma_L \exp(-2s\ell/u_p)}$ into a geometric series, as we have previously done in <u>Example 4</u>:

$$\frac{1}{1-\Gamma_G\Gamma_L \exp(-2s\ell/u_p)} = 1 + \Gamma_G\Gamma_L e^{-2s\ell/u_p} + (\Gamma_G\Gamma_L)^2 e^{-4s\ell/u_p} + (\Gamma_G\Gamma_L)^3 e^{-6s\ell/u_p}$$
$$+ \cdots\cdots\cdots .$$

This enables us to write (1-39) in the form

(1-40)

(1-40) $$V(-\ell,s) = V_g(s)\left(\frac{Z_0}{Z_0+Z_G}\right)\left(1+\Gamma_L e^{-2s\ell/v_p}\right)\left(1+\Gamma_G\Gamma_L e^{-2s\ell/v_p} + (\Gamma_G\Gamma_L)^2 e^{-4s\ell/v_p} + (\Gamma_G\Gamma_L)^3 e^{-6s\ell/v_p} + (\Gamma_G\Gamma_L)^4 e^{-8s\ell/v_p} + \ldots\right)$$

$$= V_g(s)\left(\frac{Z_0}{Z_0+Z_G}\right)\left(1+\Gamma_L(1+\Gamma_G)e^{-2s\ell/v_p} + \Gamma_G\Gamma_L^2(1+\Gamma_G)e^{-4s\ell/v_p} + \Gamma_G^2\Gamma_L^3(1+\Gamma_G)e^{-6s\ell/v_p} + \ldots\right)$$

$$= V_g(s)\left(\frac{Z_0}{Z_0+Z_G}\right)\left[1+\Gamma_L T_G e^{-2s\ell/v_p} + \Gamma_G\Gamma_L^2 T_G e^{-4s\ell/v_p} + \Gamma_G^2\Gamma_L^3 T_G e^{-6s\ell/v_p} + \ldots + \right],$$

where $T_G = 1 + \Gamma_G = \dfrac{2Z_G}{Z_0+Z_G}$ is the <u>generator-end transmission coefficient</u>. It is a measure of the amount of the wave which is <u>incident at the generator end</u> (having <u>arrived from the load</u>) and that contributes to the oscilloscope reading at $x=-\ell$. Note that the ratio of the coefficient of the exponential in the second term in parentheses to that of the first is $\Gamma_L T_G$, whereas the ratio of the third to second is $\Gamma_G\Gamma_L$; this is also the ratio of the <u>nth</u> term to the (n-1)st, for $n>3$.

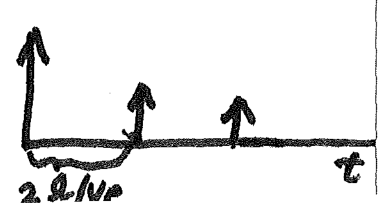
The result (1-40) is very easily interpretted physically. To facilitate this interpretation assume that Z_G and Z_L are purely resistive so that T_G, Γ_G and Γ_L are constants (independent of s). In addition, assume that $v_g(t) = V_0\,\delta(t)$ so that $V_g(s) = V_0$. Then

$$V(-\ell,s) = V_0\left(\frac{Z_0}{Z_0+Z_G}\right)\left[1+\Gamma_L T_G e^{-2s\ell/v_p} + \Gamma_G\Gamma_L^2 T_G e^{-4s\ell/v_p} + \Gamma_G^2\Gamma_L^3 T_G e^{-6s\ell/v_p} + \ldots\ldots\right],$$

and $v(-\ell,t)$ is given by taking the inverse transform of $V(-\ell,s)$ (and recalling that $\mathcal{L}^{-1}\left[Ae^{-s\tau}\right] = A\,\delta(t-\tau)$)

GEN REF
LOAD REF
FRACTION ABSORBED BY LOAD

(1-41) $$v(-\ell,t) = V_0\left(\frac{Z_0}{Z_0+Z_G}\right)\left[\delta(t) + \Gamma_L T_G\,\delta(t-2\ell/v_p) + \Gamma_G\Gamma_L^2 T_G\,\delta\!\left(t-\frac{4\ell}{v_p}\right) + \Gamma_G^2\Gamma_L^3 T_G\,\delta(t-6\ell/v_p) + \ldots\ldots\right].$$



The term $V_0\left(\frac{\quad}{Z_0+Z_G}\right)\delta(t)$ is simply the voltage divided

impulse which is the output of the generator and travels down the transmission line. It is reflected at x = 0 (with reflection coefficient Γ_L), and the fraction T_G of the reflected impulse is measured by the oscilloscope at the delayed (round-trip) time $2l/v_p$. This impulse is also reflected by the generator impedance (reflection coefficient Γ_G) and travels down the line again where it is reflected at the load and is measured $2l/v_p$ seconds later ($4l/v_p$ from the original time origin). This proces continuse indefinitely in the absence of losses in the line. Of course the higher order reflected impulses are not of the same amplitude unless $|\Gamma_L\Gamma_G| = 1$ (explain under what conditions can $|\Gamma_L\Gamma_G| = 1$, and what does this require of the system).

It is comforting to know that our mathematical maneuverings have led us to an intuitively plausible conclusion here as it did in the preceding examples. There will be other examples in which we will have no prior intuition as to what the results will be and we must then rely on the mathematics to help us develop intuitive insight into wave propagation. Finally, let us point out that if we have both boundary and distributed excitation, the net response is the superposition of the responses established by the boundary and distributed sources acting independently. This, of course, is the principle of superposition valid for linear systems.

B. Infinite lossy line. Up to now our considerations have been with the lossless transmission line, either terminated or infinitely long. There are important loss mechanisms in real transmission lines and these mechanisms have profound effect on the nature of the transient response of the line, especially if the line is very long. The simplest loss mechanisms are series resistance and shunt conductances.

Let us derive the equations for the general, lossy line by referring to Fig. 1-8 which shows an incremental section of the line.

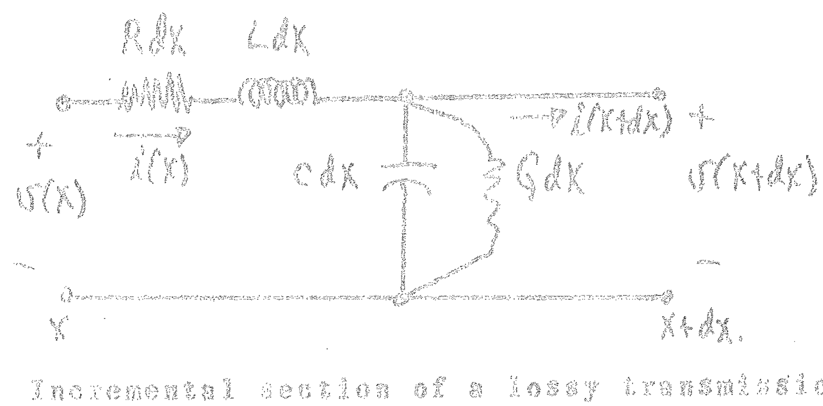


Figure 1-8. Incremental section of a lossy transmission line showing the series resistance-per-unit length, R, and shunt conductance-per-unit length, G.

Kirchoff's voltage law yields

$$v(x) = i(x)\,R\,dx + L\,dx\,\frac{\partial i(x)}{\partial t} + v(x+dx)$$

which upon rearrangement, division by dx and passage to the limit $dx \rightarrow 0$ yields

$$\frac{\partial v}{\partial x} = -iR - L\frac{\partial i}{\partial t}.$$

Continuation of Art. 3A. (p. 23) CAPACITIVE LOAD.

Let us return to (1-40) and continue to assume that $v_g(t) = V_0\delta(t)$ and that Z_G is real, but now let us relax the simplifying assumption that Z_L is resistive and let it be capacitive. Hence, $Z_L = 1/SC$, where C is the value of the capacitor, and

$\tau = Z_0 C$

$$\Gamma_L = \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0} = \frac{\frac{1}{sC} - Z_0}{1/sC + Z_0} = \frac{1 - sZ_0C}{1 + sZ_0C}.$$

Then

$$v(-l,s) = V_0\left(\frac{Z_0}{Z_0+Z_G}\right)\left[1 + T_G\frac{1-sZ_0C}{1+sZ_0C}e^{-2sl/v_p}\right.$$

$$\left. + \Gamma_G T_G\left(\frac{1-sZ_0C}{1+sZ_0C}\right)^2 e^{-4sl/v_p} + \Gamma_G^2 T_G\left(\frac{1-sZ_0C}{1+sZ_0C}\right)^3 e^{-6sl/v_p} + \cdots\right],$$

and, upon taking the inverse Laplace transform, we get

$$v(-l,t) = V_0\left(\frac{Z_0}{Z_0+Z_G}\right)\left[\delta(t) + \mathcal{L}^{-1}\left\{\frac{1-sZ_0C}{1+sZ_0C}\right\}\bigg|_{t-2l/v_p} \cdot u_{-1}(t-2l/v_p)\right.$$

$$+ \Gamma_G T_G \mathcal{L}^{-1}\left\{\left(\frac{1-sZ_0C}{1+sZ_0C}\right)^2\right\}\bigg|_{t-\frac{4l}{v_p}} \cdot u_{-1}(t-4l/v_p) + \Gamma_G^2 T_G \mathcal{L}^{-1}\left\{\left(\frac{1-sZ_0C}{1+sZ_0C}\right)^3\right\}\bigg|_{t-6l/v_p} \cdot u_{-1}(t-6l/v_p)$$

$$\left. + \cdots\cdots \right],$$

where $\mathcal{L}^{-1}\left\{\frac{1-sZ_0C}{1+sZ_0C}\right\}\bigg|_{t-2l/v_p}$ means to calculate the inverse transform of $\frac{1-sZ_0C}{1+sZ_0C}$ and set the time argument equal to $t - 2l/v_p$.

A similar meaning holds for the remaining expressions.

The calculation of the inverse transforms is straightforward but soon becomes cumbersome. We proceed to evaluate the first three. Thus,

$$\mathcal{L}^{-1}\left\{\frac{1-s z_0 C}{1+s z_0 C}\right\} = \mathcal{L}^{-1}\left\{-1+\frac{2}{1+s z_0 C}\right\} = -\delta(t)+\frac{2}{z_0 C}e^{-t/z_0 C}.$$

The second inverse transform can be computed by appealing to the convolution theorem which states that if $\mathcal{L}^{-1}\{F(s)\}=f(t)$, $\mathcal{L}^{-1}\{G(s)\}=g(t)$, then $\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)\,d\tau$. Hence,

$$\mathcal{L}^{-1}\left\{\left(\frac{1-s z_0 C}{1+s z_0 C}\right)^2\right\} = \mathcal{L}^{-1}\left\{\left(\frac{1-s z_0 C}{1+s z_0 C}\right)\left(\frac{1-s z_0 C}{1+s z_0 C}\right)\right\} = \int_0^t\left(-\delta(\tau)+\frac{2}{z_0 C}e^{-\tau/z_0 C}\right)\left(-\delta(t-\tau)+2/z_0 C\, e^{-(t-\tau)/z_0 C}\right)d\tau$$

$$= \int_0^t\left[\delta(\tau)\delta(t-\tau)-\frac{2}{z_0 C}e^{-\tau/z_0 C}\cdot\delta(t-\tau)-\frac{2}{z_0 C}\delta(\tau)e^{-(t-\tau)/z_0 C}+\frac{4}{(z_0 C)^2}e^{-t/z_0 C}\right]d\tau$$

$$= \delta(t)-\frac{4}{z_0 C}e^{-t/z_0 C}+\frac{4t}{(z_0 C)^2}e^{-t/z_0 C}$$

$$= \delta(t)-\frac{4}{z_0 C}e^{-t/z_0 C}\left(1-\frac{t}{z_0 C}\right).$$

The inverse transform of $\left(\frac{1-s z_0 C}{1+s z_0 C}\right)^3 = \left(\frac{1-s z_0 C}{1+s z_0 C}\right)\left(\frac{1-s z_0 C}{1+s z_0 C}\right)^2$ is simply obtained by convolving $-\delta(t)+\frac{2}{z_0 C}e^{-t/z_0 C}$ with the last result, $\delta(t)-\frac{4}{z_0 C}e^{-t/z_0 C}\left(1-\frac{t}{z_0 C}\right)$.

Thus,

$$\mathcal{L}^{-1}\left\{\left(\frac{1-sZ_0C}{1+sZ_0C}\right)^3\right\} = \int_0^t \left(-\delta(\tau)+\frac{2}{Z_0C}e^{-\tau/Z_0C}\right)\left(\delta(t-\tau)-\frac{4e^{-(t-\tau)/Z_0C}}{Z_0C}\cdot\left[1-\frac{t-\tau}{Z_0C}\right]\right)d\tau$$

$$= -\delta(t)+\frac{e^{-t/Z_0C}}{Z_0C}\left[6-\frac{12t}{Z_0C}+\frac{4t^2}{(Z_0C)^2}\right].$$

Hence, upon retaining the first three terms in the infinite series expansion of (1-40), we arrive at our time-response

$$v(-\ell,t) = V_0\left(\frac{Z_0}{Z_0+Z_G}\right)\left\{\delta(t)+T_G\left[-\delta\left(t-\frac{2\ell}{v_p}\right)+\frac{2}{Z_0C}e^{-(t-2\ell/v_p)/Z_0C}\cdot u_{-1}(t-2\ell/v_p)\right.\right.$$

$$+\Gamma_G T_G\left[\delta\left(t-\frac{4\ell}{v_p}\right)-\frac{4}{Z_0C}e^{-\left(t-\frac{4\ell}{v_p}\right)/Z_0C}\cdot\left(1-\frac{t-4\ell/v_p}{Z_0C}\right)\cdot u_{-1}(t-4\ell/v_p)\right]$$

(1-41)(a)

$$+\Gamma_G^2 T_G\left[-\delta\left(t-\frac{6\ell}{v_p}\right)+\frac{2}{Z_0C}e^{-(t-6\ell/v_p)/Z_0C}\left(3-\frac{6(t-6\ell/v_p)}{Z_0C}+\frac{2(t-6\ell/v_p)^2}{(Z_0C)^2}\right)\cdot u_{-1}(t-6\ell/v_p)\right]$$

$$\left.+\cdots\right].$$

Upon comparing this result with (1-41), we see that the capacitor introduces an exponentially decaying "tail series", each term of which starts $2\ell/v_p$ seconds (the round trip time) after the immediately preceding one.

If, in (1-41), we set $Z_L=0$ (short-circuited load), then $\Gamma_L=-1$ and the time-response would be the same as (1-41)(a) <u>without the exponential tail series</u>. Thus, the delta functions in (1-41)(a) arise because the capacitor terminating the transmission-line appears as a short circuit load to the transmission-line <u>at the instant of arrival of each delta function</u>. After the arrival of the delta function, the capacitor begins to discharge through the transmission-line's characteristic impedance Z_0. Hence, the appearance of the time-constant Z_0C.

Finally, we shall leave as an exercise the verification that as $C\to 0$ or ∞, (1-41)(a) reverts to (1-41) with Z_L equal respectively to $0, \infty$, and hence, $\Gamma_L=-1$ or $+1$, respectively.

Application of Kirchoff's current law implies

$$i(x) = C\,dx\,\frac{\partial v(x+dx)}{\partial t} + G\,dx\cdot v(x+dx) + i(x+dx),$$

which upon rearrangement, division by dx and passage to the limit $dx \to 0$ yields

$$\frac{\partial i}{\partial x} = -Gv - C\frac{\partial v}{\partial t}.$$

These are the two generalized transmission line equations

(1-42) (a) $\dfrac{\partial v}{\partial x} = -iR - L\dfrac{\partial i}{\partial t}$, (b) $\dfrac{\partial i}{\partial x} = -Gv - C\dfrac{\partial v}{\partial t}$.

They may be combined into the single second order equation

(1-43) $$\frac{\partial^2 v}{\partial x^2} = RGv + (RC+LG)\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2}.$$

The effect of loss is to introduce a first order time derivative term $(RC + LG)\,\partial v/\partial t$ into the wave equation. In general, there will be a term proportional to v itself, RGv, but in many cases either R or G vanishes and then so does RGv.

Laplace transforming (1-43) in the usual way gives

(1-44) $$\frac{d^2V(x,s)}{dx^2} = RGV(x,s) + (RC+LG)\big(sV(x,s) - v(x,0)\big) + LC\left(s^2V(x,s) - sv(x,0) - \frac{\partial v(x,0)}{\partial t}\right).$$

In an <u>initial value problem</u> we are given the initial data $v_0(x) = v(x,0)$, $u_0(x) = \partial v(x,0)/\partial t$ and are asked to determine the transient response. Thus, we must rewrite (1-44) as an inhomogeneous, ordinary differential equation for $V(x,s)$:

(1-45). $$\frac{d^2V}{dx^2} - \left[RG + s(RC+LG) + s^2LC\right]V(x,s) = -\left[(RC+LG)v_0(x) + sLCv_0(x) + LCu_0(x)\right].$$

In this section we will assume the line to be initially relaxed, i.e., $v_0(x) = 0$, $u_0(x) = 0$, so that the inhomogeneous term (the right-hand side) vanishes in (1-45). Thus, $V(x,s)$ satisfies

(1-46) $$\frac{d^2V}{dx^2} - \left[RG + s(RC+LG) + s^2LC\right]V = 0,$$

so that

(1-47) $$V(x,s) = Ae^{\gamma x} + Be^{-\gamma x},$$

where A and B are arbitrary constants and

(1-48) $$\gamma(s) = \left(RG + s(RC+LG) + s^2LC\right)^{1/2} = (LC)^{1/2}\left[(s+\alpha)^2 - \delta^2\right]^{1/2},$$

where $\alpha = \frac{1}{2}(R/L + G/C)$, $\delta = \frac{1}{2}(R/L - G/C)$.

One of the terms in (1-47) increases without limit as $x \to \infty$; which one it is depends on the sign of the real part of $\gamma(s)$. If we take $Re\,\gamma(s) > 0$, then the first term in (1-47) "blows up" and we must set A = 0. Thus,

$$V(x,s) = Be^{-\gamma x},$$

To calculate B, we inquire about conditions at the sending end, x = 0. Clearly the Laplace transform of the voltage at the sending end is V(0,s) and is equal to B. Therefore

$$V(x,s) = V(0,s)\,e^{-\gamma x} \quad (1\text{-}49)$$

Case a: Distortionless (though lossy) line (RC = LG).

Under the conditions RC = LG, we have, from (1-48) that $\delta = 0$ and $\gamma(s) = (LC)^{1/2}(s+\alpha)$.

Hence,

$$V(x,s) = V(0,s)\,e^{-\alpha(LC)^{1/2}x} \cdot e^{-s(LC)^{1/2}x}$$

and

$$v(x,t) = \mathcal{L}^{-1}[V(x,s)] = e^{-\alpha(LC)^{1/2}x}\,v(0, t-(LC)^{1/2}x)\,u_{-1}(t-(LC)^{1/2}x). \quad (1\text{-}50)$$

In obtaining (1-50) we have used the fact that $\mathcal{L}^{-1}[V(s)e^{-s\tau}] = v(t-\tau)u_{-1}(t-\tau)$ where $u_{-1}(t-\tau)$ is the unit step function, with the step occurring at $t = \tau$, and $v(t) = \mathcal{L}^{-1}[V(s)]$.

Thus at the point x, we get a delayed and attenuated replica of the applied signal waveform at x = 0. Because the attenuation does not affect the shape of the received signal, we call a line with RC = LG a distortionless line. Amplification easily restores the amplitude. Fig. 1-9 illustrates the distortionless phenomenon. (on page 25-a - next page).

Case b: Line with distortion - the general case. Substitution of the general expression for $\gamma(s)$, (1-48), into (1-49) gives us

$$V(x,s) = V(0,s)\,e^{-(LC)^{1/2}[(s+\alpha)^2-\delta^2]^{1/2}x}$$

Let us first determine the impulse response of the line, i.e., take the sending-end voltage to be an impulse. Then V(0,s) = 1 and the impulse response h(x,t) is determined by reference to a table of inverse transforms to be

$$h(x,t) = \mathcal{L}^{-1}\left[e^{-(LC)^{1/2}[(s+\alpha)^2-\delta^2]^{1/2}x}\right]$$

(1-51)

$$= e^{-\alpha(LC)^{1/2}x}\,\delta(t-(LC)^{1/2}x) + \left[\frac{(LC)^{1/2}\delta x}{t_1}e^{-\alpha t}\cdot I_1(\delta t_1)\right]u_{-1}(t-(LC)^{1/2}x)$$

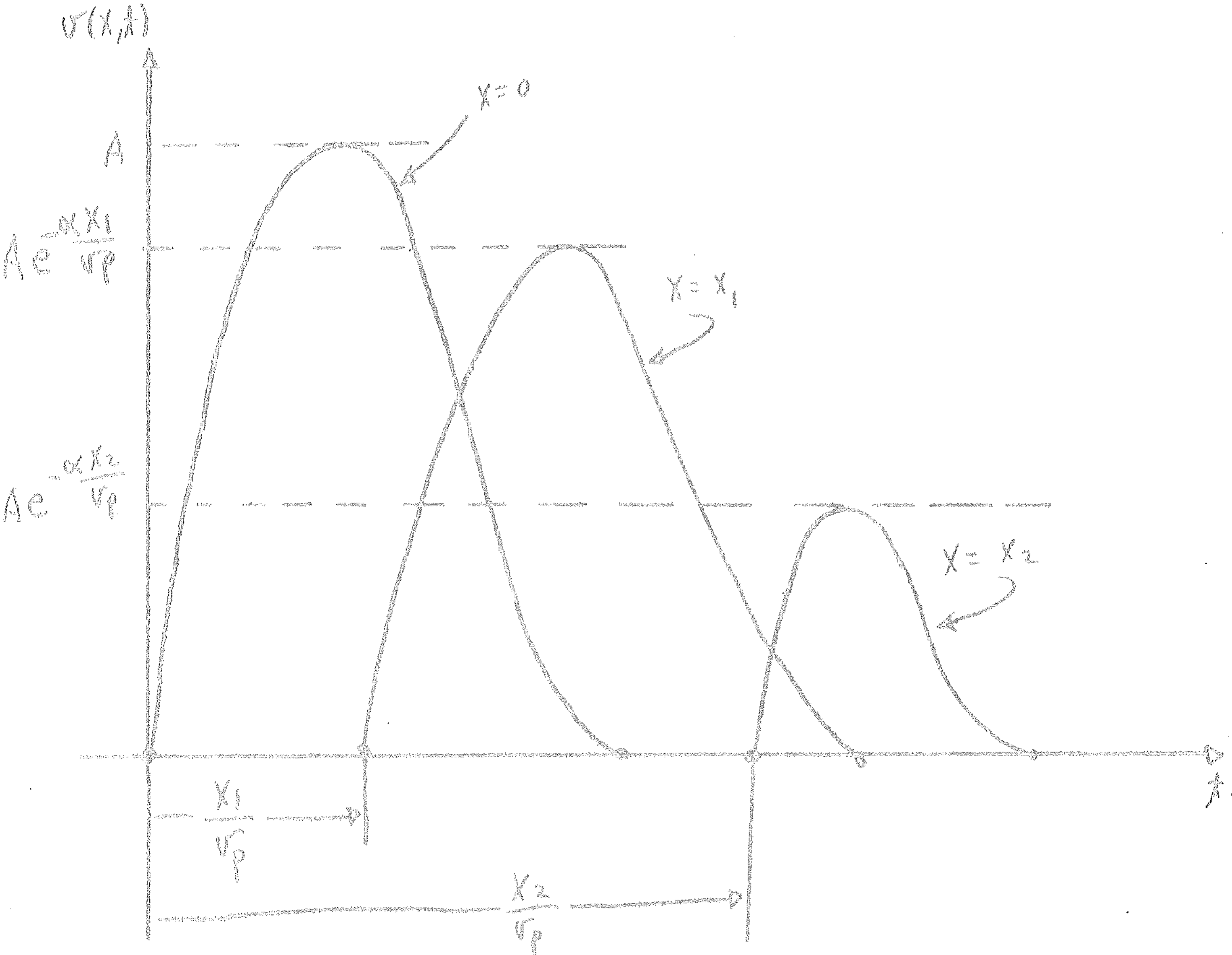


Figure 1-9. Illustrating the distortionless (though attenuated) propagation that results when RC = LG. Note that the waveforms are attenuated in amplitude by the factor $e^{-\alpha x/v_p}$ where $v_p = 1/(LC)^{1/2}$, but do not otherwise suffer a distortion.

where $t_1 = (t^2 - (LC)x^2)^{1/2}$ and $I_1(\delta t_1)$ is the <u>modified Bessel function of the first kind of order</u> 1 (Figure 1-10.)

Note in (1-51) that if $\delta = 0$, then we would have the distortionless line, previously considered. Thus, the second term (the Bessel function term) in (1-51) represents the distortion present when $\delta \neq 0$.

The interpretation of (1-51) is interesting. The <u>wavefront</u> (i.e., the δ-function) travels with velocity $1/(LC)^{1/2}$ as though the line were lossless. The attenuation of the wavefront occurs with the same spatially dependent factor, $\exp(-\alpha(LC)^{1/2}x)$, that appeared in the distortionless line. The remaining term in (1-51) represents the <u>time-decaying "tail"</u> that lingers at each point after the wavefront has passed.

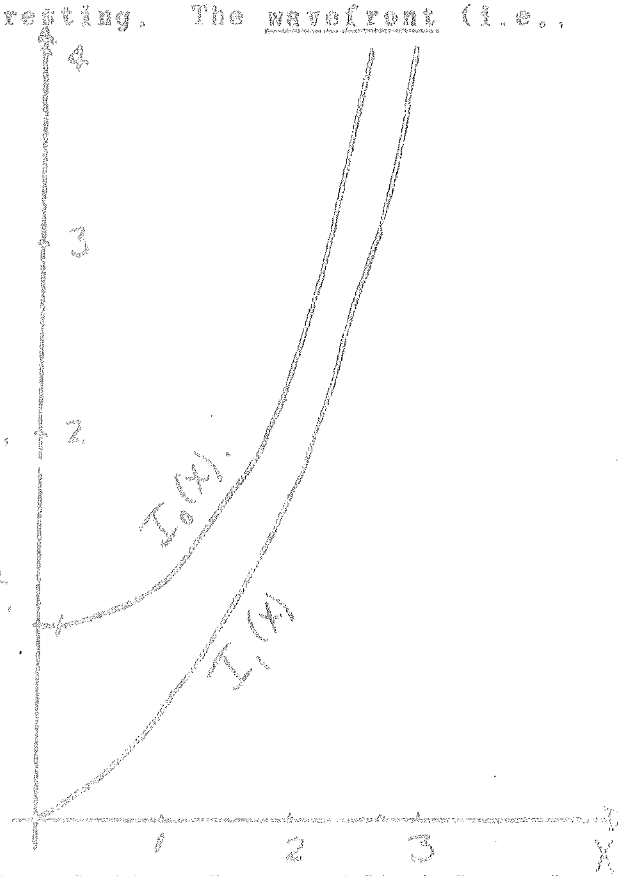


Fig. 1-10. The modified Bessel functions of the first kind, order 0 and 1, depending on the subscript.

The question may arise, why does the tail decay, when clearly in Fig. 1-10, $I_1(\delta t_1)$ increases without limit? The answer lies in the rapidity of increase of I_1 with t as $t \to \infty$. It can be shown that as $\delta t_1 \to \infty$, $I_1(\delta t_1)$ is <u>asymptotic</u> to (i.e., behaves as) the function $\frac{1}{\sqrt{2\pi\delta t}} e^{\delta t}$. Recalling our definition of δ from (1-48) we see that $\delta < \alpha$, so that $e^{-\alpha t} I_1(\delta t_1) \to 0$ as $t \to \infty$.

The current along the line can be easily derived, assuming impulsive excitation, by referring to the transformed version of (1-42)(b):

$$\frac{dI}{dx} = -(G+sC)V$$

$$= -(G+sC)e^{-(LC)^{1/2}[(s+\alpha)^2 - \delta^2]^{1/2}x}$$

Thus,

$$I(x,s) = \left[\frac{G+sC}{(LC)^{1/2}[(s+\alpha)^2 - \delta^2]^{1/2}}\right] e^{-(LC)^{1/2}[(s+\alpha)^2 - \delta^2]^{1/2} \cdot x}$$

(1-52)

$$= \left(\frac{G+sC}{R+sL}\right)^{1/2} e^{-(LC)^{1/2}[(s+\alpha)^2 - \delta^2]^{1/2} \cdot x},$$

where the latter form follows from rewriting $(LC)^{1/2}\left[(s+\alpha)^2-\delta^2\right]^{1/2}$ as $\left[(G+sC)(R+sL)\right]^{1/2}$. The <u>characteristic impedance of the general</u> (i.e., non-lossless) <u>transmission-line is</u>

$$(1\text{-}52)\,(a) \qquad Z_c(s) = \left(\frac{R+sL}{G+sC}\right)^{1/2}.$$

The inverse transform of (1-52) follows upon reference to a table of transforms

$$(1\text{-}53) \qquad i(x,t) = \left(\frac{C}{L}\right)^{1/2} e^{-\alpha(LC)^{1/2}x}\,\delta\!\left(t-(LC)^{1/2}x\right)$$

$$+\left(\frac{C}{L}\right)^{1/2}\left[e^{-\alpha t}\left\{\left(\frac{\delta t}{t_1}\right)I_1(\delta t_1) - \delta I_0(\delta t_1)\right\}\right]u_{-1}\!\left(t-(LC)^{1/2}x\right).$$

I_0 is the <u>modified Bessel function of the first kind of order zero</u> and is plotted in Fig. 1-10.

Knowing the impulse response, $h(x,t)$, we can find the response to an arbitrary, sending-end signal $v(0,t) = v_0(t)$ by appealing to the <u>convolution integral</u>

$$(1\text{-}54) \qquad v(x,t) = \int_0^t v_0(\tau)h(x,t-\tau)\,d\tau = \int_0^t v_0(t-\tau)\,h(x,\tau)\,d\tau.$$

Upon substitution of (1-51), this becomes

$$(1\text{-}55) \qquad v(x,t) = u_{-1}\!\left(t-(LC)^{1/2}x\right)\left\{e^{-\alpha(LC)^{1/2}x}\,v_0\!\left(t-(LC)^{1/2}x\right)\right.$$

$$\left.+(LC)^{1/2}\delta x\int_{(LC)^{1/2}x}^{t} v_0(t-\tau)e^{-\alpha\tau}\,\frac{I_1(\delta t_1)}{t_1}\,d\tau\right\}.$$

Again, the first term is just a spatially damped, delayed replica of the sending-end signal waveform. The remainder (the "tail"), for a general input $v_0(t)$ can generally only be computed numerically. The convolution integral can, of course, be applied to (1-53) to determine the current response to a non-impulsive voltage source.

4. <u>Transient Wave Propagation in a Waveguide.</u>

Distortion in a transmission-line is caused by the presence of losses in the conductor or dielectric separating the conductors. This was brought out in the last section when we showed that if $\delta = \frac{1}{2}\left(\frac{R}{L}-\frac{G}{C}\right) \neq 0$ then distortion occurred.

We now wish to consider a lossless transmission line which also introduces distortion (only now we will call it dispersion). The transmission line that we will derive serves as a model for the common rectangular, electromagnetic waveguide that is used in microwave systems. All waveguiding systems are dispersive to a certain extent, whether lossy or lossless, so the model under consideration will be useful for gaining insight into the nature of dispersion.

It is known in the theory of electromagnetic waveguides that there are two fundamental types of modes which can propagate: transverse electric (TE) and transverse magnetic (TM). Though the waveguide is a distributed system, not containing real lumped elements, it can be represented, as far as propagation characteristics pertaining to either the TE or TM modes are concerned, by the equivalent transmission line circuits containing distributed inductors and capacitors (Fig. 1-11).

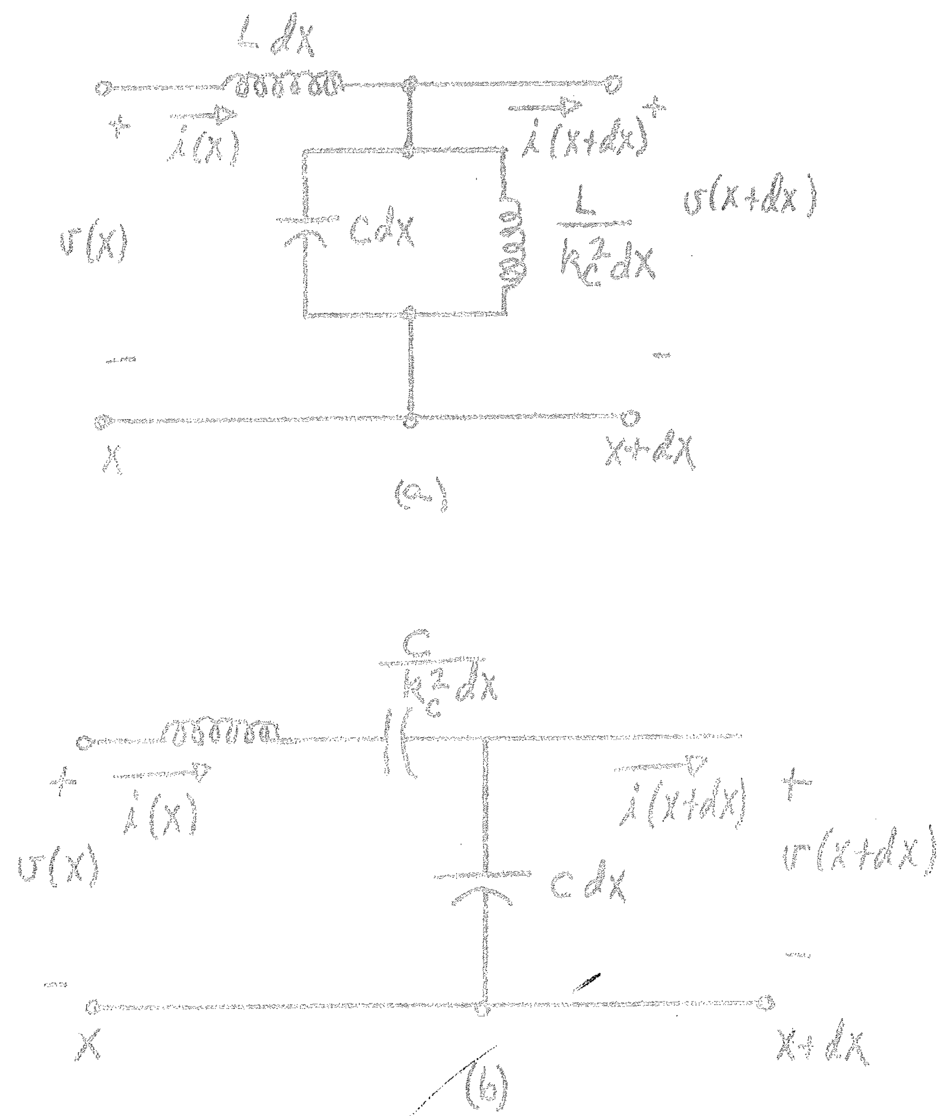


Figure 1-11. Equivalent transmission line circuits for a waveguide supporting a transverse electric (TE) wave (a) or transverse magnetic (TM) wave (b).

*S. Ramo, J.R. Whinnery and T. Van Duzer, "Fields and Waves in Communication Electronics," Chapt. 7, John Wiley and Sons, 1965.

In Figure 1-11, L and C are, respectively, inductance and capacitance per-unit-length, while k_c is called the "cut-off wavenumber" and has the units of $(\text{length})^{-1}$, typical of any wavenumber. k_c is determined by the dimensions of the rectangular cross-section of the guide.

Let us apply Kirchoff's laws to the Laplace transformed version of the incremental section of Figure 1-11(a). The corresponding analysis for the TM wave (Fig. 1-11(b)) will be left as a problem. We note that Cdx is the <u>total</u> shunt capacitance lying between x and x+dx and $L/k_c^2\,dx$ is the <u>total shunt</u> inductance. The total <u>series</u> inductance is, of course, Ldx.

Kirchoff's current law therefore yields

$$I(x,s) = I(x+dx,s) + \left(sC\,dx + \frac{k_c^2\,dx}{sL}\right)V(x+dx,s).$$

Hence,

$$\frac{I(x+dx,s) - I(x,s)}{dx} = -\left(sC + \frac{k_c^2}{sL}\right)V(x+dx,s),$$

so that when we pass to the limit $dx \to 0$, we arrive at

(1-56)(a) $$\frac{dI(x,s)}{dx} = -\left(sC + \frac{k_c^2}{sL}\right)V(x,s)$$

~~Kirchoff's voltage law, on the other hand, yields the usual~~

~~(1-56)(a)~~

Kirchoff's voltage law, on the other hand, yields the usual

(1-56)(b) $$\frac{dV(x,s)}{dx} = -sL\,I(x,s).$$

The obvious difference between (1-56) and the equations for the "standard" lossless transmission line ((1-15), with $v_0(x) = 0$, $i_0(x) = 0$) is the presence of the frequency dependent term k_c^2/sL. This term plays a major role in distinguishing a lossless transmission line from a lossless waveguide. Incidentally, we note that because there are no lossy elements, such as series resistors or shunt conductances, in Fig. 1-11, the resulting waveguide is lossless.

Equations (1-56) can be combined to yield a single second order equation for either V(x,s) or I(x,s). Thus, upon differentiating (1-56)(b) and substituting (1-56)(a), we obtain

$$\text{(1-57)} \qquad \frac{d^2 V(x,s)}{dx^2} = (s^2 LC + k_c^2)\, V(x,s) \leftarrow \text{NEW!}$$

$$= \frac{1}{v_p^2}\left(s^2 + \omega_c^2\right) V(x,s).$$

As usual, we have called $v_p = 1/\sqrt{LC}$ and have defined the <u>cut-off frequency</u>, $\omega_c = k_c v_p$.

Arguing as we did in our discussion culminating in (1-49), we get for the positive going wave in the infinite waveguide

$$V(x,s) = V(0,s)\, e^{-(s^2+\omega_c^2)^{1/2} x/v_p},$$

(±) EXPONENT MUST BE NEGATIVE FOR PROPAGATION TO RIGHT

where V(0,s) is the Laplace transform of the sending-end voltage v(0,t). In comparing this result with the corresponding expression, for a lossy transmission-line, (the general case), we see that $\alpha = 0$ and $\delta^2 = -\omega_c^2$ for the waveguide (lossless transmission-line model).

As before, the impulse response is given upon setting V(0,s)=1:

$$h(x,t) = \mathcal{L}^{-1}\left[e^{-(LC)^{1/2}(s^2+\omega_c^2)^{1/2} x}\right]$$

$$\text{(1-58)} \qquad = \delta\left(t-(LC)^{1/2}x\right) - x\omega_c (LC)^{1/2}\, \frac{J_1(\omega_c x_1)}{x_1}\, u_{-1}\left(t-(LC)^{1/2}x\right). \rightarrow \text{RINGING DISPERSIVE TAIL}$$

↑ DELAYED REPLICA OF INPUT

Here $t_1 = (t^2 - (LC)x^2)^{1/2}$, as before, and $J_1(\omega_c t_1)$ is the <u>Bessel function of the first kind, order unity.</u> Its relation to the <u>modified</u> Bessel function of the first kind, order unity, is

$$\text{(1-59)} \qquad I_1(x) = (j)^{-1} J_1(jx), \qquad j = \sqrt{-1}.$$

Eqn. (1-58) follows formally, therefore, from (1-51) by setting $\alpha = 0$ and $\delta = j\omega_c$. $J_1(x)$ is sketched in Fig. 1-12. Note its oscillatory nature and decreasing amplitude. In fact, the <u>asymptotic form</u> of $J_1(\omega_c t_1)$ is

$$\text{(1-60)} \qquad J_1(\omega_c t_1) \xrightarrow{\omega_c t_1 \to \infty} \left(\frac{2}{\pi \omega_c t_1}\right)^{1/2} \cos\left(\omega_c t_1 - 3\pi/4\right)$$

$f = \frac{d\theta}{dt}$

$\omega(t) = \frac{d}{dt}\omega_c\left(t^2 - x^2/v_p^2\right)^{1/2}$

$= \frac{\omega_c t}{\left(t^2 - x^2/v_p^2\right)^{1/2}}$

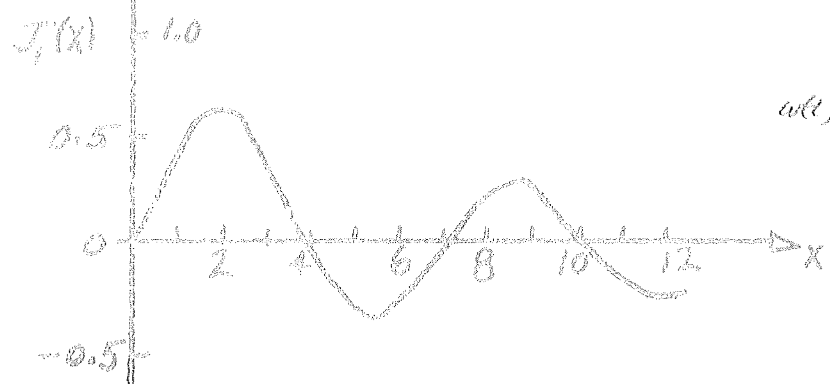


Fig. 1-12 Bessel function of the first kind, order unity.

[illegible] of the impulse response of a lossless waveguide is a fundamental difference from that of the lossy transmission-line. Clearly, from (1-58) and (1-60) the decay of h(x,t) with time is not exponential, as one would expect of a lossy system, but is much slower.

A good question is, why is there a decay at all? The answer lies in the fact that an impulse function is a <u>broadband function</u>, meaning that it generates, equally, all frequency components. Not all of the components travel at the same phase or group velocity, however. In fact, the variation of group velocity with ω will be determined in the next chapter to be $v_g = v_p[1-(\omega_c/\omega)^2]^{1/2}$.
This variation of velocities of different frequency components, of course, is precisely what is meant by "dispersion."

There is, therefore, a <u>destructive interference</u> established as $t \to \infty$ which accounts for the decay of the transient ringing. Note that the <u>asymptotic frequency</u> i.e., the frequency appearing in (1-60), is ω_c the characteristic (cut off) frequency of the guide. A more thorough analysis will be given in the next chapter.

The response to any other input voltage $v(0,t) = v_0(t)$ is given by the convolution integral (see (1-54) and (1-55)):

$$(1\text{-}61)\qquad v(x,t) = \left[v_0(t-(LC)^{1/2}x) - (LC)^{1/2}\omega_c x \int_{(LC)^{1/2}x}^{t} v_0(t-\tau)\frac{J_1(\omega_c t_1)}{t_1}\,d\tau\right] u_{-1}(t-(LC)^{1/2}x).$$

If $\omega_c = 0$, i.e., if the wave were propagating in unbounded space, rather than in the waveguide, the tail (second term) would vanish and we would recover the normal response of the "standard" lossless transmission line.

In order to understand why ω_c is called the cut-off frequency in the first place, return to the expression for the Laplace transformed voltage at point x, $V(x,s) = V(0,s)\exp[-\frac{(s^2+\omega_c^2)^{1/2}}{v_p}x]$,
and let $s = j\omega$, so that we have the corresponding sinusoidal steady-state response

$$V(x,j\omega) = V(0,j\omega)\, e^{-j(\omega^2-\omega_c^2)^{1/2} x/v_p}.$$

For frequencies above cut-off, $\omega > \omega_c$, the propagation constant, $\frac{(\omega^2-\omega_c^2)^{1/2}}{v_p}$ is real, implying wave propagation and energy transfer down the waveguide (or equivalent transmission line), whereas if $\omega < \omega_c$, the propagation constant is a pure imaginary and the resulting exponential function decays with distance

$$V(x,j\omega) = V(0,j\omega)\, e^{-(\omega_c^2-\omega^2)^{1/2} x/v_p}.$$

Such a decaying wave is said to be <u>evanescent</u>; there is no wave propagation, nor power flow. Hence, the wave is said to be "cut-off" and ω_c is the corresponding cut-off frequency. We emphasize that evanescence is a condition of wave decay <u>not due to losses</u> -in our waveguide, and the corresponding transmission line models of Fig. 1-11 are lossless. In an evanescent mode there is no phase shift along the guide length (the x-axis); all points along the guides axis are in phase. This is in the same way that a rigid shaft differs from a flexible one! Motion at different points of the flexible shaft are not in phase--a wave motion results.

[illegible] and re-examine it in light of the equivalent circuit of Fig. 1-11(a). The impulse response in (1-58) consists first of all of an impulse and then (after the impulse has arrived) the tail occurs. The initial response (i.e., the impulse at $t = (LC)^{1/2} x = x/v_p$) is that which would have occurred for a non-dispersive, lossless delay line, i.e., if the shunt inductance in Fig. 1-11(a) were absent.

The initial value theorem of Laplace transforms, however, states that it is the higher (infinite) frequencies which govern the initial response of a system. At such higher (infinite) frequencies the shunt inductor has infinite impedance (or zero admittance), thereby appearing to be "transparent." That is, at extremely high frequencies, the shunt branch in Fig. 1-11(a) consists of a single capacitor, which means that the equivalent incremental circuit is a series inductor and a shunt capacitor — precisely that of a lossless, non-dispersive delay line.

Another very useful interpretation of the cut-off frequency, ω_c, is that it is the anti-resonant frequency of the shunt branch in Fig. 1-11(a). That is, the shunt admittance-per-unit-length is given by $Y(s) = (sC + k_c^2/sL)$. This function has a zero (i.e., appears to be an open circuit) when $s^2 LC = -k_c^2 = -\omega_c^2 LC$ or $s = \pm j\omega_c$. At this frequency the transmission line appears to be a chain of inductors connected in series. Obviously such an arrangement cannot propagate waves; the waves are said to be "cut-off."

5. Transient and Steady-State Response of a Laser.

The theory that we have developed in the preceding examples using the Laplace transform is applicable to the understanding of laser action, at least as far as the wave characteristics of the laser are concerned. LASER is an acronym (like RADAR, SONAR, etc) for light amplification by the stimulated emission of radiation. We cannot go into a discussion of stimulated emission because it involves notions of atomic transitions, notions that require an understanding of modern physics and quantum mechanics.

A simplified, transmission-line, model of a laser is shown in Fig. 1-13(a). A voltage wave, V_i, is incident from an infinite, lossless line with characteristic impedance Z_0 and phase velocity v_p, upon an "active" line of length l. The active line is a model for a system consisting of a host crystal, say Al_2O_3, into which are doped active impurity ions, such as triply ionized chromium, C_r^{3+}, the combination being called, in this case, ruby. Hence, this is a simple model for a solid-state, ruby laser. It could serve equally well for a gaseous (for example helium-neon) laser.

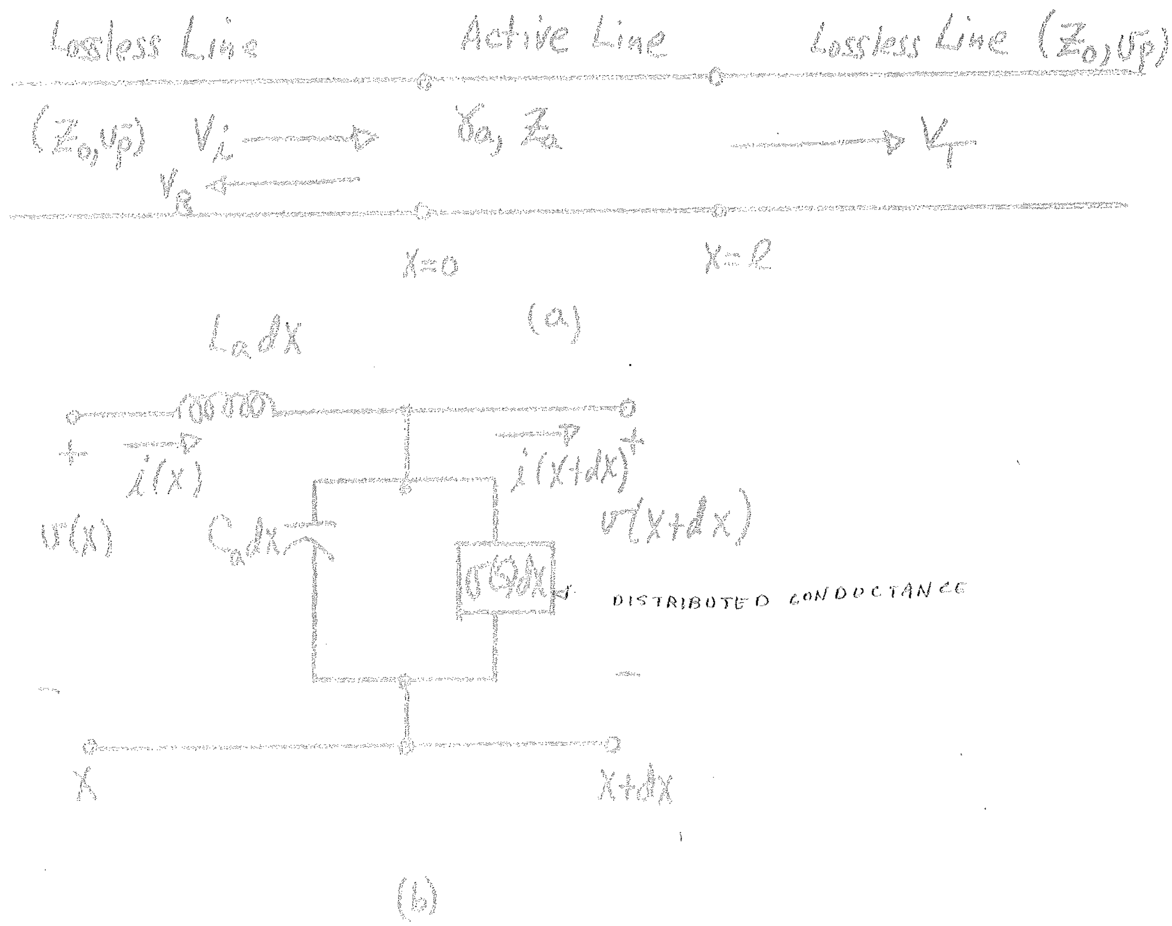


Figure 1-13. A transmission-line model of a laser (a) consisting of two infinite, lossless lines sandwiched around an active line of length l, (b) an incremental section of the active line.

Our job is to calculate the reflected voltage wave V_r and the wave, V_t, transmitted into another infinite, lossless line, identical to the first. We shall see under what conditions the overall system behaves as a stable optical amplifier or as an optical oscillator. This latter condition is what one normally refers to as a laser.

The incremental model for the active transmission-line is shown in Fig. 1-13(b). It consists of the usual series inductance-per-unit length, L_a, and shunt capacitance-per-unit length C_a. In addition, as in the transmission-line model of a waveguide, Fig. 1-11(a), there is another shunt branch giving rise to an equivalent frequency dependent admittance-per-unit length, $\sigma(\Omega)$. $\sigma(\Omega)$ originates in the atomic transitions of the chromium ions, in the case alluded to above, or in the excited neon atoms in a He-Ne laser. Its precise form will be given later.

The Laplace transmission-line equations for the active line are easily obtained using Kirchoff's laws; they are

$$(1\text{-}62)\quad \frac{dV(x,s)}{dx} = -sL_a I(x,s), \quad \frac{dI(x,s)}{dx} = -(sC_a + \sigma(s))V(x,s).$$

These equations combine to yield

$$(1\text{-}63)\quad \frac{d^2V}{dx^2} = \left[L_a C_a s^2 + L_a s\sigma(s)\right]V = \gamma_a^2(s)V.$$

The solutions for V and I follow from (1-63) and the first of (1-62)

$$(1\text{-}64)\quad V(x,s) = C_1 e^{-\gamma_a x} + C_2 e^{\gamma_a x}$$

$$I(x,s) = \frac{1}{Z_a(s)}\left[C_1 e^{-\gamma_a x} - C_2 e^{\gamma_a x}\right],$$

where $Z_a(s)$ is the characteristic impedance of the active medium

$$Z_a(s) = \left(\frac{L_a s}{C_a s + \sigma(s)}\right)^{1/2}.$$

Let $V_i(s)$ and $I_i(s)$ be the Laplace transforms of the incident voltage and current at x = 0, $V_R(s)$ and $I_R(s)$ be the Laplace transforms of the reflected voltage and current at x = 0 and $V_T(s)$, $I_T(s)$ the Laplace transforms of the transmitted voltage and current at x = l. Then our boundary conditions require that

CONTINUITY V X=0 ; X=ℓ

CONTINUITY I X=0 ; X=ℓ

$$(1\text{-}65)\quad \left.\begin{aligned} C_1 + C_2 &= V_i(s) + V_R(s) \\ \frac{1}{Z_a(s)}(C_1 - C_2) &= I_i(s) + I_R(s) \end{aligned}\right\} x=0\ ; \quad \begin{aligned} C_1 e^{-\gamma_a l} + C_2 e^{\gamma_a l} &= V_{T\,\text{TRANSMITTED}} \\ \frac{1}{Z_a(s)}\left[C_1 e^{-\gamma_a l} - C_2 e^{\gamma_a l}\right] &= I_T \end{aligned}\ x=l$$

In addition to (1-65) we have the usual relations for the characteristic impedance of the lossless lines:

$$(1\text{-}66)\quad \frac{V_i}{I_i} = \frac{V_T}{I_T} = -\frac{V_R}{I_R} = Z_0.$$

Equations (1-65) and (1-66) are seven in number, in the seven unknowns V_R, I_i, I_R, V_T, I_T, C_1 and C_2 (we assume $V_i(s)$ is given). Hence we can solve for the ratio $V_T(s)/V_i(s) \equiv Y_T(s)$ straight-forwardly to obtain

$$(1\text{-}67)\quad \frac{V_T(s)}{V_i(s)} \equiv Y_T(s) = \frac{1}{\cosh\gamma_a l + A\sinh\gamma_a l}, \text{ (TRANSFER FUNCTION)}$$

where $A = \frac{1}{2}\left(\frac{Z_a}{Z_0} + \frac{Z_0}{Z_a}\right).$

$$(1\text{-}68)\quad \frac{V_R(s)}{V_i(s)} = Y_R(s) = \frac{B\sinh \gamma_a \ell}{\cosh \gamma_a \ell + A\sinh \gamma_a \ell},$$

where $B = \frac{1}{2}\left(\frac{Z_a}{Z_0} - \frac{Z_0}{Z_a}\right)$

In practical situations $|\sigma(s)| \ll |s| C_a$, so that Z_a becomes

$$Z_a \approx \left(\frac{L_a}{C_a}\right)^{1/2}$$

which is the characteristic impedance of the host material in the absence of any active impurity ions. Applying this result to the definition of A we see that $A > 1$.

Using the same relation between $|\sigma(s)|$ and $|s| C_a$, we get for $\gamma_a(s)$ from (1-63)

$$(1\text{-}69)\quad \gamma_a(s) = \left[L_a C_a s^2 + L_a s\,\sigma(s)\right]^{1/2} = (L_a C_a)^{1/2} s\left[1 + \frac{\sigma(s)}{sC_a}\right]^{1/2}$$

$$\approx s/v_{pa} + \frac{1}{2}\left(\frac{L_a}{C_a}\right)^{1/2}\sigma(s) = \frac{s}{v_{pa}} + \alpha(s),$$

where $v_{pa} = 1/(L_a C_a)^{1/2}$ is the phase velocity of the host material in the absence of the impurity ions, and $\alpha(s) = \frac{1}{2}\left(\frac{L_a}{C_a}\right)^{1/2}\sigma(s)$.

If $V_i(t)$ is a switched on, complex sinusoid

$$V_i(t) = \begin{cases} 0, & t < 0 \\ V_0 e^{j\omega t}, & t > 0 \end{cases}$$

then

$$V_i(s) = \mathcal{L}\left[V_i(t)\right] = \frac{V_0}{s - j\omega}.$$

Hence, from (1-67)

FORCING FUNCTION (YIELDS STEADY STATE RESP.) — TRANSFER FUNCTION (YIELDS TRANSIENT RESPONSE)

$$(1\text{-}70)\quad V_T(s) = \frac{V_0}{s - j\omega}\;\frac{1}{\cosh \gamma_a \ell + A\sinh \gamma_a \ell}.$$

In order to determine $V_T(t) = \mathcal{L}^{-1}\left[V_T(s)\right]$ we expand (1-70) into partial fractions

$$V_T(s) = \frac{A_0}{s - j\omega} + \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_q}{s - s_q} + \cdots,$$

where

the A's are the residues of (1-70) at its poles; $j\omega$ and the $\{s_i\}$ are the poles. The inverse transform of each factor is an exponential, so that

$$(1\text{-}71)\quad V_T(t) = A_0 e^{j\omega t} + A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots + A_q e^{s_q t} + \cdots.$$

In principle, we can calculate the residues by using the rules of Laplace transforms. In fact

$$(1\text{-}72) \qquad A_0 = \lim_{s \to j\omega} (s - j\omega) V_T(s) = \frac{V_0}{\cosh \gamma_a(j\omega) l + A \sinh \gamma_a(j\omega) l} .$$

We are not so much interested in the values of the other residues as we are in knowing the location of the <u>poles</u> $s_1, s_2, \ldots\ldots$ The pole s_n of $V_T(s)$ is the root of the characteristic equation

$$(1\text{-}73) \qquad \cosh \gamma_a(s_n) l + A \sinh \gamma_a(s_n) l = 0$$

This equation is equivalent to

$$\coth \gamma_a(s_n) l = \frac{e^{\gamma_a(s_n) l} + e^{-\gamma_a(s_n) l}}{e^{\gamma_a(s_n) l} - e^{-\gamma_a(s_n) l}} = -A,$$

which, in turn, may be written

$$(1\text{-}74) \qquad e^{2\gamma_a(s_n) l} = \frac{A-1}{A+1} = \left(\frac{Z_a - Z_0}{Z_a + Z_0} \right)^2 = \Gamma^2,$$

where we have used the definition of A. Γ is the <u>reflection coefficient</u> at either boundary surface, x = 0 and x = l.

We define L_r, the <u>reflection loss at either boundary</u> by $\Gamma = e^{-L_r}$

so that (1-74) becomes

$$e^{2\gamma_a(s_n) l} = e^{-2L_r} = e^{-2L_r} \cdot e^{j2n\pi}, \quad n = 0, \pm 1, \pm 2, \ldots\ldots .$$

Therefore, equating exponents

$$\gamma_a(s_n) l = -L_r + jn\pi .$$

From (1-69), however, $\gamma_a(s_n) = s_n / v_{pa} + \alpha(s_n)$,
so that we obtain an implicit expression for s_n upon substitution into the preceding equation:

$$(1\text{-}75) \qquad s_n = -\frac{v_{pa}}{l} \left\{ l\alpha(s_n) + L_r \right\} + j\omega_n .$$

where $\omega_n = n\pi v_{pa}/\ell$ is the nth resonant angular frequency of the line lying between $x = 0$ and $x = 1$. The nth resonant frequency is given by $f_n = \omega_n/2\pi = n v_{pa}/2\ell$. Note that $2\ell/v_{pa}$ is the total round-trip transit time for a wave to go from $x = 0$ to $x = 1$ and back. We also note that there are infinitely many resonant frequencies. This is typical of distributed systems as contrasted to finite, lumped systems.

$\alpha(s_n)$ is a complex number whose imaginary part is much smaller than ω_n, so that the imaginary part may be ignored. Calling $\alpha_n = Re\,\alpha(s_n)$, we may write s_n in its real and imaginary parts

$$(1\text{-}76) \qquad s_n = \frac{v_{pa}}{\ell}\left(\alpha_n \ell - L_r\right) + j\omega_n .$$

Substitution of (1-76) into (1-71) gives us our final form for the transmitted voltage at $x = 1$:

$$(1\text{-}77) \qquad V_T(t) = \frac{V_0 e^{j\omega t}}{\cosh \gamma_a(j\omega)\ell + \Delta \sinh \gamma_a(j\omega)\ell} + \sum_{n=-\infty}^{\infty} A_n e^{\frac{v_{pa}}{\ell}(\ell\alpha_n - L_r)t} e^{j\omega_n t} .$$

The first term of (1-77) is the sinusoidal, steady-state response of the laser to the applied signal. The remaining terms are the transient response which oscillate at the resonant frequencies established by the dimension (l) of the active line. These terms either decay or grow with time, depending on the sign of $(\ell\alpha_n - L_r)$. If there is sufficient gain in the atomic system (the active line), it is possible for $\ell\alpha_n$ to be greater than L_r, for certain values of n. This means that the gain at these poles, s_n, exceeds the transmission losses at the boundaries, and, thus, the system becomes unstable in the sense that initial conditions create an increasing transient output even in the absence of the input signal V_0.
-reflection

This is the desired case for a laser oscillator. For a laser amplifier, on the other hand, we wish to have $\alpha_n \ell < L_r$, so that transients decay with time, and we are left with only the amplified replica of the input term, i.e., we are left with only the first term of (1-77). This is a stable response.

We shall discuss the solutions in the stable and unstable regions separately.

A. Solutions in Stable Region: Consider the sinusoidal, steady-state terms in (1-77). From (1-69) and upon taking $\alpha(j\omega)$ to be real (i.e., neglecting its imaginary part compared to $j\omega/v_{pa} = j\beta$) and equal to

$$(1\text{-}78) \qquad \alpha(j\omega) = \frac{K\Delta\omega}{(\omega-\omega_0)^2 + (\Delta\omega)^2}$$

where K is a constant, $\Delta\omega$ is the bandwidth of the active ions and ω_0 is the frequency separation of the energy-levels of the active ions (or the "resonant frequency" of the ion), we get

$$(1\text{-}79) \qquad \left|\frac{V_T(t)}{V_0 e^{j\omega t}}\right| = \left|\frac{V_T(j\omega)}{V_i(j\omega)}\right| = \left|Y_T(j\omega)\right| = \left[\frac{1}{(\cosh K\ell - \Delta \sinh K\ell)^2 + \beta^2 \sin^2 \beta\ell}\right]^{\frac{1}{2}} ,$$

where $\delta = \frac{1}{2}\left(\frac{1}{\tau_b} - \frac{1}{\tau_a}\right)$.

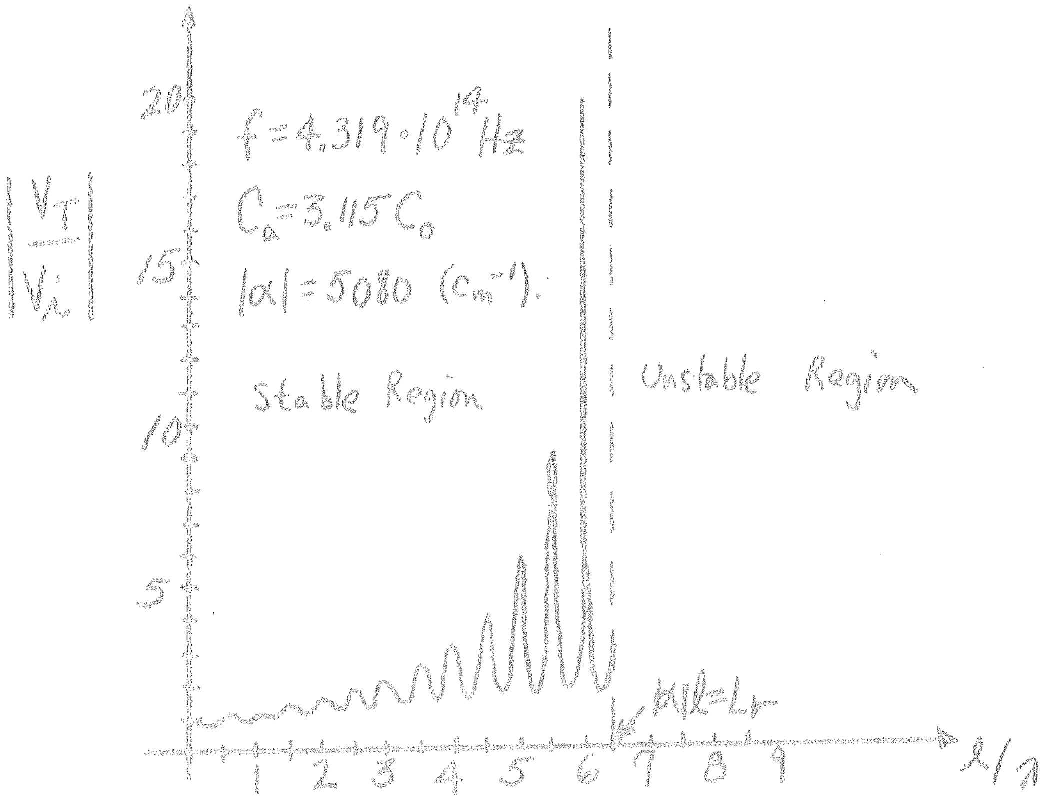


Figure 1-14. Sinusoidal, steady-state gain of a laser amplifier versus normalized length, l/λ.

This function, (1-79), is plotted versus l/λ for a set of typical parameter values in Fig. 1-14. When $l/\lambda = 6.42$, the gain becomes infinite and we leave the stable region, in which $\alpha l < L_r$ in (1-76), and enter the unstable region where $\alpha l > L_r$.

8. <u>Solutions in Unstable Region</u>: The transition region between stable and unstable operation is given by $(\alpha_n l - L_r) = 0$ (see (1-76)). When $\alpha_n \cdot l < L_r$, then the transient terms of (1-77) die out in time. Conversely, when $\alpha_n l > L_r$, the oscillations build up in time.

Notice that when $\alpha_n l = L_r$, the poles are located on the $j\omega$-axis, i.e., they are pure imaginaries (see 1-76)). Hence, we are interested in the case $\alpha(j\omega_n) l = L_r$. Substituting (1-78) into this equation gives us the equation for determining which modes (frequencies) build up in time

$$\alpha(j\omega_n)\, l \geq L_r \Longrightarrow \frac{K\, \Delta\omega\, l}{(\omega_n - \omega_0)^2 + (\Delta\omega)^2} \geq L_r . \tag{1-80}$$

We can give this equation a more graphical interpretation in Fig. 1-15. Note that the number of modes (ω_n's) which are excited depends upon the atomic bandwidth $\Delta\omega$ of the active lenses as well as on the loss-level, L_r.

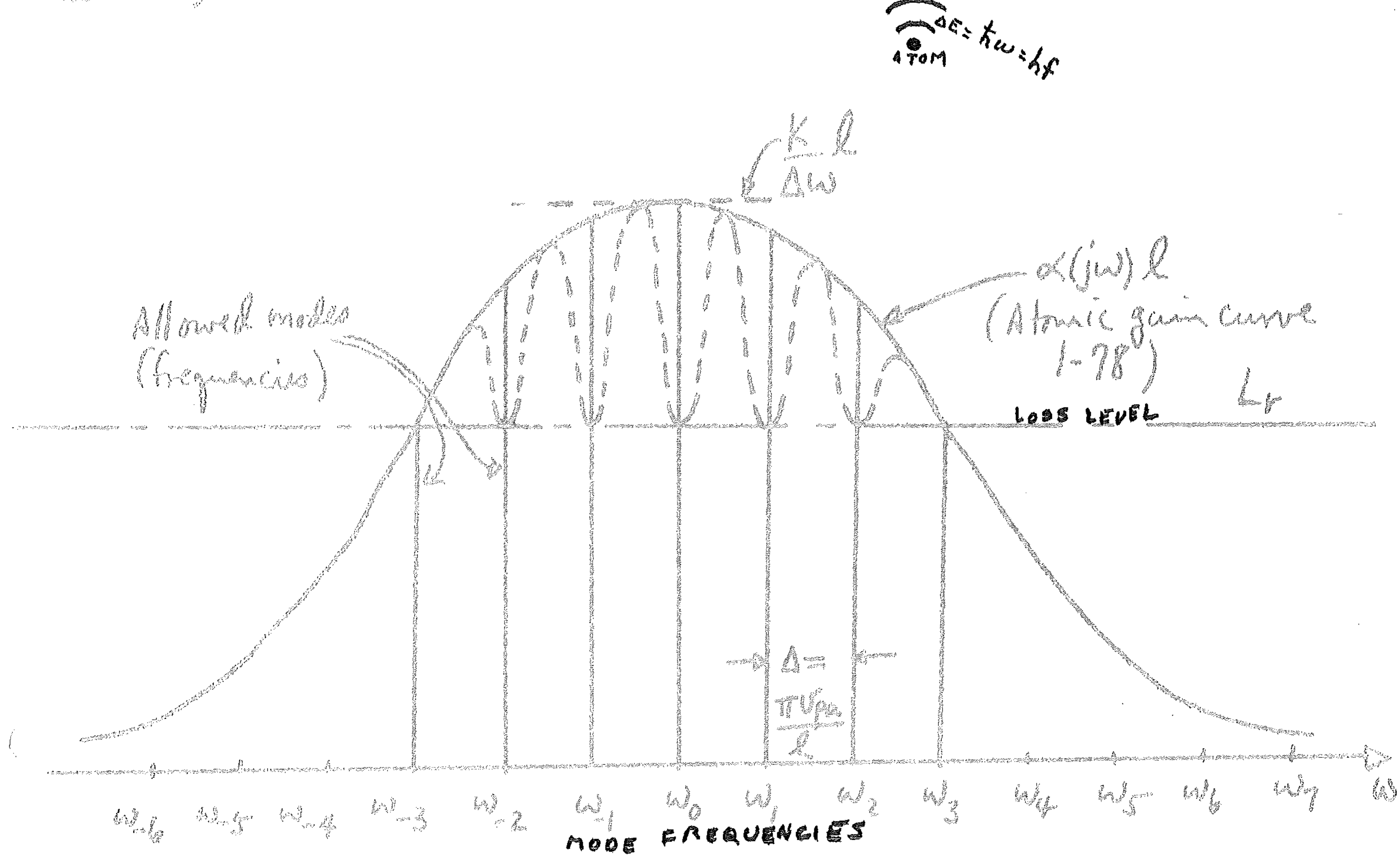


Fig. 1-15. The modes which build up in time to produce steady-state oscillations must initially satisfy $\alpha(j\omega_n)l > L_r$. Thus, those modes whose resonant frequencies are $\omega_{-3}, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \omega_3$ will achieve steady-state oscillations. The dotted curves are examples of "hole burning". The <u>oscillation bandwidth</u> is $(\omega_3 - \omega_{-3})$.

From Fig. 1-15, we see that the normal manner of operation of a laser oscillator (or simply, "laser") is "multimode", i.e., there are several modes oscillating simultaneously. As these modes increase in intensity during their transient buildup, non-linear effects in the atomic system reduce the gain at the frequencies of these modes. This saturation phenomenon is called "hole burning" and is what enables the system to achieve steady-state oscillations. The "burned holes" are shown dotted in Fig. 1-15.

The modes shown in Fig. 1-15 differ in the number of half-wavelengths of light in the length l. Using the data shown in Fig. 1-14, we have

$$v_{pa} = \frac{1}{(L_o C_a)^{1/2}} = \frac{1}{(\mu_0 \varepsilon_0)^{1/2}(3.115)^{1/2}} \approx 1.7 \cdot 10^8 \text{ m/sec},$$

where we have taken for the numerical value of μ_0, $4\pi\cdot10^{-7}$, the magnetic permeability of free space and $3.45\epsilon_0$, the dielectric constant of free space for ϵ_a. At the frequency $f = 4.34\cdot10^{14}$ Hz, the wavelength of (times 3.45) the light in the active medium is

$$\lambda = \frac{1.7\cdot10^{8}}{4.3\cdot10^{14}} \approx 0.4\cdot10^{-6}\ m = 4000\text{Å} \quad (\text{Å} = \text{Ångstrom} = 10^{-10}\ m).$$

If a typical optical cavity length, l, is of the order of 0.2m, then the number of half-wavelenths of light at the frequency $4.3\cdot10^{14}$ Hz is of the order of 10^6.

To obtain laser operation at a single frequency (or in a single mode) it is usually necessary to design the laser active medium length, l, to be sufficiently short so that $v_{pm}/2l$, the spacing between adjacent mode frequencies, is greater than the oscillation bandwidth (see Fig. 1-15). An alternative procedure is to design a complex optical resonator (which is modeled by the active-medium transmission-line) which has high loss for all modes, except the favored one, within the oscillation bandwidth.

Sometimes it is desirable to operate in a multimode manner if the modes are "locked", for such mode locked sources are useful in high-data-rate optical pulse-code modulation (PCM) systems.

In order to understand what is meant by "mode-locking" let us return to (1-77) and consider only the laser oscillating terms in the summation. Since we are assuming that the oscillations are in the steady state (for those modes within the oscillation bandwidth) we can set $l\alpha_n = L_r$ for those modes. Hence, the laser output at x = l is given by

$$\text{(1-81)} \qquad V_T(t) = \sum_{n=-N}^{N} A_n e^{j(\omega_0 + n\Delta)t},$$

where N is the total number of modes within the oscillating bandwidth, ω_0 is the atomic resonant frequency (the optical center frequency) and $\Delta = \pi v_{pm}/l$. Thus, the only difference between (1-77) and (1-81) is that we have renumbered our modes starting with -N and ending at +N and have set $\omega_n = \omega_0 + n\Delta$.

The $\{A_n\}$ appearing in (1-81) are complex phasors, giving the magnitudes and phases of the various modes. In general, due to random perturbations within the optical medium these amplitudes and phases vary with time. By mode-locking is meant ensuring that all of the phases are fixed (non-random) and are known relative to each other and that the magnitudes are fixed in time. Returning to (1-81), let us factor the term $e^{j\omega_0 t}$ from the summation

$$\text{(1-82)} \qquad V_T(t) = e^{j\omega_0 t} \sum_{n=-N}^{N} A_n e^{jn\Delta t}.$$

This is the form of an <u>amplitude modulated carrier wave</u>, the carrier frequency being ω_0 and the summation being the <u>envelope</u> (or amplitude) of the resulting wave.

We see immediately that the envelope is in the form of a truncated Fourier series, so that it is periodic with period $T = 2\pi/\Delta = 2\pi/\nu_m$. One possible form, which is an idealized version of an experimentally verified envelope, is given in Fig. 1-16, the classical periodic square-wave pulse train.

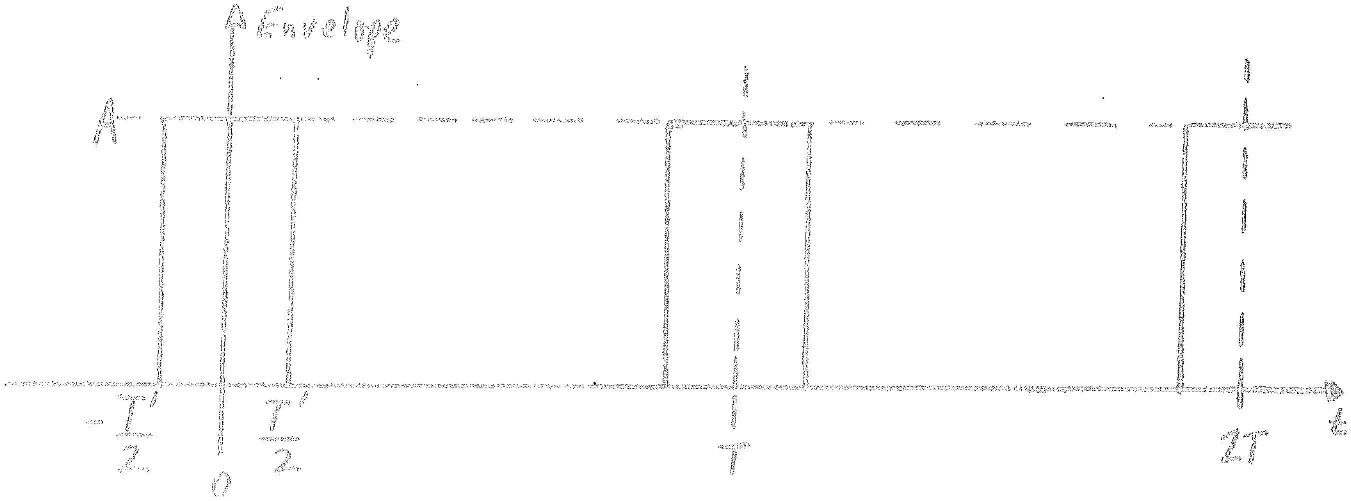


Figure 1-16. An envelope, consisting of a periodic pulse train of pulse width T' and period T.

The Fourier coefficients of the periodic pulse train are easily calculated* to be

$$AT' \frac{\sin(n\pi T'/T)}{(n\pi T'/T)} .$$

These coefficients are to be taken equal to the A_n of (1-82). In order to determine the number of laser modes necessary to produce a reasonable approximation to Fig. 1-16, we return to the expression for the Fourier coefficients just stated and observe that the coefficients are all real (phase angle is either 0 or π) and decrease in amplitude as $\frac{1}{n}$. The principle harmonics (or modes) present lie within the major lobe of the $\frac{\sin x}{x}$ curve. This lobe, by definition, lies between the two zeroes nearest the origin in frequency space, namely where $N\pi T'/T = \pm\pi$. Hence, $N = T/T'$, and we conclude that the number of modes present is given approximately by the ratio of the pulse-repetition rate (or period, T) to the pulse width T'.

Under typical experimental conditions, $T \sim 14 . 10^{-9}$ sec and $T' \sim 330 . 10^{-12}$ sec, so that $N \sim 40$.

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* See any basic circuits text, e.g., E. M. Sabbagh, "Circuit Theory", Chapt. 25, The Ronald Press, 1959.

Thus, when the physical configuration of the laser can be arranged so that the A_q in (1-82) are all real and roughly of a $\frac{\sin x}{x}$ nature, i.e., when the output frequency spectrum is discrete and approximately $\frac{\sin x}{x}$, then the laser is said to be mode locked to produce a periodic pulse train envelope. Other envelope configurations versus time are also theoretically possible, and one could conceivably even produce an FM-modulated envelope. We should recall that in the example discussed here the pulse train of Fig. 1-16 amplitude modulates a carrier of frequency ω_0, so that the actual time waveform is that of Fig. 1-16 with oscillations at the optical frequency ω_0 within each pulse.

A thorough discussion of mode-locking of lasers can be found in P. W. Smith, "Mode-Locking of Lasers", Proc. IEEE, Vol. 58, No. 9, Sept. 1970, pp. 1342-1357. Further reference should be made to the Special Issue on Optical Communication, Proc. IEEE, Vol. 18, No. 10, Oct. 1970.

6. The RC Transmission Line. When the parameters of the generalized transmission-line equations, (1-42), are such that G = 0, L = 0, these equations become

$$(1\text{-}83)\quad (a)\ \frac{\partial v}{\partial x} = -iR, \qquad (b)\ \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t},$$

and the resulting transmission-line is called an RC transmission-line (RC line) because the only surviving parameters are the series resistance-per-unit length, R, and shunt capacitance-per-unit length, C.

Just as the lossless transmission-line and its equations, (1-1), modeled simple wave propagation, so does the RC line and (1-83) model a distinct type of dynamical response called diffusion; in addition, heat flow can also be modeled by the RC line.

Upon differentiation of (1-83)(a) with respect to x and substitution of (1-83)(b) into the result, there obtains

$$(1\text{-}84)\quad \frac{\partial^2 v}{\partial x^2} = -R\frac{\partial i}{\partial x} = RC\frac{\partial v}{\partial t},$$

the classical diffusion or heat-flow equation. Note that it differs from the classical wave equation (1-2) in having only a first-order time derivative on the right-hand side rather than a second order derivative. Equation (1-84) could, of course, have been determined directly from (1-43) by letting L = 0, G = 0.

Let us consider the response of a semi-infinite RC line to a unit step at x = 0. The analysis begins with the Laplace transformed version of (1-84), which, in the absence of initial conditions is the same as (1-46) with L = 0, G = 0:

$$(1\text{-}85)\quad \frac{d^2 V(x,s)}{dx^2} - sRC\,V(x,s) = 0.$$

The solution of (1-85), valid in the region $x \geq 0$, is given by (1-49), upon setting $V(0,s) = 1/s$ and $\gamma(s) = (sRC)^{1/2}$. This expression for $\gamma(s)$ follows from (1-48) with $G = 0$, $L = 0$. Thus,

$$V(x,s) = \frac{1}{s} e^{-(sRC)^{1/2} x} \quad (1\text{-}86)$$

An entry in a table of Laplace transforms is

$$\mathcal{L}^{-1}\left[e^{-a\sqrt{s}}\right] = \frac{a}{2} \cdot \frac{e^{-a^2/4t}}{(\pi t^3)^{1/2}} \quad (1\text{-}87)$$

Therefore, by recognizing that multiplication by $1/s$ corresponds to integration from 0 to t, we find by using (1-86) and (1-87) that

$$v(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s} e^{-\sqrt{sRC}\, x}\right] = \int_0^t \frac{\sqrt{RC}\, x}{2\pi^{1/2}} \cdot \frac{e^{-RCx^2/4\xi}}{\xi^{3/2}} \, d\xi, \quad (1\text{-}88)$$

where ξ is a dummy variable of integration.

In order to simplify the integral appearing in (1-88), we make the change of variable $RCx^2/4\xi = \tau^2$, so that $d\tau = -\frac{1}{4}\sqrt{\frac{RC}{\xi^3}}\, x\, d\xi$.

Hence, the integral becomes

$$\int_0^t \frac{\sqrt{RC}\, x}{2\pi^{1/2}} \cdot \frac{e^{-RCx^2/4\xi}}{\xi^{3/2}} \, d\xi = -\frac{2}{\sqrt{\pi}} \int_{+\infty}^{\frac{\sqrt{RC}\,x}{2t^{1/2}}} e^{-\tau^2} d\tau = \frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{RC}{t}}\left(\frac{x}{2}\right)}^{\infty} e^{-\tau^2} d\tau .$$

The latter integral can be written as the difference of two integrals, one extending from 0 to $+\infty$ and the other from 0 to $\frac{x}{2}\sqrt{\frac{RC}{t}}$:

$$\frac{2}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-\tau^2} d\tau - \int_0^{\frac{x}{2}\sqrt{\frac{RC}{t}}} e^{-\tau^2} d\tau\right].$$

The first term can be simplified by the definite integral identity $\int_0^{\infty} e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{2}$. The second integral cannot be simplified further; it is tabulated, however, and given the name <u>error function</u>, $\operatorname{erf}\left[\frac{x}{2}\sqrt{\frac{RC}{t}}\right]$. That is

$$\operatorname{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-\tau^2} d\tau .$$

Our final result for the step response of the RC transmission-line thus, becomes

$$v(x,t) = 1 - \text{erf}\left[\frac{x}{2}\sqrt{\frac{RC}{t}}\right] \tag{1-89}$$

At the points x_1 and x_2 on the line the voltage as a function of time is sketched in Fig. 1-17. Note that the voltage increases more slowly in time the further away from the source the point of observation. In addition, we see that there is no delay in time before the voltage "wave" arrives at a given point, but the voltage does build up sluggishly. This is in marked contrast with the result (1-55) which implied that as long as L, C ≠ 0, there is no step response at point x until the time $x\sqrt{LC}$, i.e., the wave travels with the finite speed $v_p = 1/\sqrt{LC}$. When L = 0, we expect some form of instantaneous response because then $v_p \to \infty$. Under this condition we do not really have wave propagation, but rather "diffusion".

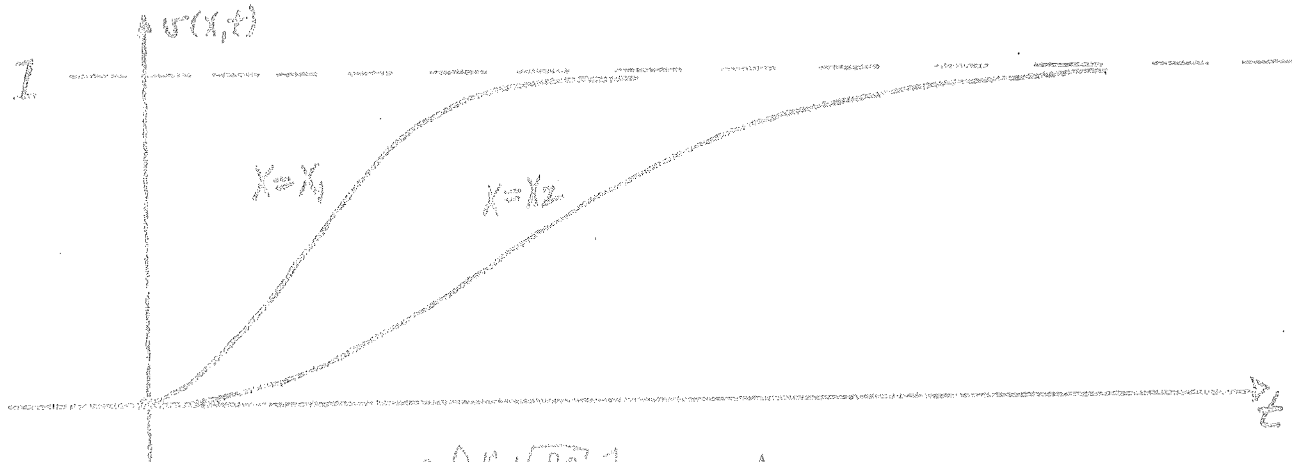

Figure 1-17. $v(x,t) = 1 - \text{erf}\left[\frac{x}{2}\sqrt{\frac{RC}{t}}\right]$ vs t at two points x_1, x_2, with $x_1 < x_2$, of an RC-line.

This diffusion (or sluggishness) manifested itself in the early transatlantic submarine cables and had to be corrected by inserting more inductance (oddly enough) in the form of periodically spaced loading coils in the attempt to produce the distortionless line condition RC = LG. Now electronic filters called "equalizers" replace the loading coils.

The current response to the unit step can be easily found by utilizing (1-83)(a) and (1-89):

$$i = -\frac{1}{R}\frac{\partial v}{\partial x} = \frac{1}{R}\frac{\partial}{\partial x}\,\text{erf}\left[\frac{x}{2}\sqrt{\frac{RC}{t}}\right]$$

$$= \frac{2}{\sqrt{\pi}R}\,\frac{\partial}{\partial x}\left(\int_0^{\frac{x}{2}\sqrt{\frac{RC}{t}}} e^{-r^2}\,dr\right).$$

parameter (such as x) appearing in the upper limit is to replace the variable of integration (τ, in this case) by the upper limit and multiplying the result by the derivative of the upper limit with respect to the parameter. Thus

$$\frac{\partial}{\partial x}\int_0^{\frac{x}{2}\sqrt{\frac{RC}{t}}} e^{-\tau^2}\,d\tau = e^{-RCx^2/4t}\cdot\frac{1}{2}\sqrt{\frac{RC}{t}}\,.$$

Hence, upon using this in the expression for i(x,t), we obtain

$$\text{(1-90)}\qquad i(x,t) = \sqrt{\frac{C}{\pi R t}}\; e^{-RCx^2/4t}\,.$$

This result is sketched vs. t and x in Fig. 1-18. The diffusion of the current away from the source located at x = 0 is apparent in Fig. 1-18(b). Again, note the instantaneous response at any point x. A final interesting observation is that the areas under the curves in Fig. (1-18)(b), i.e., $\int_0^\infty i(x,t)\,dx$, is independent of t:

$$\int_0^\infty \sqrt{\frac{C}{\pi R t}}\; e^{-RCx^2/4t}\,dx = \frac{1}{R}\int_0^\infty \sqrt{\frac{RC}{\pi t}}\; e^{-RCx^2/4t}\,dx = \frac{1}{R}\,.$$

This result appears to have dimensions of meters/ohm, but it really is volts-$\frac{\text{meter}}{\text{ohm}}$ = amperes x meters, as it should, because the 1 in the numerator stands for 1 volt.

The functional form of (1-90) is that of a delta function (or impulse) of current located at x = 0 and switched on at t = 0. Then the delta diffuses to produce the usual Gaussian (or "normal") curve.

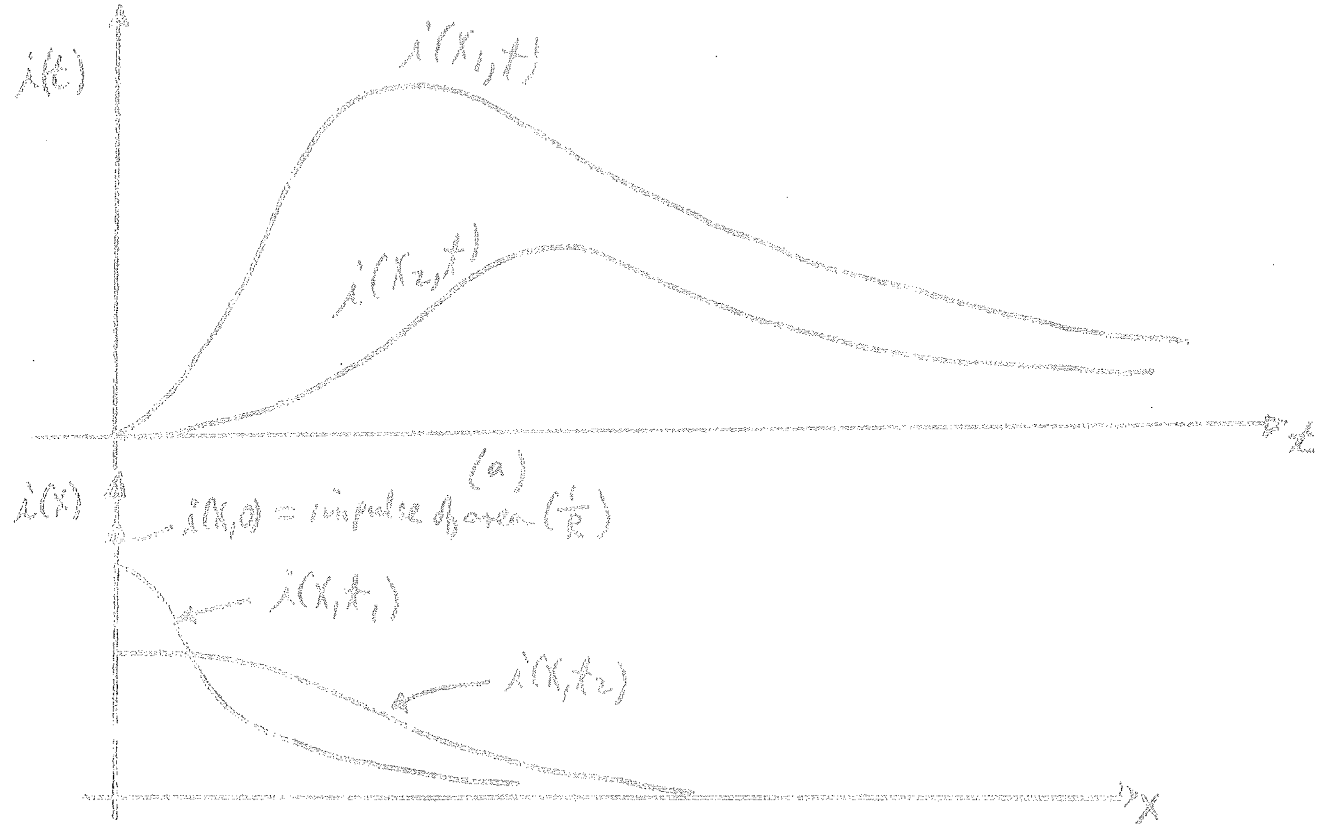

Figure 1-18. I(x;t) vs. t, x as a parameter (a) and vs. x, t as a parameter (b). Note the impulse or delta function at x=0 diffusing into the usual Gaussian (or "normal") curves for t>0, (b). In (a) $x_1 < x_2$ and in (b) $0 < t_1 < t_2$

[illegible]
from hot to cold, i.e., in the direction of decreasing temperature. Thus, if we let q stand for <u>heat flux</u>, having dimensions of (watts/area), T for temperature (°) and σ_T the <u>thermal conductivity</u> (watts/length x temp), a constant for a given material, Newton's law in differential form becomes

(1-91) (a) $$q = -\sigma_T \frac{\partial T}{\partial x} .$$

This is one of the two equations necessary to determine q and T. The second equation is one of energy conservation and is related to the second law of thermodynamics. In order to derive it, refer to Fig. 1-19.

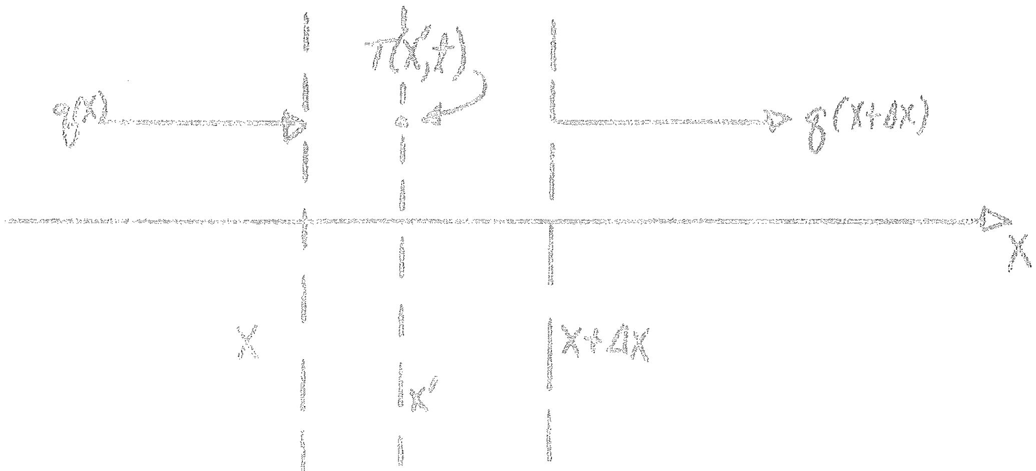

Figure 1-19. Deriving the heat energy conservation equation.

The net <u>outflow</u> of heat power is $[q(x+\Delta x) - q(x)]A$, where A is the cross-sectional area of the region between x and x + Δx. If there are no heat sources within this region, energy conservation demands that the net outflow of heat power be matched by a corresponding decrease in the rate of heat energy stored within the region. In terms of temperature, this rate decrease is given by

$$-C \Delta x A \frac{\partial T}{\partial t},$$

where C is the <u>heat capacity</u> (= specific heat x density) of the material. It has the dimensions of $\frac{\text{watt-sec}}{\text{volume-Temp}}$. $\Delta x \cdot A$ is, of course, the volume of material between x and x + Δ x having cross section A.

Upon equating the net outflow of heat power to the rate of decrease of stored energy, we get

$$[q(x+\Delta x) - q(x)]A = -C \Delta x A \frac{\partial T}{\partial t},$$

which, upon division by the volume Δx A and subsequent passage to the limit $\Delta x \to 0$, yields the required second differential equation

(1-91) (b) $$\frac{\partial q}{\partial x} = -C \frac{\partial T}{\partial t}.$$

Equations [illegible] are analogous to [illegible] with v analogous to T, i to q, C to c and $\frac{1}{R}$ to σ_T. Thus, we can model many heat flow problems in terms of RC transmission-lines. We shall leave specific examples to the problems at the end of the chapter.

8. Diffusion of Electrons and Holes in Semiconductors. Closely related to heat transfer is the transfer of particles by diffusion. Such transfer is vital to the transient and steady-state performance of solid-state, i.e., semiconductor, devices.

In a semi-conductor electric current is carried by two species of charged "particles", electrons and holes. The electron, of course, is a well known classical particle that carries a negative charge. A hole, on the other hand, is really the absence of an electron, but, at the same time, has many of the attributes of a positively charged particle. A thorough discussion of holes awaits your study of solid-state electronics.* Suffice it to say that the charge on the hole is exactly the negative of that on an electron.

Being particles in thermal agitation, electrons and holes don't like to remain concentrated, but would rather diffuse into regions of space in which the concentration is less, thereby filling space with equal densities of particles. Diffusion from one region of space to another naturally involves motion, which, because the moving particles are charged, constitutes an electric current. Thus there is a contribution to total current flow which is due to diffusion, and this contribution is called the diffusion current.

If we let J_e, J_h, respectively, stand for electron and hole current densities (amperes/meter^2), e be the (+) charge on a hole (equal to the negative of the electronic charge), D_e, D_h, be, respectively the electronic and hole diffusion constants, and n, h, the concentrations (no. of particles per unit volume) of electrons and holes, respectively, then the diffusion current contribution is given by

$$J_e\Big|_{diffusion} = e D_e \frac{\partial n}{\partial x}$$

$$J_h\Big|_{diffusion} = -e D_h \frac{\partial h}{\partial x}$$

In order to understand the difference in signs, consider Fig. 1-20. If $\partial n/\partial x < 0$, then electrons will diffuse to the right (direction of increasing x), which motion, because the electrons are negatively charged, constitutes a current flow to the left; i.e., in the direction of negative x. Thus, the sign of J_e/diffusion is identical to that of $\partial n/\partial x$. A similar argument applied to hole indicates the correctness of the (-) sign in the expression for $J_h|_{diffusion}$.

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* See for example, Shyh Wang, Solid-State Electronics, McGraw-Hill Book Co., 1966.

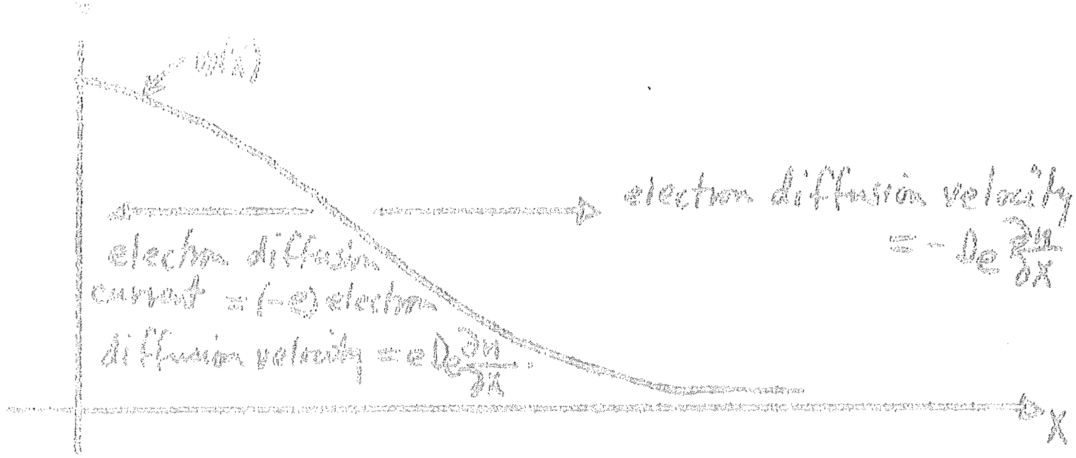

Figure 1-20. Illustrating $J_{e/diffusion} = eD_e \frac{\partial n}{\partial x}$.

In addition to the diffusion current there exists a drift current due to the presence of an electric field $\mathcal{E}$ (volts/meter) in the semiconductor. This contribution, for both electrons and holes, is

$$J_e|_{drift} = e\mu_e n\mathcal{E} = \sigma_e \mathcal{E}$$

$$J_h|_{drift} = e\mu_h h\mathcal{E} = \sigma_h \mathcal{E},$$

where μ_e, μ_h are, respectively, the electron and hole mobilities and σ_e, σ_h, are the electron and hole conductivities, respectively. There exists an interesting relationship, called the Einstein relation, between mobility and diffusion

$$D_{e,h} = \frac{kT}{e}\mu_{e,h},$$

where k is Boltzmann's constant and T is the temperature.

The total current is the sum of the diffusion and drift contributions

(1-92)
$$\text{(a)}\quad J_e = eD_e \frac{\partial n}{\partial x} + e\mu_e \mathcal{E} n.$$

$$\text{(b)}\quad J_h = -eD_h \frac{\partial h}{\partial x} + e\mu_h \mathcal{E} h.$$

The equation of current continuity supplies us with a second equation for electrons and holes. Thus, consider Fig. 1-21 as applied to electrons only (the result for holes is analogous). Our reasoning is identical to that used in the derivation of (1-91)(b).

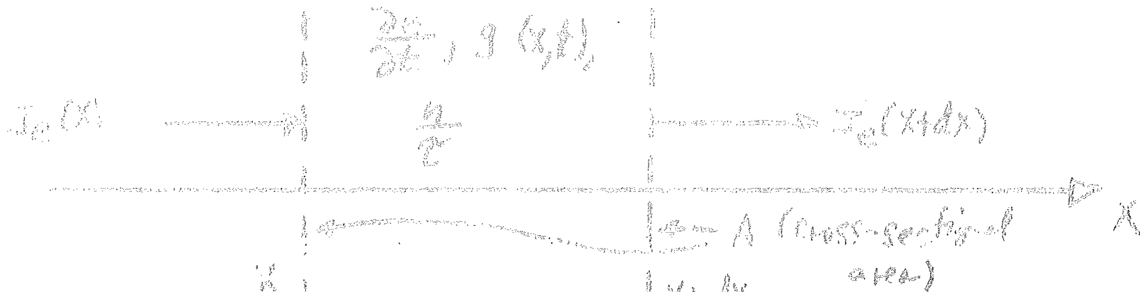

Figure 1-21. Derivation of the continuity equation for electrons. g (x,t) is the external source rate and τ is the recombination time of hole-electron pairs.

Let $\partial n/\partial t$ be the rate-of-change of stored charge (per unit volume) within the region, g (x,t) the rate of <u>production</u> (per unit volume) of new electrons (and holes, for that matter, because electrons and holes are produced in pairs) due, for example, to light absorbed within the region, and, finally, n/τ, the rate of recombination of electrons (per unit volume) with holes, τ, being the electron recombination time. The number of electrons per unit time <u>leaving</u> the region due to current flow is given by $-\frac{1}{e}\left[J_e(x+\Delta x)-J_e(x)\right]A$. Thus, conservation of charge demands that

$$\frac{\partial n}{\partial t}\cdot\Delta x A=\frac{1}{e}\left[J_e(x+\Delta x)-J_e(x)\right]A+g\,\Delta x A-\frac{n}{\tau}\Delta x A.$$

Upon dividing by $\Delta x\cdot A$ and passage to the limit $\Delta x\to 0$, we obtain the equation of electron charge continuity

(1-93) (a) $$\frac{\partial n}{\partial t}=\frac{1}{e}\frac{\partial J_e}{\partial x}+g-\frac{n}{\tau}.$$

The corresponding equation for holes is

(1-93) (b) $$\frac{\partial h}{\partial t}=-\frac{1}{e}\frac{\partial J_h}{\partial x}+g-\frac{h}{\tau}.$$

Again, note that the difference in sign in the corresponding term of the two equations, (1-93), is due to the difference in sign carried by the charge on the electron and hole.

Under "small-injection" conditions it is true that n = h, so that upon subtraction of (1-93) (b) from (a), it follows that $\frac{\partial}{\partial x}(J_e+J_h)=0$ or that $J_{total}=J_e+J_h$ is uniform with respect to the spatial variable x.

In order that we may show how (1-92) and (1-93) combine into a single diffusion equation (for electrons and holes), let us set $\mathcal{E}=0$ so that the drift term in the current equations vanishes. Then, upon differentiation of (1-92)(a) with respect to x, we get

$$\frac{\partial J_e}{\partial x}=eD_e\frac{\partial^2 n}{\partial x^2},$$

which, when substituted into (1-93)(a) yields

(1-94) $$\frac{\partial n}{\partial t}=D_e\frac{\partial^2 n}{\partial x^2}+g-\frac{n}{\tau}.$$

This is a <u>generalized</u> <u>diffusion</u> equation in that it involves a source term g(x,t) and a recombination term (or sink) n/τ. There exists an analogous electrical transmission-line for (1-92), (1-93). Let us consider the equations for the holes and the incremental section of transmission-line of Fig. 1-22.

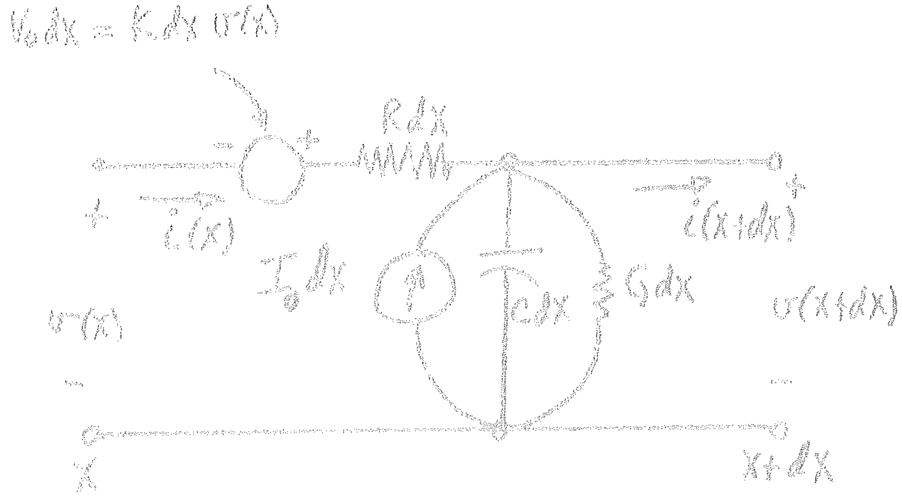

Figure 1-22. An incremental section of an RCG transmission-line that includes a series-distributed voltage-controlled voltage source and a shunt-distributed independent current source.

Kirchoff's voltage law yields

$$v(x) = -V_0 dx + i(x) R dx + v(x+dx)$$
$$= -K\, v(x) dx + i(x) R dx + v(x+dx).$$

This becomes, after rearrangement

$$i(x) = -\frac{1}{R}\frac{\partial v}{\partial x} + \frac{K}{R} v(x),$$

which is clearly analogous to (1-92)(b) with $i \sim J_n$, $v \sim n$, $R \sim \frac{1}{eD_n}$,

$$K \sim \frac{\mu_n \mathcal{E}}{D_n},$$

Similarly, Kirchoff's current law is

$$i(x) + I_0 dx = C dx \frac{\partial v(x+dx)}{\partial t} + G\, dx\, v(x+dx) + i(x+dx).$$

Rearrangement yields

$$\frac{\partial v}{\partial t} = \frac{I_0}{C} - \frac{1}{C}\frac{\partial i}{\partial x} - \frac{G}{C}\, v(x).$$

This is analogous to (1-93)(b), using the same analogies as before. Note that $\mathcal{E} \sim \mathcal{E}/G$.

[illegible] Let us now use the equations that we developed in the last section to discuss the transient diffusion of electrons in a semiconductor diode as it is switched from one bias state to another. Such an analysis is obviously useful to the circuit designer who is using these devices under transient conditions.

The system is depicted in Fig. 1-23. An ideal voltage source $v_b(t)$ applies a step-voltage of magnitude V_b directly across the diode. A magnified version of the diode is shown in (b) of the figure. The diode consists of a p-type semiconductor in contact with an n-type (the base). A p-n junction is formed at $x = 0$, and the base is terminated at $x = w$ in an ohmic contact.

A semiconductor that is doped with impurities such that the principal (or majority) charge carriers are <u>electrons</u> is said to be <u>n-type</u>; one doped such that the principal charge carriers are <u>holes</u> is said to be <u>p-type</u>. The <u>minority</u> carriers in an n-type semiconductor are holes and in a p-type, electrons.

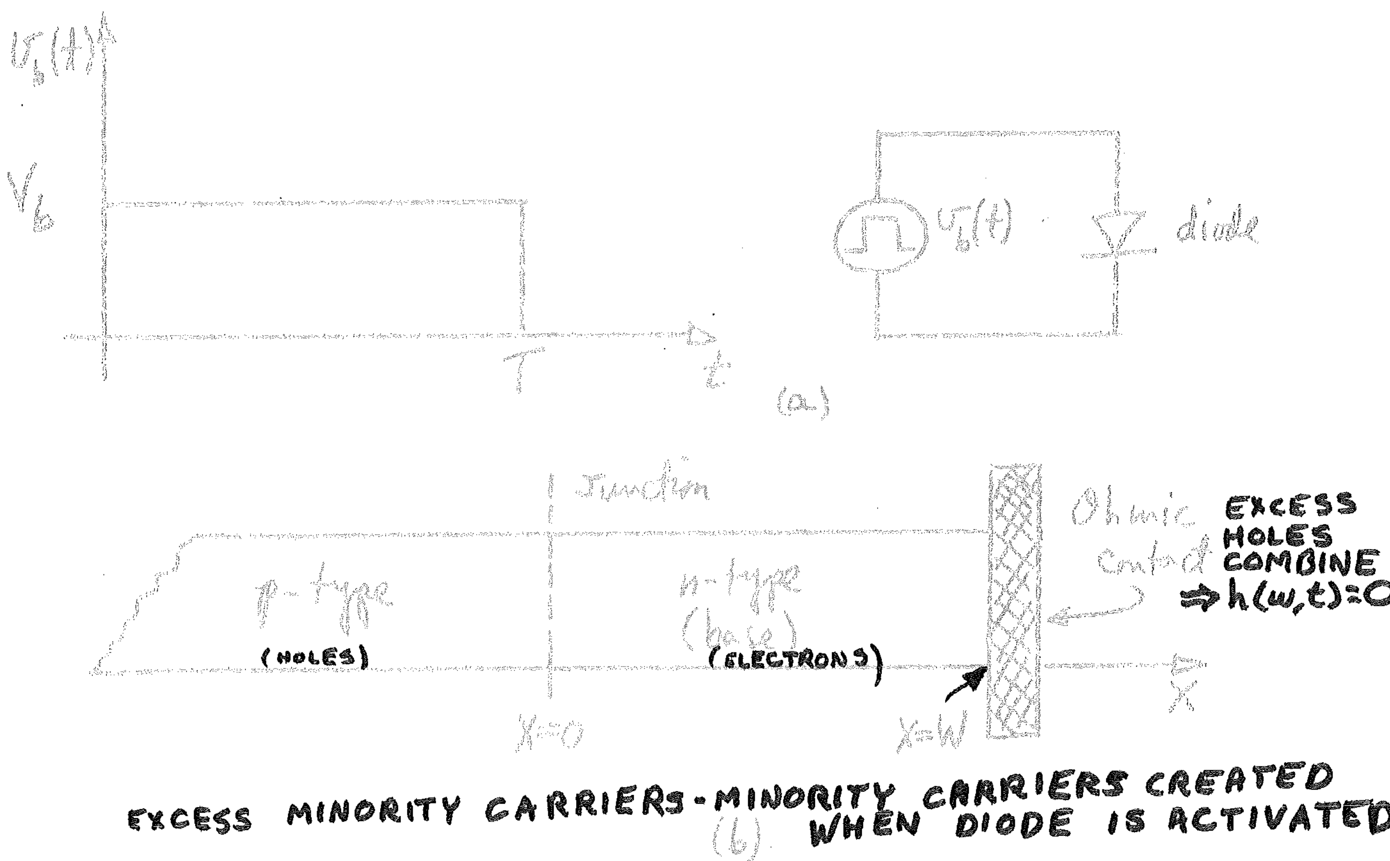

Figure 1-23. Relevant to calculating the switching response of a semiconductor diode. A magnified version of the diode, showing the p-n junction, the n-type base region, and the ohmic contact, is shown in (b).

Whenever two dissimilar materials, such as a p- and n-type semiconductor, or a semiconductor and a metal, are brought into contact such that a junction is formed, that junction becomes rectifying in the electrical sense of passing current one way easily but with difficulty in the reverse direction. Thus, we speak of a p-n junction rectifier, or diode. On the other hand, if suitable precautions are taken, the semiconductor-metal junction can be made non-rectifying. Such a junction is said to be an "ohmic" contact.

There are two conditions of interest in a semiconductor (or, for that matter, any physical system) - equilibrium and non-equilibrium. A semiconductor in equilibrium, i.e., with no applied bias voltages or light, etc., cannot conduct current. Current results only when there is an excess of minority carriers in the appropriate region of the diode. Thus, when a diode is biased, a situation is said to have been created whereby excess minority carriers are injected across the junction. This is a distinctly non-equilibrium condition.

In modeling the diode transient behavior we will assume that the excess minority carriers (holes) at the junction $x = 0$ in the base region (Fig. 1-23) instantly respond to the bias voltage (which can be assumed to appear entirely across the junction for low injection conditions) in accordance with

(1-95)(a) $$p(0+,t) = \frac{n_i^2}{N_D}\left[\exp\left(\frac{e v_b(t)}{kT}\right) - 1\right].$$ BOUNDRY CONDITION AT JUNCTION

Here n_i is called the intrinsic electron density in the n-type (base) region and N_D is the donor impurity density, i.e., the density of impurity atoms capable of donating electrons for conduction purposes, e is the electronic charge, k is Boltzmann's constant and T the temperature. When $v_b(t) = 0$, $p = 0$, which implies that the excess minority carrier density vanishes when the system returns to equilibrium upon removal of the bias voltage.

Eqn. (1-95)(a) is one boundary condition for p. The second is that at the ohmic contact, $x = w$, the excess minority carrier density vanishes

(1-95)(b) $$p(w,t) = 0 \quad .$$

The proof of (1-95) rests on a thorough study of semiconductor theory and cannot be given here.

Our analysis of the transient behavior of the diode is based on the solution of the diffusion equation for holes (the excess minority carriers in the base region) under the conditions of vanishing electric field, $\mathscr{E}$, in the base region, $g = 0$ and no recombination ($\tau = \infty$). The former condition follows from our assumption that the entire voltage drop, $v_b(t)$, is across the junction, there being no drop, hence, no field, in either the p- or n-type regions. The latter condition is due to the fact that the base is assumed sufficiently narrow, i.e., W is small, so that no excess minority holes recombine with electrons during the trip from $x = 0$ to $x = w$. Such a diode is called narrow base.

The appropriate equations are (1-92)(b), which under the assumptions stated, reads $J_h = -e D_h \partial h/\partial x$ and (1-93)(b) $\partial h/\partial t = -\frac{1}{e}\partial J_h/\partial x$.
These two equations combine to yield the diffusion equation

$$\frac{\partial h}{\partial t} = D_h \frac{\partial^2 h}{\partial x^2} .$$

(A). Switch-on Transient. At t=0 the diode is forward biased (the positive voltage contact is made with the p-type material) to V_b volts and held fixed. Initially, there was no bias voltage applied. Hence, for the narrow base diode, with no recombination, our problem is to solve the system

(1-96)(a) $\dfrac{\partial h}{\partial t} = D_h \dfrac{\partial^2 h}{\partial x^2}$; (b) boundary conditions; $h(0+,t) = \dfrac{n_i^2}{N_D}\left\{\exp\left(\dfrac{eV_b}{kT}\right)-1\right\}u_{-1}(t)$
$= p_{n0}\, u_{-1}(t)$

(c) initial condition, $h(w,t) = 0$, all t.

$$h(x,0) = 0, \quad 0 < x \le w.$$

In the absence of non-zero initial conditions, the Laplace transform of (1-96)(a) is the same as (1-85) with D_h replacing RC:

$$\frac{d^2 H(x,s)}{dx^2} - \frac{s}{D_h} H(x,s) = 0$$

where $H(x,s) = \mathcal{L}[h(x,t)]$. The solution of this equation is

(1-97) $\quad H(x,s) = A e^{-\beta x} + B e^{\beta x}, \qquad \beta = \left(\dfrac{s}{D_h}\right)^{1/2},$

and A, B are arbitrary constants. In (1-86) we retained only a single exponential term because the region under consideration was infinitely long and the second exponential term exploded at x = ∞. Here, obviously, we have a finite region, h ence two boundary conditions and two arbitrary constants.

Upon taking the Laplace transforms of the boundary conditions, (1-96)(b), we get

$$H(0+,s) = \frac{p_{n0}}{s}, \qquad H(w,s) = 0.$$

Using these results in (1-97) we obtain two equations in the two unknowns A, B:

$$\frac{p_{n0}}{s} = A + B$$
$$0 = A e^{-\beta w} + B e^{\beta w},$$

which when solved yield

$$A = \frac{p_{n0}}{2s}\frac{e^{\beta w}}{\sinh \beta w}, \qquad B = -\frac{p_{n0}}{2s}\frac{e^{-\beta w}}{\sinh \beta w}.$$

Thus, H(x,s) is

$$H(x,s) = \frac{p_{no}}{2s}\left[\frac{e^{\beta w}\cdot e^{-\beta x}}{\sinh \beta w} - \frac{e^{-\beta w}\cdot e^{\beta x}}{\sinh \beta w}\right] \tag{1-98}$$

FORCING FUNCTION ↓

$$= \frac{p_{no}}{s}\cdot\frac{\sinh \beta(w-x)}{\sinh \beta w}$$ ← SYSTEM TRANSFER FUNCTION

We can immediately determine the steady-state hole distribution by applying the final value theorem to (1-98). We recall that the final value theorem states that as long as the Laplace transform, F(s), of f(t) has no poles on the $j\omega$-axis (other than S=0), then $f(\infty) = \lim_{s\to 0} sF(s)$. In applying this result to (1-98) we see that

$$sH(x,s) = p_{no}\frac{\sinh \beta(w-x)}{\sinh \beta w} = p_{no}\frac{\sinh\left(\frac{s}{D_h}\right)^{1/2}(w-x)}{\sinh\left(\frac{s}{D_h}\right)^{1/2} w}.$$

As $s\to 0$, the arguments of each of the sinh functions approaches zero. The sinh function, like the sin, approaches its argument, for small values of the argument. Hence, $\sinh\left(\frac{s}{D_h}\right)^{1/2}(w-x) \to \left(\frac{s}{D_h}\right)^{1/2}(w-x)$ and $\sinh\left(\frac{s}{D_h}\right)^{1/2} w \to \left(\frac{s}{D_h}\right)^{1/2} w$.

$\sinh j\theta = j\sin\theta$

Thus,

$$h(x,+\infty) = \lim_{s\to 0} sH(x,s) = p_{no}\left(\frac{w-x}{w}\right), \tag{1-99}$$

a function that decreases linearly from p_{no}, at x=0, to 0 at x=w. The steady-state hole diffusion-current follows from $J_h = -eD_h\,\partial h/\partial x$, and is equal to

$$i_h(x,+\infty) = AJ_h(x,+\infty) = \frac{eAD_h p_{no}}{w}, \tag{1-100}$$

where A is the cross-sectional area of the diode base region.

The Laplace transform of the hole diffusion-current at the junction, x=0, follows upon differentiation of (1-98) with respect to x:

$$I_h(0,s) = -eAD_h\left.\frac{dH(x,s)}{dx}\right|_{x=0} = eAp_{no}\left(\frac{D_h}{s}\right)^{1/2}\frac{\cosh \beta w}{\sinh \beta w}. \tag{1-101}$$

If we apply the final value theorem to (1-101) we obtain the same result (1-100), as we should. The initial current (as $t\to 0+$) can be obtained from (1-101) by appealing to the initial value theorem, which states that $f(0+) = \lim_{s\to\infty} sF(s)$.

Hence, the initial value of hole diffusion-current at the junction is

$$(1\text{-}102) \qquad i_h(0,0+) = \lim_{s\to\infty} eAp_{no}\, s\left(\frac{D_h}{s}\right)^{1/2} \frac{\cosh \beta W}{\sinh \beta W}$$

$$\xrightarrow[s\to\infty]{} eAp_{no}(sD_h)^{1/2}$$

$$\xrightarrow[s\to\infty]{} \infty .$$

Note that $\displaystyle \lim_{s\to\infty} \frac{\cosh\left(\frac{s}{D_h}\right)^{1/2} W}{\sinh\left(\frac{s}{D_h}\right)^{1/2} W} = \lim_{s\to\infty} \frac{e^{\left(\frac{s}{D_h}\right)^{1/2} W} + e^{-\left(\frac{s}{D_h}\right)^{1/2} W}}{e^{(s/D_h)^{1/2} W} - e^{-(s/D_h)^{1/2} W}} = 1.$

Therefore, we conclude that the hole diffusion-current at the junction initially jumps to infinity after switch-on and then decays to the steady-state value $eAD_h p_{no}/W$ (Figure 1-24)

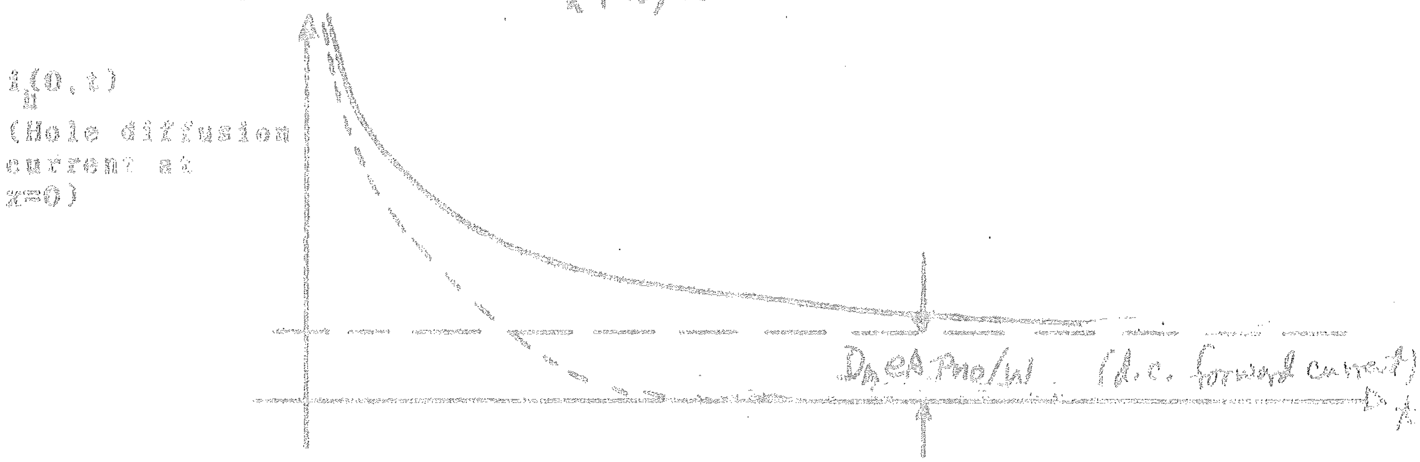

Figure 1-24. Transient response of the hole diffusion-current at the junction due to a step of voltage: switch-on transient. The dashed and dotted curves are explained in the text.

This behavior is roughly what one would expect from a charge-storage system, such as a capacitor, in that the current jumps to infinity in response to a suddenly changed voltage and then decays to a steady-state value.

Let us go back to (1-98) and examine, qualitatively, the nature of h(x,t), as a function of x. We have already observed that $\lim_{t\to\infty} h(x,t) = p_{no}\left(\frac{W-x}{W}\right)$. The initial value theorem, when applied to (1-98), implies that

$$\lim_{t\to 0+} h(x,t) = \lim_{s\to\infty} p_{no} \frac{\sinh\left(\frac{s}{D_h}\right)^{1/2}(W-x)}{\sinh\left(\frac{s}{D_h}\right)^{1/2} W} = 0, \quad x \neq 0$$

$$= p_{no}, \quad x = 0,$$

a discontinuous function of x. We have sketched the qualitative response in Fig. 1-25. The total number of excess charges (holes) is given by $A\int_0^{W} h(x,t)\,dx$; i.e., it is proportional to the area under the curves corresponding to the different times. We have

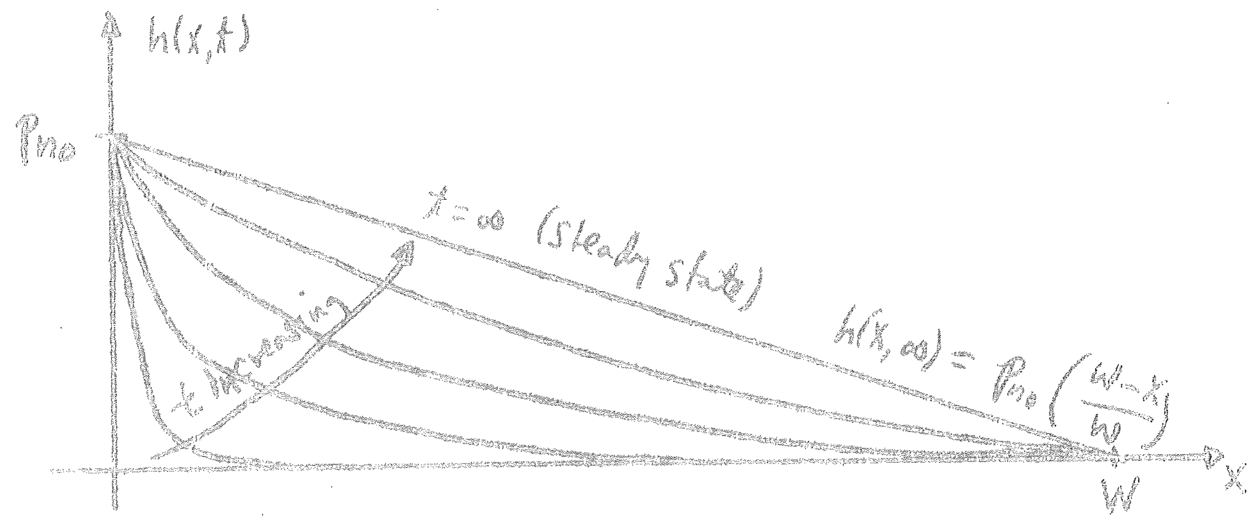

Figure 1-25. The build-up of excess holes, h, in the n-type base region during the switch-on transient.

surmised a monotone character for current vs. time or h vs. x and t. We must now give a more accurate account of the variation of these variables with x and t.

Return to (1-98) as our starting point. In order to determine the time response we must determine the inverse Laplace transform

$$h(x,t) = \mathcal{L}^{-1}\left[H(x,s)\right] = \mathcal{L}^{-1}\left[\frac{p_{no}}{s}\cdot\frac{\sinh\left(\frac{s}{D_h}\right)^{1/2}(W-x)}{\sinh\left(\frac{s}{D_h}\right)^{1/2}W}\right].$$

Our first job is to evaluate the poles. Obviously, $s = 0$ is one; the others are given by the zeroes of the denominator $\sinh\left(\frac{s}{D_h}\right)^{1/2}W$, except for $s = 0$, because the numerator also vanishes at $s = 0$. That is, $s = 0$ is a pole contributed by the term $\frac{1}{s}$ and not by the sinh function.

From the definition of $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$, we see that the only real zero of $\sinh\theta$ is $\theta = 0$. There are imaginary zeros, however,-- infinitely many! Note that $\sinh j\theta = \frac{\exp(j\theta)-\exp(-j\theta)}{2} = j\sin\theta$, and $\sin\theta$ has zeros at $\theta = \pm n\pi$, $n = 1, 2, \cdots$. as well as the aforementioned zero at $\theta = 0$. Thus

$$\sinh\left(\frac{s_n}{D_h}\right)^{1/2}W = 0 \Longrightarrow \left(\frac{s_n}{D_h}\right)^{1/2}W = \pm jn\pi, \quad n = 1, 2, \cdots.$$

and it follows that the zeros are given by

$$s_n = -n^2\pi^2\, D_h/W^2, \qquad n = 1, 2, \cdots;$$

they all lie on the negative real axis in the complex s-plane.

We can expand H(x,s) into partial fractions

(1-103)
$$H(x,s) = \frac{A_0}{s} + \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n} + \cdots$$
$$= \frac{A_0}{s} + \frac{A_1}{s+\pi^2 D_h/w^2} + \frac{A_2}{s+4\pi^2 D_h/w^2} + \cdots + \frac{A_n}{s+n^2\pi^2 D_h/w^2} + \cdots,$$

where the residues $\{A_n\}$ are given by the formula

(1-104)
$$A_n = \lim_{s \to s_n} (s-s_n) H(x,s) = \lim_{s \to s_n} (s-s_n) \cdot \frac{p_{n0}}{s} \cdot \frac{\sinh\left(\frac{s}{D_h}\right)^{1/2}(w-x)}{\sinh\left(\frac{s}{D_h}\right)^{1/2} w}.$$

Note that $\sinh\left(\frac{s_n}{D_h}\right)^{1/2} w = 0$ because the $\{s_n\}$ are zeros (by definition) of the denominator. Hence, the factor

$$\frac{s-s_n}{\sinh\left(\frac{s}{D_h}\right)^{1/2} w}$$ is indeterminate as $s \to s_n$.

The indeterminary may be resolved by using L'Hospital's rule

$$\lim_{s \to s_n} \frac{s-s_n}{\sinh\left(\frac{s}{D_h}\right)^{1/2} w} = \lim_{s \to s_n} \frac{1}{\frac{d}{ds}\left[\sinh\left(\frac{s}{D_h}\right)^{1/2} w\right]}$$
$$= (s_n D_h)^{1/2} \frac{2}{w \cosh\left(\frac{s_n}{D_h}\right)^{1/2} w}.$$

Thus,

(1-105)
$$A_n = \frac{p_{n0}}{s_n}\left(\frac{2(s_n D_h)^{1/2}}{w}\right)\frac{\sinh\left(\frac{s_n}{D_h}\right)^{1/2}(w-x)}{\cosh\left(\frac{s_n}{D_h}\right)^{1/2} w} = -\frac{2 p_{n0}}{n\pi}\sin\frac{n\pi x}{w}, \quad n \neq 0.$$
$$A_0 = p_{n0}\left(\frac{w-x}{w}\right).$$

The expression for A_n follows (after some algebra) from the results for s_n and $(s_n)^{1/2}$. The steps can be done as an exercise.

Now, when the results, (1-105), are substituted into (1-103) and the inverse transform is taken (recalling that $\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$, and $\mathcal{L}^{-1}$[sum of terms] = sum of L^{-1}[term]) we arrive at the result

(1-106)
$$h(x,t) = \mathcal{L}^{-1}\left[H(x,s)\right] = p_{n0}\left(\frac{w-x}{w}\right)u_{-1}(t) - 2 p_{n0}\sum_{n=1}^{\infty} e^{-D_h\left(\frac{n\pi}{w}\right)^2 t}\,\frac{\sin(n\pi x/w)}{n\pi}$$

As usual, in a distributed system there are infinitely many time-constants, but only a few are significant. As n increases, the time constants decrease as $1/n^2$. Hence, the tenth time constant is one-percent of the first (and largest). It follows that one may need to keep only the first four or five terms in order to get an accurate estimate of (1-106).

An interesting result is obtained when we recall that $h(x,0)=0$ for all x was our initial condition (1-96)(c). When this is used in (1-106) we have

$$(1\text{-}107) \qquad p_{no}\left(\frac{W-x}{W}\right) = 2p_{no}\sum_{n=1}^{\infty} \frac{\sin(n\pi x/W)}{n\pi},$$

which is the Fourier series expansion of $p_{no}\left(\frac{W-x}{W}\right)$. Note that at x=0, the series converges to zero. This is a typical phenomenon of Fourier series of discontinuous functions.

The hole diffusion-current at the junction is

$$(1\text{-}108) \qquad i_h(0,t) = -eAD_h\frac{\partial h}{\partial x}\bigg|_{x=0} = p_{no}\frac{eAD_h}{W}u_{-1}(t) + 2p_{no}\frac{eAD_h}{W}\sum_{n=1}^{\infty} e^{-D_h\left(\frac{n\pi}{W}\right)^2 t}.$$

The first term is the dashed step-function, shown in Fig. 1-24, and the second term, the infinite series, is the dotted curve. As time passes the infinite series behaves asymptotically like $\frac{2p_{no}eAD_h}{W}e^{-D_h\left(\frac{\pi}{W}\right)^2 t}$, the dominant time-constant term.

(8). Switch-off Transient. After the system has achieved the steady-state implied in (1-99), (1-100) and Figures (1-24, 1-25), the source voltage $v_b(t)$ in Fig. 1-23 is switched to zero. We must calculate the resulting time response of either $h(x,t)$ or $i_h(x,t)$.

Because $v_b(t) = 0$ in (1-95)(a) it follows that the excess hole concentration at the junction, x=0, instantly reduces to zero and remains at that value. The initial distribution of holes is given by (1-99); the condition at x=W, (1-95)(b) continues to be valid. Hence, our mathematical model reduces to the solution of the following initial-boundary value problem for the diffusion equation

$$(1\text{-}109) \qquad (a)\ \frac{\partial h}{\partial t} = D_h\frac{\partial^2 h}{\partial x^2} \qquad (b)\ \text{boundary conditions: } h(0,t) = h(W,t) = 0 \text{ for all } t > 0.$$

$$(c)\ \text{initial condition. } h(x,0+) = p_{no}\left[\frac{W-x}{W}\right],\quad 0 < x \le W.$$

Because we have a non-zero initial condition, the differential equation for the Laplace transform of $h(x,t)$ is

$$(1\text{-}110) \qquad \frac{d^2H(x,s)}{dx^2} - \left(\frac{s}{D_h}\right)H(x,s) = -\frac{h(x,0+)}{D_h} = -\frac{p_{no}}{D_h}\left[\frac{W-x}{W}\right].$$

While we could apply the Green's function technique previously described in connection with initial value problems, the right-hand side of (1-110) is so simple that we may solve for H(x,s) and satisfy the boundary conditions more easily.

The total solution of (1-110) is the sum of a complementary function and a particular integral. The complementary function is $Ae^{-\beta x} + Be^{\beta x}$, $\beta = \left(\frac{s}{D_h}\right)^{1/2}$, as in (1-97), with A and B arbitrary. A particular integral satisfying (1-110) is $\frac{p_{no}}{s}\left[\frac{W-x}{W}\right]$, as can be easily verified upon substitution into (1-110). Thus, the solution for H(x,s) is

$$(1\text{-}111) \quad H(x,s) = Ae^{-\beta x} + Be^{\beta x} + \frac{p_{no}}{s}\left[\frac{W-x}{W}\right].$$

The boundary conditions (1-109)(b) yield

$$H(0,s) = 0 = A + B + p_{no}/s$$
$$H(W,s) = 0 = Ae^{-\beta W} + Be^{\beta W},$$

from which we find that

$$A = -\frac{p_{no}}{s}\cdot\frac{e^{\beta W}}{2\sinh\beta W}, \quad B = \frac{p_{no}}{s}\cdot\frac{e^{-\beta W}}{2\sinh\beta W},$$

and, hence

$$(1\text{-}112) \quad H(x,s) = \frac{p_{no}}{s}\left[\left(\frac{W-x}{W}\right) - \frac{\sinh\beta(W-x)}{\sinh\beta W}\right], \quad \beta = \left(\frac{s}{D_h}\right)^{1/2}.$$

The inverse transform of $\frac{p_{no}}{s}\left[\frac{W-x}{W}\right] = p_{no}\left[\frac{W-x}{W}\right]u_{-1}(t)$,

while from our previous work we found that

$$\mathcal{L}^{-1}\left[\frac{p_{no}}{s}\cdot\frac{\sinh\beta(W-x)}{\sinh\beta W}\right] = p_{no}\left(\frac{W-x}{W}\right)u_{-1}(t) - 2p_{no}\sum_{n=1}^{\infty} e^{-D_h\left(\frac{n\pi}{W}\right)^2 t}\,\frac{\sin(n\pi x/W)}{n\pi}.$$

Thus, the inverse transform of (1-112), being the difference of these two expressions, is given by

$$(1\text{-}113) \quad \mathcal{L}^{-1}\left[H(x,s)\right] = h(x,t) = 2p_{no}\sum_{n=1}^{\infty} e^{-D_h\left(\frac{n\pi}{W}\right)^2 t}\,\frac{\sin(n\pi x/W)}{n\pi}.$$

This result, which is sketched in Fig. 1-26(a), should be compared with the corresponding result for switch-on, Fig. 1-25. The hole diffusion-current at any point x is given by

$$(1\text{-}114) \quad i_h(x,t) = -eAD_h\frac{\partial h}{\partial x} = -\frac{2eAD_h p_{no}}{W}\sum_{n=1}^{\infty} e^{-D_h\left(\frac{n\pi}{W}\right)^2 t}\cos\left(\frac{n\pi x}{W}\right).$$

At $x = 0$, the current becomes

$$(1\text{-}115) \quad i_h(0,t) = -\frac{2eAD_h}{W} p_{no} \sum_{n=1}^{\infty} e^{-D_h\left(\frac{n\pi}{W}\right)^2 t},$$

and is sketched in Fig. 1-26(b). The total (switch-on and switch-off) hole diffusion current at $x=0$ is shown in Fig. 1-26(c).

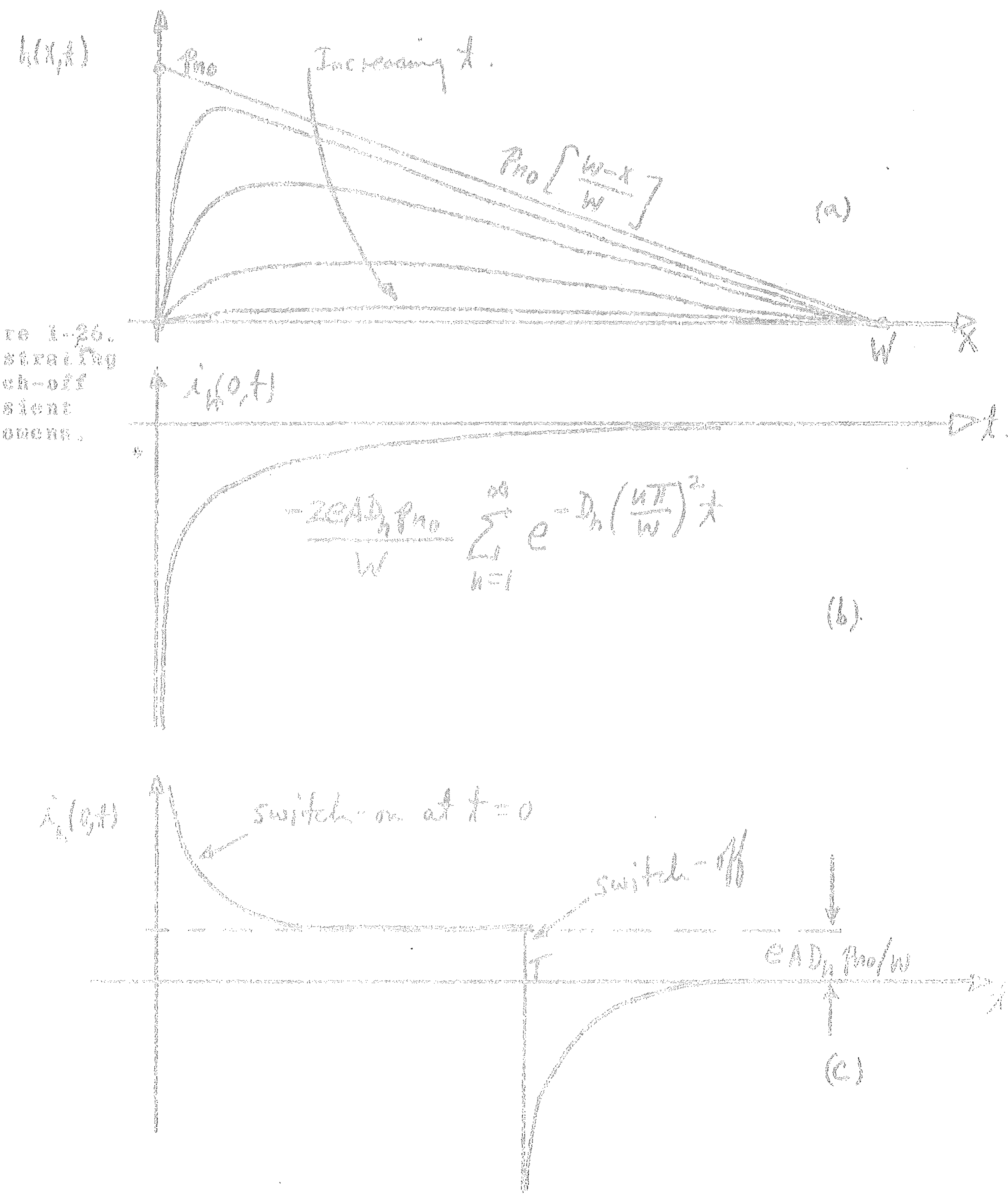

Figure 1-26. Illustrating switch-off transient phenomena.

Due to external resistances, the switch-off current does not drop to infinity, but drops and remains constant at a finite negative value for a short length of time, then rises to zero, as indicated by the tail end of Figure I-26(c).

10. Transient Response of a Wide-Base Diode.*

A wide-base diode has the same configuration as Fig. I-23(b) except that the base region is so wide that excess holes will recombine with excess electrons before reaching the base contact. Effectively, then, the base contact, in the mathematical model, is removed to ∞ ($W = \infty$ in Fig. I-23(b)) and the appropriate generalized diffusion equation is (I-94) without the source term and written for holes,

$$\frac{\partial h}{\partial t} = D_h \frac{\partial^2 h}{\partial x^2} - \frac{h}{\tau}.$$

Here is the problem that we pose. Assume that in the semi-infinite base region of Fig. I-27 there exists an initial distribution of holes, $h(x,0+)=f(x)$, $x>0$, and that there also exists a junction bias voltage, $v_b(t)$, which produces an excess of holes at the junction $x=0$ in accordance with (I-95)(a). Call this excess-hole time-function $g(t)$. Thus, $h(0,t)=g(t)$, $t>0$. The problem, then, is to solve the diffusion equation

$$\frac{\partial h}{\partial t} = D_h \frac{\partial^2 h}{\partial x^2} - \frac{h}{\tau}$$

in the base region, $x>0$, and for $t>0$, subject to these initial and boundary conditions. We have only listed a boundary condition at $x=0$; the boundary condition at $x=\infty$ is, of course, $h(x,t)\to 0$ $x\to\infty$, because all of the excess holes have recombined before reaching $x=\infty$.

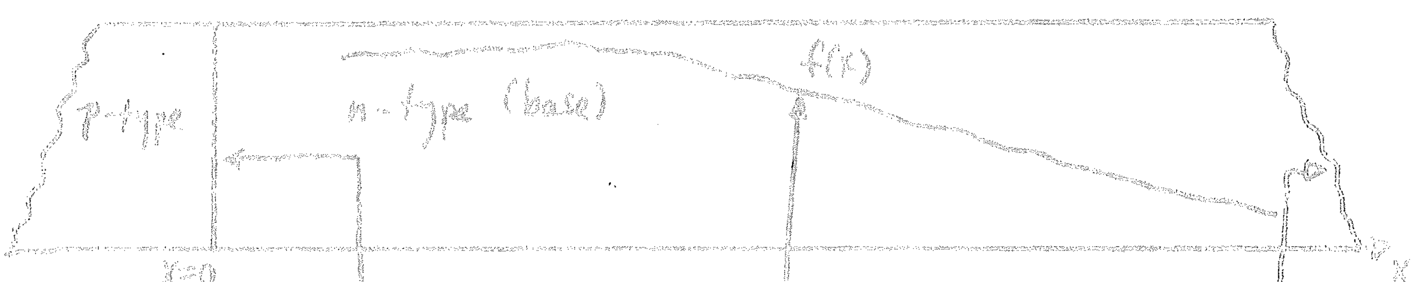

Figure I-27. Illustrating the initial and boundary conditions relevant in solving the transient p-n junction response for a wide-base diode.

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* See B. Lax and S. F. Neustadter "Transient Response of a p-n Junction", J. App. Phys., Vol. 25, pp. 1148-1154, 1954.

[illegible] define our problem:

(1-115) to solve: $$\frac{\partial h}{\partial t} = D_h \frac{\partial^2 h}{\partial x^2} - \frac{h}{\tau}$$

boundary conditions: $h(0,t) = g(t)$, $h(\infty,t) = 0$, for all $t \geq 0$

initial condition: $h(x,0+) = f(x)$, for all $x \geq 0$.

This is a "mixed" problem, i.e., one involving both a non-zero initial condition as well as boundary conditions. We argue that because the differential equation plus boundary and initial conditions are all linear (i.e., this is a linear system), the total response will be the superposition of the responses of the individual <u>initial-value</u> problem, taking the boundary conditions to be zero, i.e., setting g(t)=0, and the <u>boundary value problem</u>, taking the initial condition to be zero, i.e., setting f(x)=0.

Before solving either of these sub-problems, let us simplify the differential equation. Let

(1-116) $$z(x,t) = e^{t/\tau}\, h(x,t).$$

Then $$\frac{\partial z}{\partial t} = \frac{1}{\tau} e^{t/\tau} h(x,t) + e^{t/\tau} \frac{\partial h}{\partial t}$$

$$\frac{\partial^2 z}{\partial x^2} = e^{t/\tau} \frac{\partial^2 h}{\partial x^2}.$$

Multiplying the second equation by $-D_h$ and adding to the first yields

(1-117) $$\frac{\partial z}{\partial t} - D_h \frac{\partial^2 z}{\partial x^2} = e^{t/\tau}\left[\frac{h}{\tau} + \frac{\partial h}{\partial t} - D_h \frac{\partial^2 h}{\partial x^2}\right] = 0,$$

where the last equality follows from the original generalized diffusion equation satisfied by h(x,t), (1-115). Thus, we have shown that the function z(x,t), defined by (1-116), satisfies the conventional diffusion equation without the recombination term.

Let us now proceed to solve the individual boundary-and initial-value problems, whose individual solutions will add to the total response.

A. Boundary-value problem. We must solve

$$D_h \frac{\partial^2 z_1}{\partial x^2} = \frac{\partial z_1}{\partial t}, \quad z_1(0,t) = e^{t/\tau} \cdot g(t).$$

This problem can be solved using the same approach taken in section 6 to solve for the step response of an infinite RC-line. In (1-88) we found that step response to be

$$v(x,t) = \int_0^t \left(\frac{RC}{\pi}\right)^{1/2} \cdot \frac{x}{2} \cdot \frac{e^{-RCx^2/4\rho}}{\rho^{3/2}} \cdot d\rho.$$

The impulse response for the same line is given by the derivative of the step response, because the impulse is the derivative of a step. Hence, the impulse response is simply the integrand of the above expression with t replacing ρ

$$\left(\frac{RC}{\pi}\right)^{1/2} \cdot \frac{x}{2} \cdot \frac{e^{-RCx^2/4t}}{t^{3/2}}.$$

It follows, therefore, by the convolution integral, that the response of the RC-line to an arbitrary input, say $e^{t/\tau} \cdot g(t)$, is given by

$$\left(\frac{RC}{\pi}\right)^{1/2} \cdot \frac{x}{2} \cdot \int_0^t e^{\rho/\tau} g(\rho) \frac{e^{-RCx^2/4(t-\rho)}}{(t-\rho)^{3/2}} \, d\rho.$$

If, therefore, we simply replace RC by $1/D_h$ (in order to establish the correct analogy), we have the solution of the boundary-value problem of hole diffusion

$$(1\text{-}118) \quad z_1(x,t) = \frac{x}{2(\pi D_h)^{1/2}} \int_0^t e^{\rho/\tau} g(\rho) \frac{e^{-\frac{x^2}{4D_h(t-\rho)}}}{(t-\rho)^{3/2}} \, d\rho.$$

B. Initial-value problem. We must solve $\frac{\partial z_2}{\partial t} = D_h \frac{\partial^2 z_2}{\partial x^2}$, $z_2(x,0+) = f(x)$.

Note that $z(x,0+) = h(x,0+) = f(x)$ by virtue of (1-116).

Let us take the Laplace transform of this equation and utilize the initial condition

$$\frac{\partial^2 Z_2(x,s)}{\partial x^2} = \frac{s}{D_h} Z_2(x,s) - \frac{f(x)}{D_h}$$

which is a specialized form of (1-45). This equation is to be solved assuming homogeneous boundary conditions, i.e., $Z_2(0,s) = Z_2(\infty,s) = 0$.

The solution of the above inhomogeneous equation will be in the form of an integral involving a Green's function (recall example 3, section 2). Had f(x) been given as a simple function, we might have been able to guess a particular integral without the need of a Green's function. The Green's function approach, however, is the most general form of solution of such an equation; in addition, our method of solution will involve the interesting application of images, with which we already have a nodding acquaintance (section 1).

Thus, we postulate

$$(1\text{-}119) \quad \bar{Z}_2(x,s) = -\frac{1}{D_h}\int_0^{\infty} G(x|x')\, f(x')\,dx',$$

where the Green's function $G(x/x')$ satisfies

$$(1\text{-}120) \quad (a) \quad \frac{d^2G(x|x')}{dx^2} - \frac{s}{D_h}G(x|x') = \delta(x-x')$$

$$(b) \quad G(0|x') = G(+\infty|x') = 0.$$

The interpretation of (1-120) (a) is that the Green's function is the response at x to a unit point source at x'. The range of integration in (1-119) is $[0,\infty)$ because we are interested in the hole distribution in the n-type region only.

We proceed to determine $G(x/x')$. It will facilitate matters to assume that the region of interest is the entire x-axis $(-\infty,\infty)$; we will consider the effects of the boundary $x = 0$ later.

Therefore, we wish to find the response at x due to a unit point source at x', when x and x' can lie anywhere on the entire x-axis. The response must satisfy (1-22) (b), (d), i.e., the derivative of $G(x|x')$ must be discontinuous at $x = x'$, with unity value for the discontinuity, but $G(x|x')$ must be continuous at $x = x'$.

Appropriate solutions for (1-120)(a), because $\delta(x-x')=0$, $x \neq x'$, are

$$G(x|x') = A_+ e^{-\left(\frac{s}{D_h}\right)^{1/2}x}, \qquad x' < x < \infty$$

$$= A_- e^{\left(\frac{s}{D_h}\right)^{1/2}x}, \qquad -\infty < x < x',$$

where the exponentials are chosen so that $G(x|x') \to 0$ as $x \to \pm\infty$.

Continuity of $G(x|x')$ at $x \to x'$ requires that

$$A_+ e^{-\left(\frac{s}{D_h}\right)^{1/2}x'} = A_- e^{(s/D_h)^{1/2}x'},$$

while the discontinuity in derivative at $x = x'$ implies that

$$\left(\frac{s}{D_h}\right)^{1/2}\left[A_+ e^{-\left(\frac{s}{D_h}\right)^{1/2}x'} + A_- e^{\left(\frac{s}{D_h}\right)^{1/2}x'}\right] = -1.$$

Upon solving these two equations for A_+ and A_-, there results

$$A_+ = -\frac{1}{2}\left(\frac{D_h}{s}\right)^{1/2} e^{(s/D_h)^{1/2}x'}$$

$$A_- = -\frac{1}{2}\left(\frac{D_h}{s}\right)^{1/2} e^{-(s/D_h)^{1/2}x'},$$

which yields for our solution in the infinite region

$$(1\text{-}121) \quad G(x|x') = -\frac{1}{2}\left(\frac{D_h}{s}\right)^{1/2} e^{-(s/D_h)^{1/2}(x-x')} \quad , \; x' < x < \infty$$

$$= -\frac{1}{2}\left(\frac{D_h}{s}\right)^{1/2} e^{(s/D_h)^{1/2}(x-x')} \quad , \; -\infty < x < x' .$$

Of course, we haven't satisfied our boundary condition at x=0 (1-120(b)); how could we have, we deliberately removed the boundary by expanding our u-type region to the entire x-axis?

Equation (1-121) is the solution for a point source located at x' (which could be anywhere). Now for each point x', and its corresponding unit point source, let us consider the image point in the origin, i.e., -x', with its corresponding negative unit point source (Fig. 1-28).

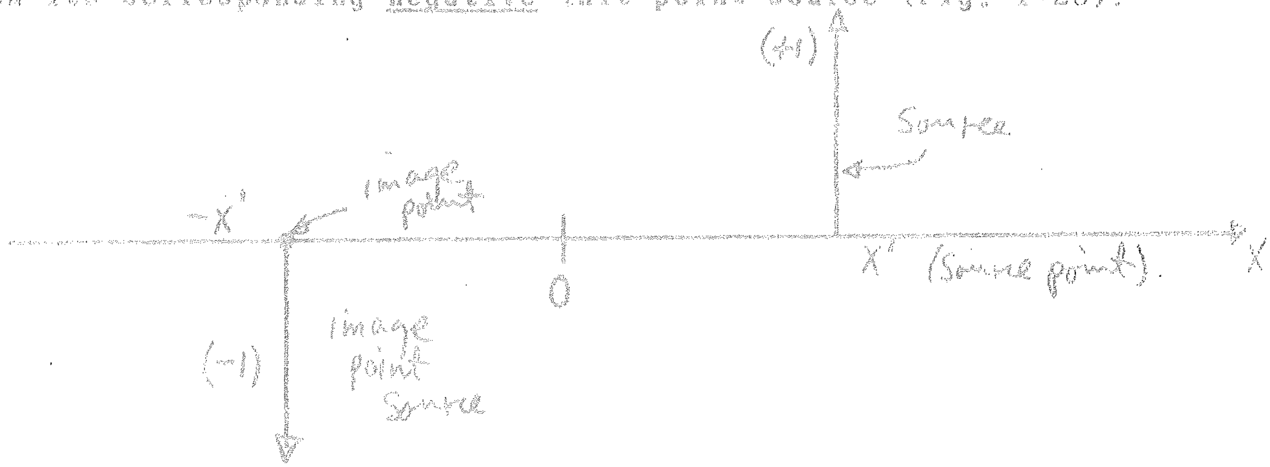

Figure 1-28. Illustrating sources, source-points and their images in the origin.

The response to the image point source is obtained from (1-121) by changing the sign there and replacing x' by -x':

$$(1\text{-}122) \quad G(x|x') = \frac{1}{2}\left(\frac{D_h}{s}\right)^{1/2} e^{-(s/D_h)^{1/2}(x+x')} \quad , \; -x' < x < \infty$$

$$= \frac{1}{2}\left(\frac{D_h}{s}\right)^{1/2} e^{(s/D_h)^{1/2}(x+x')} \quad , \; -\infty < x < -x' .$$

The point x = 0 is in the range x < x' in (1-121) and -x' < x in (1-122). Hence, the net response at x = 0 is the superposition of the functions defined over these ranges in (1-121) and (1-122):

$$(1\text{-}123) \quad G(0|x') = -\frac{1}{2}\left(\frac{D_h}{s}\right)^{1/2} e^{-(s/D_h)x'} + \frac{1}{2}\left(\frac{D_h}{s}\right)^{1/2} e^{-(s/D_h)^{1/2}x'}$$

$$\equiv 0 \, ,$$

Thus, by utilizing an image source located in the region x < 0, we are able by superposition to satisfy the boundary conditions (1-120)(b).

Now, before we go any further in our development of the Green's function for the half-line $0 < x < \infty$, let us return to (1-119) and see how it will be used. The time-response $z_2(x,t)$ is given by the inverse transform of (1-119). Because the only place in the right-hand side of (1-119) that an "s" appears is in $G(x|x')$, it follows that $z_2(x,t)$ will be given in terms of the inverse Laplace transform of $G(x|x')$

(1-124) $$z_2(x,t) = \mathcal{L}^{-1}\{z_2(x,s)\} = -\frac{1}{D_4}\int_0^\infty \mathcal{L}^{-1}[G(x|x')]\, f(x')\,dx'.$$

Hence, it is necessary for us to know the inverse transform of (1-121) and (1-122) in order to calculate $z_2(x,t)$. The inverse transform of (1-121) valid for either $x < x'$ or $x > x'$ is

$$-\left(\frac{D_4}{4\pi t}\right)^{1/2} e^{-(x-x')^2/4D_4 t},$$

and that of (1-122) valid for either $x < -x'$ or $x > -x'$ is

$$\left(\frac{D_4}{4\pi t}\right)^{1/2} e^{-(x+x')^2/4D_4 t}.$$

Therefore, the overall <u>time-dependent Green's function</u> is the superposition of these two expressions

$$-\left(\frac{D_4}{4\pi t}\right)^{1/2} \left[e^{-(x-x')^2/4D_4 t} - e^{-(x+x')^2/4D_4 t}\right],$$

and (1-124) because

(1-125) $$z_2(x,t) = \int_0^\infty \left[\frac{e^{-(x-x')^2/4D_4 t} - e^{-(x+x')^2/4D_4 t}}{(4\pi D_4 t)^{1/2}}\right] f(x')\,dx'.$$

Though we have constructed (1-125) as the solution of our initial-value problem, let us verify that it does indeed satisfy the initial condition $\lim_{t \to 0+} z_2(x,t) = f(x)$.

First, let us prove an interesting property of the function $\dfrac{e^{-(x-x')^2/4D_4 t}}{(4\pi D_4 t)^{1/2}}$ as $t \to 0$, and that is that

(1-126) $$\lim_{t \to 0} \frac{e^{-(x-x')^2/4D_4 t}}{(4\pi D_4 t)^{1/2}} = \delta(x-x').$$

From a table of definite integrals we have

$$\int_{-\infty}^{\infty} \frac{e^{-(x-x')^2/(4D_h t)}}{(4\pi D_h t)^{1/2}}\,dx = 1,$$

for any value of t. Hence, in the limit $t \to 0$, we at least will satisfy the unit area condition $\int_{-\infty}^{\infty} \delta(x-x')\,dx' = 1$.

Also, if $x \neq x'$, then

$$\lim_{t\to 0} \frac{e^{-(x-x')^2/(4D_h t)}}{(4\pi D_h t)^{1/2}} = 0,$$

because the exponential goes to zero faster than $\left(\frac{1}{t}\right)^{1/2}$ goes to infinity as $t \to 0$. Finally, if $x = x'$, then

$$\lim_{t\to 0} \frac{e^{-(x-x')^2/(4D_h t)}}{(4\pi D_h t)^{1/2}} = \lim_{t\to 0} \frac{1}{(4\pi D_h t)^{1/2}} = \infty .$$

The three conditions just listed, then, demonstrate the correctness of (1-126). Actually, by simply sketching

$$\frac{e^{-(x-x')^2/4D_h t}}{(4\pi D_h t)^{1/2}}$$

as a function of x for smaller and smaller values of t is sufficient to convince one of the delta function nature of the function.

Armed with (1-126), we return to (1-125) and compute its value in the limit as $t \to 0+$:

$$\lim_{t\to 0+} z_2(x,t) = \int_0^{\infty} \lim_{t\to 0+}\left[\frac{e^{-(x-x')^2/(4D_h t)}}{(4\pi D_h t)^{1/2}} - \frac{e^{-(x+x')^2/(4D_h t)}}{(4\pi D_h t)^{1/2}}\right] f(x')\,dx'$$

(1-127)

$$= \int_0^{\infty} \left(\delta(x-x') - \delta(x+x')\right) f(x')\,dx'$$

$$= f(x).$$

Note that $\delta(x-x')$ is located at $x' = x > 0$, whereas $\delta(x+x')$ is located at $x' = -x < 0$. Hence, only the first delta function is covered in the range of integration $0 < x' < \infty$, and only it contributes, therefore, to the result. Finally, it is a simple matter to verify that $z_2(x,t) = 0$ at $x = 0$ in (1-125).

The total z(x,t) is given by the sum of (1-118) and (1-125)

$$(1\text{-}128)\quad z(x,t) = \frac{x}{2(\pi D_h)^{1/2}} \int_0^t e^{\xi/\tau} g(\xi) \frac{e^{-x^2/4D_h(t-\xi)}}{(t-\xi)^{3/2}} d\xi$$

$$+ \int_0^\infty \left[\frac{e^{-(x-x')^2/4D_h t} - e^{-(x+x')^2/4D_h t}}{(4\pi D_h t)^{1/2}} \right] f(x') dx'.$$

h(x,t), finally, is given by

$$(1\text{-}129)\quad h(x,t) = e^{-t/\tau} z(x,t) = \frac{x}{2(\pi D_h)^{1/2}} \int_0^t e^{-(t-\xi)/\tau} g(\xi) \frac{e^{-x^2/(4D_h(t-\xi))}}{(t-\xi)^{3/2}} d\xi$$

$$+ e^{-t/\tau} \int_0^\infty \left[\frac{e^{-(x-x')^2/4D_h t} - e^{-(x+x')^2/4D_h t}}{(4\pi D_h t)^{1/2}} \right] f(x') dx'.$$

The method of solution, whereby initial conditions (such as f(x)) and boundary forcing conditions (such as g(x)) are treated as independent variables which are the forcing functions in their respective <u>homogeneous problems</u> (problems "A" and "B", above) is very general and applicable in any distributed system, not just diffusion problems. In addition, the method of images, which led to the solution of the initial value problem, is also a powerful method for solving and gaining insight into a wide class of fields and waves problems.

APPENDIX A: The Inversion Integral for the Laplace Transform.*

Though we assume a familiarity with only the elementary concepts of the Laplace transform and have used ~~only~~ tables in performing inversions, it is to our advantage to study the theory of the inversion integral for Laplace transforms in order to gain insight into the manner of calculating these inverse transforms which are not listed in tables and also to appreciate ~~some~~ some of the finer points of the theory.

There are many texts which expound the theory of the Laplace transform and its relationship to complex variables. Our definition of the transform and the theorem, concerning the inversion integral, which follows it are taken from J. W. Dettman, Applied Complex Variables, The Macmillan Co., 1965, ch. 9.

Definition of the Laplace Transform: Let f(t) be a complex-valued function of the real variable t which is zero for t negative. Let f(t) be piecewise continuous in any finite interval and $|f(t)| \le Ke^{bt}$, $0 \le t < \infty$. Then the Laplace transform of f(t) is

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt. \tag{A-1}$$

Theorem on the Inversion Integral: Let f(t) be a complex-valued function of the real variable t which is zero for t negative. Let f(t) be piecewise continuous and have a piecewise continuous derivative in any finite interval. Let $|f(t)| \le ke^{bt}$, $0 \le t < \infty$. Then the Laplace transform of f(t) is analytic for $b < \mathrm{Re}(s)$ and

$$\frac{f(t_+) + f(t_-)}{2} = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)e^{st}\,ds, \tag{A-2}$$

where the integration is carried out along a line $\sigma_0 + j\omega$, $-\infty < \omega < \infty$, with $b < \sigma_0$.

* This Appendix presupposes a knowledge of complex variable theory

Example 1. Find the inverse transform of $\frac{1}{s}$.

Solution: $F(s) = \frac{1}{s}$ has a pole at $s = 0$. Hence, the line of integration must be to the right of the imaginary axis (Figure A-1).

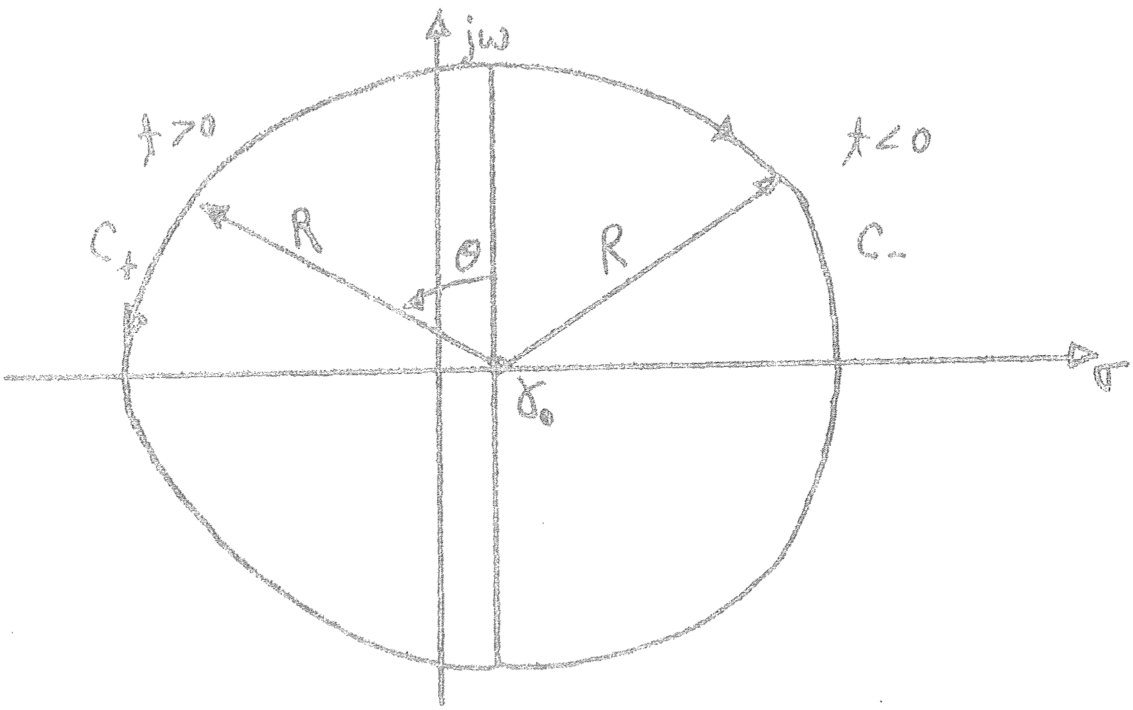

Figure A-1. Illustrating the contour for evaluating the inversion integral (A-2) when $F(s) = 1/s$.

We use Cauchy's residue theorem to evaluate (A-2). In order to apply the residue theorem we must have a closed contour, whereas the straight line, $\sigma_0 + j\omega$, $-\infty < \omega < \infty$, is obviously not closed. We close the contour by either of two large semi-circles, one for $t > 0$ and the other for $t < 0$, as shown in Figure A-1.

Thus, upon applying the residue theorem to the figure, we have

(A-3) $$\frac{1}{2\pi j}\int_{\sigma_0 - jR}^{\sigma_0 + jR} \frac{e^{st}}{s}\,ds + \frac{1}{2\pi j}\int_{C_+} \frac{e^{st}}{s}\,ds = \text{Residue at } s=0 = e^{st}\Big|_{s=0} = 1$$

If we can show that $\lim_{R\to\infty}\int_{C_+}\frac{e^{st}}{s}\,ds \to 0$ for $t>0$ then we will have shown that

$$\frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{s}\,ds = \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1, \quad t > 0.$$

In order to show that the integral over the semicircular contour vanishes as R goes to infinity, start by letting $s = \sigma_0 + Re^{j\theta}$, $\pi/2 \le \theta \le 3\pi/2$.

Then

$$\left|\int_{C_+}\frac{e^{st}}{s}\,ds\right| = \left|\int_{\pi/2}^{3\pi/2}\frac{e^{\sigma_0 t}e^{Rt\cos\theta}e^{jRt\sin\theta}}{(\sigma_0+Re^{j\theta})}\cdot jRe^{j\theta}\,d\theta\right|$$

$$\le e^{\sigma_0 t}\int_{\pi/2}^{3\pi/2}\frac{e^{Rt\cos\theta}}{R}\,R\,d\theta = e^{\sigma_0 t}\int_{\pi/2}^{3\pi/2}e^{Rt\cos\theta}\,d\theta = 2e^{\sigma_0 t}\int_{\pi/2}^{\pi}e^{Rt\cos\theta}\,d\theta.$$

Consider the graph of the cosine function between $\pi/2$ and $3\pi/2$ (Figure A-2). The cosine function is less than or

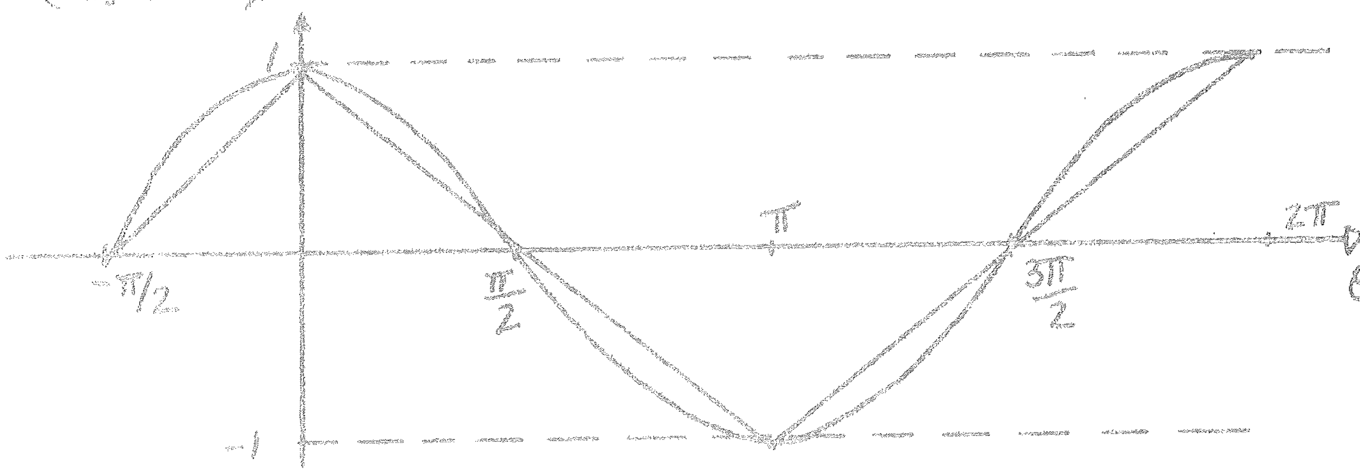

Figure A-2. The cosine function.

equal to the linear function $\left(1-\frac{2}{\pi}\theta\right)$ for $\pi/2 \le \theta \le \pi$.

Hence, the last integral above is less than or equal to

$$2e^{\sigma_0 t}\int_{\pi/2}^{\pi}e^{Rt}\cdot e^{-Rt\frac{2}{\pi}\theta}\,d\theta = e^{\sigma_0 t}e^{Rt}\cdot\pi\left[\frac{e^{-2Rt}-e^{-Rt}}{-Rt}\right]$$

$$= \pi\frac{e^{\sigma_0 t}}{t}\left[\frac{1-e^{-Rt}}{R}\right] \longrightarrow 0 \quad \text{as } R\to\infty \text{ and } t>0.$$

Thus, we have shown that $\left|\int_{C+} \frac{e^{st}}{s}\, ds\right| \to 0$ as $R \to \infty$, $t > 0$,

which indicates that

$$\frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{s}\, ds = \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1, \quad t > 0.$$

For $t < 0$, obviously, we cannot use the semi-circle C+ because the integral along C+ vanishes only for $t > 0$. (Recall the factor e^{-Rt} in the estimate). We can, however, use the contour C−, lying in the right-hand plane.

Cauchy's residue theorem now reads

$$\text{(A-4)} \quad \frac{1}{2\pi j}\int_{\sigma_0 - jR}^{\sigma_0 + jR} \frac{e^{st}}{s}\, ds + \frac{1}{2\pi j}\int_{C-} \frac{e^{st}}{s}\, ds = 0. \qquad \text{(no poles enclosed).}$$

Let us now show that $\int_{C-} \frac{e^{st}}{s}\, ds$ vanishes as $R \to \infty$.

We proceed precisely as before except that now θ varies between $\pi/2$ and $-\pi/2$, and over the range $-\pi/2 \le \theta \le 0$, $\cos\theta \ge (1 + \frac{2}{\pi}\theta)$. Thus,

$$\left|\int_{C-} \frac{e^{st}}{s}\, ds\right| \le e^{\sigma_0 t}\int_{-\pi/2}^{\pi/2} e^{Rt\cos\theta}\, d\theta \le 2e^{\sigma_0 t}\int_{-\pi/2}^{0} e^{Rt} e^{Rt\cdot\frac{2}{\pi}\theta}\, d\theta$$

$$= e^{\sigma_0 t} e^{Rt}\pi\left[\frac{1 - e^{-Rt}}{Rt}\right] = \frac{\pi e^{\sigma_0 t}}{t}\left[\frac{e^{Rt} - 1}{R}\right] \to 0 \text{ as } R \to \infty, \ t < 0.$$

Thus, we have shown that

$$\therefore \quad \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{s}\, ds = 0, \quad t < 0.$$

Both of these results may be subsummed under the single equation

$$\text{(A-5)} \quad \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{s}\, ds = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = u_{-1}(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$$

Of course, we knew all along that the inverse transform of $1/s$ was the unit step function, $u_{-1}(t)$. Let us next calculate the inversion integral for $t = 0$.

$$\frac{1}{2\pi j}\int_{\sigma_0-jR}^{\sigma_0+jR}\frac{e^{st}}{s}ds\Bigg|_{t=0} = \frac{1}{2\pi j}\int_{\sigma_0-jR}^{\sigma_0+jR}\frac{ds}{s} = \frac{1}{2\pi j}\ln\left[\frac{\sigma_0+jR}{\sigma_0-jR}\right].$$

write $\left(\frac{\sigma_0+jR}{\sigma_0-jR}\right) = \left|\frac{\sigma_0+jR}{\sigma_0-jR}\right| e^{j\tan^{-1}\left(\frac{\sigma_0+jR}{\sigma_0-jR}\right)}$,

The ratio of the magnitude of a complex number to that of its conjugate, is, of course, unity, while if we check Figure A-1, we see that as $R \to \infty$ along the straight line path $\sigma_0 + jR$ then

$$\tan^{-1}\left(\frac{\sigma_0+jR}{\sigma_0-jR}\right) = \pi. \quad \text{Hence, } \left(\frac{\sigma_0+jR}{\sigma_0-jR}\right) = e^{j\pi} \text{ and}$$

$$\frac{1}{2\pi j}\ln\left(\frac{\sigma_0+jR}{\sigma_0-jR}\right) = \frac{1}{2\pi j}\ln(e^{j\pi}) = \frac{j\pi}{j2\pi} = \frac{1}{2}.$$

This result confirms the statement of the theorem that at a discontinuity in f(t) (here, at t=0), the inversion integral converges to the arithmetic mean of the discontinuity. In the present example $u_{-1}(0+) = 1$, $u_{-1}(0-) = 0$, hence,

$$\frac{u_{-1}(0+) + u_{-1}(0-)}{2} = \frac{1}{2}.$$

Example 2. Apply the inversion integral to the transforms

$$V_T(s) = \frac{V_0}{s-j\omega_s} \cdot \frac{1}{\cosh\gamma_a l + A\sinh\gamma_a l}$$

$$V_R(s) = \frac{V_0}{s-j\omega_s} \cdot \frac{B\sinh\gamma_a l}{\cosh\gamma_a l + A\sinh\gamma_a l}$$

which appeared in the transient response of the laser.

Solution. The closed contour to which Cauchy's residue theorem is applied is shown in Fig. A-3.

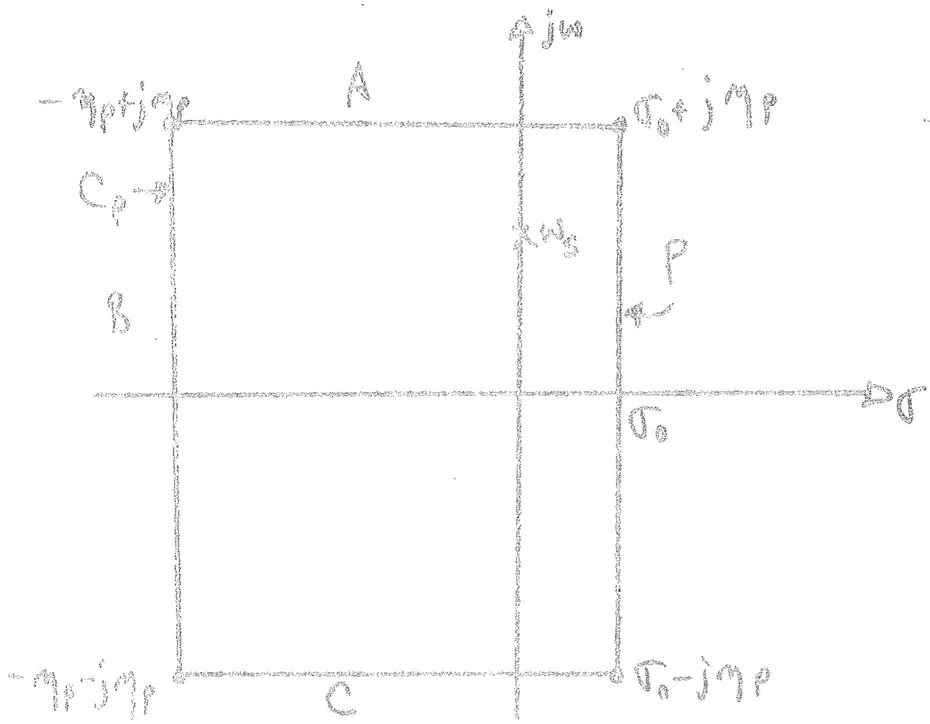

Figure A-3. The rectangular contour for evaluating the inversion integral for $V_T(s)$ and $V_Q(s)$ which appeared in the analysis of the laser.

We are going to estimate the behavior of the integrand along the path C_p in order to find out under what conditions the integral along C_p vanishes, thereby enabling us to equate the inversion integral along the straight-line path, P_2 (the value of which integral being the inverse Laplace transform), to the sum of the residues of the integrand, which will justify our procedure in the text.

Consider, first, $V_T(s)$. The expression $\gamma_a(s)$ appearing in $V_T(s)$ and $V_R(s)$ is equal to $s/v_{pa} + \alpha(s)$, where $v_{pa} = (1/L_a C_a)^{1/2}$

is the phase velocity of the host material in the absence of the active impurity ions (see (1-69)). Because $\alpha(s) \to 0$ as $s \to \infty$ (see 1-78), it follows that $\gamma_a(s) \to s/v_{pa}$ as $s \to \infty$. Then

$$\lim_{\eta_p \to \infty} \int_{C_p} \frac{V_0}{s - j\omega_s} \cdot \frac{e^{st}\, ds}{(\cosh \gamma_a l + A \sinh \gamma_a l)} \to \lim_{\eta_p \to \infty} \int_{C_p} \frac{V_0}{s - j\omega_s} \cdot \frac{e^{st}\, ds}{\left(\cosh \frac{sl}{v_{pa}} + A \sinh \frac{sl}{v_{pa}}\right)}$$

Along path A (Fig. A-3), $s = \sigma + j\eta_p$, σ varying from σ_0 to $-\eta_p$ and $ds = d\sigma$. The integral along A becomes

$$\lim_{\eta_p \to \infty} \int_{\sigma_0}^{-\eta_p} \frac{V_0}{[\sigma + j(\eta_p - \omega_s)]} \cdot \frac{e^{\sigma t} \cdot e^{j\eta_p t} \, d\sigma}{\left[\cosh[\sigma + j\eta_p]\frac{\ell}{v_{pa}} + A \sinh(\sigma + j\eta_p)\frac{\ell}{v_{pa}}\right]}.$$

Recalling that $\cosh(\sigma + j\eta_p)\frac{\ell}{v_{pa}} = \frac{e^{(\sigma + j\eta_p)\frac{\ell}{v_{pa}}} + e^{-(\sigma + j\eta_p)\ell/v_{pa}}}{2}$,

$$\sinh(\sigma + j\eta_p)\frac{\ell}{v_{pa}} = \frac{e^{(\sigma + j\eta_p)\ell/v_{pa}} - e^{-(\sigma + j\eta_p)\ell/v_{pa}}}{2},$$

it follows that the right-hand factor in the integrand above can be written

$$\frac{e^{(\sigma + j\eta_p)(t - \ell/v_{pa})}}{\frac{1}{2}(1+A) - \frac{1}{2}(A-1)\, e^{-(\sigma + j\eta_p)\cdot \frac{2\ell}{v_{pa}}}}.$$

The denominator of this result can be sketched using a "phasor" (or complex number) diagram, assuming A to be a real constant (independent of s) (recall that $A = \frac{1}{2}\left(Z_0/Z_a + Z_a/Z_0\right)$ and $Z_a = \left(\frac{L_a}{C_a}\right)^{1/2}$) (Fig. A-4):

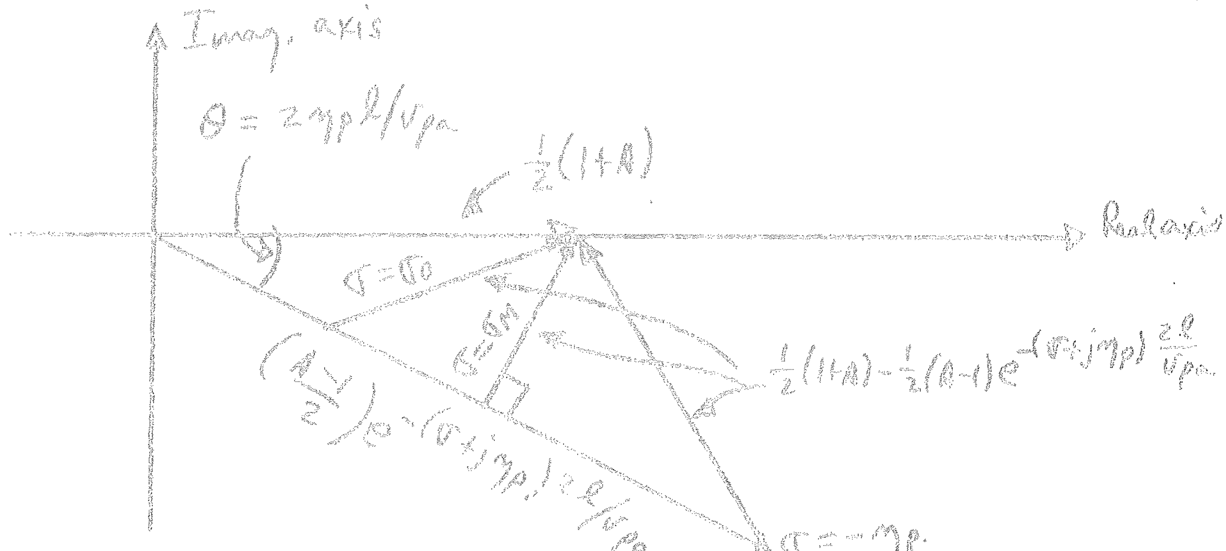

Figure A-4. Phasor diagram used to estimate the value of $\sigma = \sigma_M$ at which $\frac{1}{2}(1+A) - \frac{1}{2}(A-1)e^{-(\sigma + j\eta_p)2\ell/v_{pa}}$ has its minimum magnitude.

Utilizing this figure, we have

$$\left|\frac{e^{(\sigma+j\eta_p)(t-\ell/v_{pa})}}{\frac{1}{2}(1+A)-\frac{1}{2}(A-1)e^{-(\sigma+j\eta_p)2\ell/v_{pa}}}\right| = \frac{e^{\sigma(t-\ell/v_{pa})}}{\left|\frac{1}{2}(1+A)-\frac{1}{2}(A-1)e^{-(\sigma_n+j\eta_p)2\ell/v_{pa}}\right|}$$

$$= \frac{e^{\sigma(t-\ell/v_{pa})}}{\Delta}$$

where Δ is independent of σ. Thus

$$\lim_{\eta_p\to\infty}\left|\int_{\sigma_0}^{-\eta_p}\frac{V_0}{\sigma+j(\eta_p-\omega_s)}\cdot\frac{e^{\sigma t}\cdot e^{j\eta_p t}\,d\sigma}{\cosh(\sigma+j\eta_p)\ell/v_{pa}+A\sinh(\sigma+j\eta_p)\ell/v_{pa}}\right|$$

(A-6)

$$\leq \lim_{\eta_p\to\infty}\int_{-\eta_p}^{\sigma_0}\frac{V_0\,e^{\sigma(t-\ell/v_{pa})}}{(\eta_p-\omega_s)\Delta}\,d\sigma$$

$$= \lim_{\eta_p\to\infty} V_0\,\frac{e^{\sigma_0(t-\ell/v_{pa})}-e^{-\eta_p(t-\ell/v_{pa})}}{(\eta_p-\omega_s)\Delta}$$

$$= 0\ , \text{ if } t > \ell/v_{pa}.$$

A similar analysis holds for paths B and C in Fig. A-3.

Thus, we can close the contour in the left-half plane $\sigma < \sigma_0$, i.e., to the left of the line, P, for $t > \ell/v_{pa}$, and the contribution to the closed contour integral along C_p vanishes. Hence, the value of the integral along P, which is the inverse Laplace transform, is equal to the sum of residues of the integrand.

For $t < \ell/v_{pa}$, we close the contour in the right-half plane, $\sigma > \sigma_0$, and then see that $\sigma = \sigma_0$ gives the minimum value for Δ. The only other difference now is that the sign of η_p in the limit of integration in (A-6) (and, hence, the exponential term in the same equation) changes, meaning that the integral vanishes if $t < \ell/v_{pa}$. Because there are no poles lying to the right of $\sigma = \sigma_0$ (remember that σ_0 must lie to the right of all singularities), and because the infinite contour, C_p, lying to the right of σ_0 contributes zero for $t < \ell/v_{pa}$, it follows that the value of the integral along P, i.e., the inverse Laplace transform, vanishes for $t < \ell/v_{pa}$. This makes sense physically because it implies that the response at $x = \ell$ due to a switched on excitation at $t = 0$ and $x = 0$, is null until time $t = \ell/v_{pa}$, the time it takes a wave traveling with phase velocity v_{pa} to travel the distance ℓ.

When we consider the reflected wave with transform

$$V_R(s) = \frac{V_0}{s - j\omega_s} \cdot \frac{B \sinh \gamma_a \ell}{\cosh \gamma_a \ell + A \sinh \gamma_a \ell} \xrightarrow[s \to \infty]{} \frac{V_0}{s - j\omega_s} \cdot \frac{B\left[1 - e^{-(\sigma + j\eta_p)2\ell/v_{pa}}\right]}{(1+A) - (A-1)e^{-(\sigma + j\eta_p)2\ell/v_{pa}}}$$

and try to obtain estimates for the integral on segment A of C_p we end up with two terms, corresponding to the two terms in the numerator of $V_R(s)$, above

(A-7)

(a) $$\left| \int_{\sigma_0}^{-\eta_p} \frac{V_0 B \, d\sigma \, e^{\sigma t}}{(\eta_p - \omega_s)\Delta} \right|$$

(b) $$\left| \int_{\sigma_0}^{-\eta_p} \frac{V_0 B e^{\sigma(t - 2\ell/v_{pa})}}{(\eta_p - \omega_s)\Delta} \, d\sigma \right| ,$$

Each of these expressions is similar to the right-hand expression in the inequality (estimate) of (A-6), except that (A-7)(b) has the delayed time parameter $t - 2\ell/v_{pa}$. By the arguments of (A-6), therefore, it follows that the (a) term above vanishes along the left-half-plane contour for $t > 0$ and the (b) term for $t > 2\ell/v_{pa}$. Thus, the residues of that part of the integrand of $V_R(s)$ that contributed to (A-7)(a) produce the inverse transform of a reflected wave starting at $t = 0$, whereas those that contributed to (A-7)(b) produce a reflected wave starting at $t = 2\ell/v_{pa}$. By closing the contour in the right-half plane it follows that wave (a), above, vanishes for $t < 0$ and wave (b) vanishes for $t < 2\ell/v_{pa}$.

The physical origin of these waves is readily explained: the wave corresponding to (A-7) (a) arises from the direct reflection off the face at $x = 0$ of the incident pulse there, while (A-7)(b) corresponds to the wave reflected at $x = \ell$ reappearing at $x = 0$ and propagating in the (-)x direction. This latter wave may be called a multiply reflected wave.

Example 2.* . Evaluate the inversion integral for the dispersive transmission-line analogue of a waveguide (Section 4).

Solution: We have found the solution of (1-57) corresponding to a positive-going wave in an infinite line to be

$$V(x,s) = V(0,s)\, e^{-(s^2+\omega_c^2)^{1/2} x/v_p} \tag{A-8}$$

where $V(0,s)$ is the Laplace transform of the sending-end voltage $v(0,t)$. Hence, the instantaneous voltage at any point x is given by the inverse transform of (A-8)

$$v(x,t) = \frac{1}{2\pi j}\int_{\sigma_0-j\infty}^{\sigma_0+j\infty} V(0,s)\exp\left[st - \frac{x}{v_p}(s^2+\omega_c^2)^{1/2}\right] ds. \tag{A-9}$$

σ_0, of course, lies to the right of any singularities of the integrand.

As $s \to \infty$, the exponential function approaches $\exp s[t - x/v_p]$. If we close the contour by the semi-circular arc C− of Figure A-1, then as $R \to \infty$, the contribution of the integral along C− vanishes if $t - x/v_p < 0$, i.e., $t < x/v_p$. Therefore, because there are no singularities enclosed to the right of σ_0, the value of the integral (A-9) is zero, which indicates that $v(x,t) = 0$, $t < x/v_p$. Thus, no signal can arrive at any point x with a speed greater than v_p, or in a time less than x/v_p. This is in agreement with our earlier results (1-58) and (1-61).

For $t - x/v_p > 0$ the contour will have to be closed in the left-half plane along C+ as shown in Fig. A-1. In this case, however, there will be singularities enclosed by the contour.

Generally there are two types of singularities encountered: (1) poles of $V(0,s)$ and (2) branch points $s = \pm j\omega_c$ associated with the double-valued function $(s^2+\omega_c^2)^{1/2}$. We are going to show that each type of singularity gives rise to a solution type—the poles to a "steady-state" or "forced" response and the branch points, with their associated branch-cuts, to a transient or "free" response. The forced

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* Our treatment follows closely that of R. E. Haskell and C. T. Case, "Transient Signal Propagation in Lossless, Isotropic Plasmas (Vol. I)", Air Force Cambridge Research Lab. report AFCRL-66-234, April, 1966.

response is characteristic of the input forcing function v(0,t), while the free response is characteristic of the unforced system.

Our first job is to introduce branch cuts for $(s^2+\omega_c^2)^{1/2}$ and thereby define the two-sheeted Reimann surface for this function. On one sheet (the "upper" sheet) we want the function $(s^2+\omega_c^2)^{1/2}$ to behave as +s as $s \to \infty$, because that implies a "casual" solution for (A-9), i.e., the solution vanishes identically for $t < x/vp$ (recall our arguments concerning the location of the infinite semi-circle for $t < x/vp$ and $t > x/vp$).

On the second, or "lower", sheet of the Riemann surface the function $(s^2+\omega_c^2)^{1/2} \to -s$ as $s \to \infty$. This sheet corresponds to the negative branch of the square-root function. Our interest is in the upper sheet, corresponding to the positive root of $(s^2+\omega_c^2)^{1/2}$.

In order to properly define the two sheets we write $(s^2+\omega_c^2)^{1/2}$ as $(s-j\omega_c)^{1/2} \cdot (s+j\omega_c)^{1/2}$, where $(s-j\omega_c)^{1/2} = \sqrt{\rho_1}\, e^{j\varphi_1/2}$ and $(s+j\omega_c)^{1/2} = \sqrt{\rho_2}\, e^{j\varphi_2/2}$ (Figure A-5).

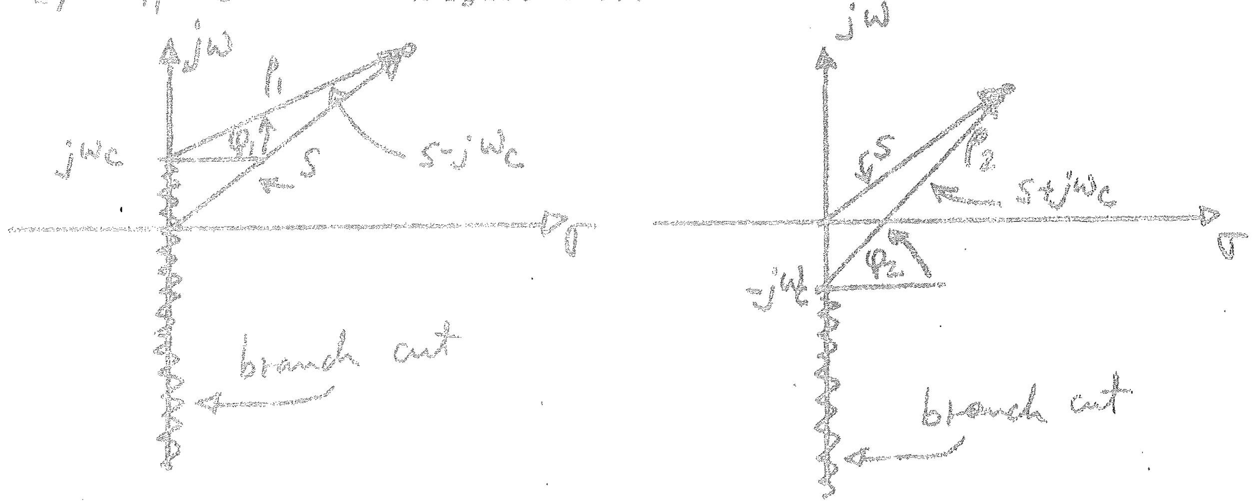


Figure A-5. The complex frequency plane showing branch cuts for $(s-j\omega_c)^{1/2}$ and $(s+j\omega_c)^{1/2}$.

On the upper sheet of both $(s-j\omega_c)^{1/2}$ and $(s+j\omega_c)^{1/2}$, φ_1 and φ_2 both lie in the semi-closed interval $(-\pi/2, 3\pi/2)$, which excludes $3\pi/2$. On the lower sheet of both $(s-j\omega_c)^{1/2}$ and $(s+j\omega_c)^{1/2}$, φ_1 & φ_2 both lie in the semi-closed interval $(3\pi/2, 7\pi/2)$ which includes $\frac{3\pi}{2}$ and excludes $7\pi/2$.

Therefore on the upper sheet, $(s^2+\omega_c^2)^{1/2}$ is defined by

(A-10) (a) $$(s^2+\omega_c^2)^{1/2} = (\rho_1\rho_2)^{1/2}\, e^{j(\varphi_1+\varphi_2)/2}, \quad -\frac{\pi}{2} \le \varphi_1, \varphi_2 < \frac{3\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} \le \varphi_1, \varphi_2 < 7\pi/2$$

and on the lower sheet

(A-10 (b) $$(s^2+\omega_c^2)^{1/2} = (\rho_1\rho_2)^{1/2}\, e^{j(\varphi_1+\varphi_2)/2}, \quad -\frac{\pi}{2} \le \varphi_1 < 3\pi/2, \quad \frac{3\pi}{2} \le \varphi_2 < \frac{7\pi}{2}$$

$$\text{or} \quad \frac{3\pi}{2} \le \varphi_1 < \frac{7\pi}{2}, \quad -\frac{\pi}{2} \le \varphi_2 < \frac{3\pi}{2}.$$

Note that the upper sheet of $(s^2+\omega_c^2)^{1/2}$ is given by the product of $(s-j\omega_c)^{1/2}$ with $(s+j\omega_c)^{1/2}$ when both of these factors lie in either of their own upper or lower sheets. The lower sheet of $(s^2+\omega_c^2)^{1/2}$ is given by the product of $(s-j\omega_c)^{1/2}$ with $(s+j\omega_c)^{1/2}$ when each factor lies in the opposite sheet, i.e., either $(s-j\omega_c)^{1/2}$ lies in its upper sheet and $(s+j\omega_c)^{1/2}$ in its lower, or vice versa.

The sign distributions of the real and imaginary parts of $(s^2+\omega_c^2)^{1/2}$ are shown in Figure A-6 and are derived from

(A-11) $$(s^2+\omega_c^2)^{1/2} = (\rho_1\rho_2)^{1/2}\, e^{j\frac{(\varphi_1+\varphi_2)}{2}} = (\rho_1\rho_2)^{1/2}\left\{\cos\frac{(\varphi_1+\varphi_2)}{2} + j\sin\frac{(\varphi_1+\varphi_2)}{2}\right\}.$$

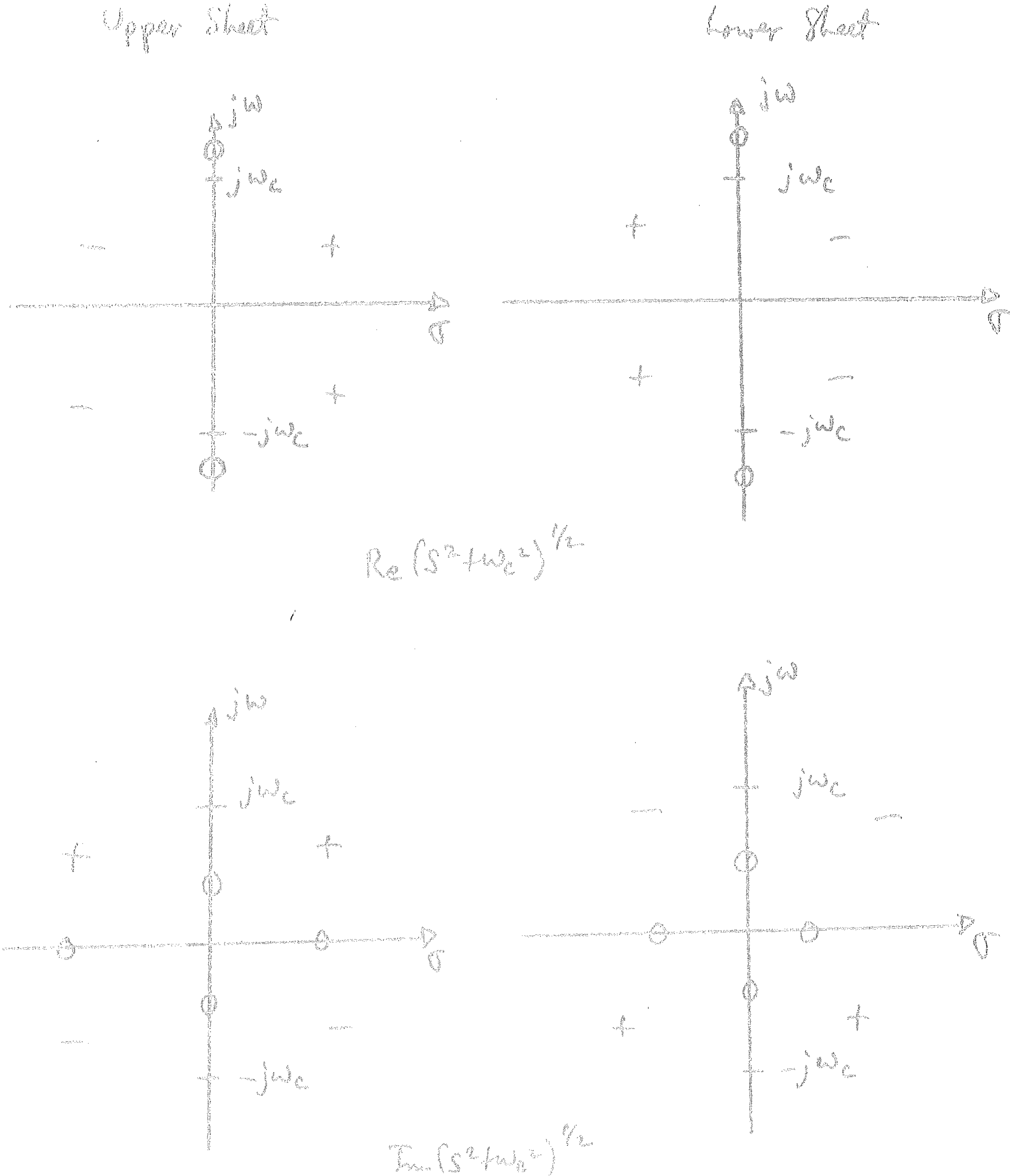


Figure A-6. Sign distributions of the real and imaginary parts of $(s^2+\omega_c^2)^{1/2}$ on each sheet of its Riemann surface. "0" indicates the value zero.

As suspected, the lower sheet sign distribution is just the negative of the upper.

Now that we have described the Riemman surface, let us integrate (A-9) over the upper sheet, letting $V(0,s) = \dfrac{V_0 \omega_0}{s^2 + \omega_0^2}$, i.e., taking the sending end voltage to be a sine wave of frequency ω_0, switched on at $t = 0$. The closed contour of integration plus the branch cut connecting $\pm j\omega_c$ and the poles $\pm j\omega_0$ are shown in Figure A-7. Note that the branch cut lies <u>outside</u> the closed path of integration. This is necessary because Cauchy's residue theorem applies only to isolated singularities and not to enclosed branch points and their associated cuts.

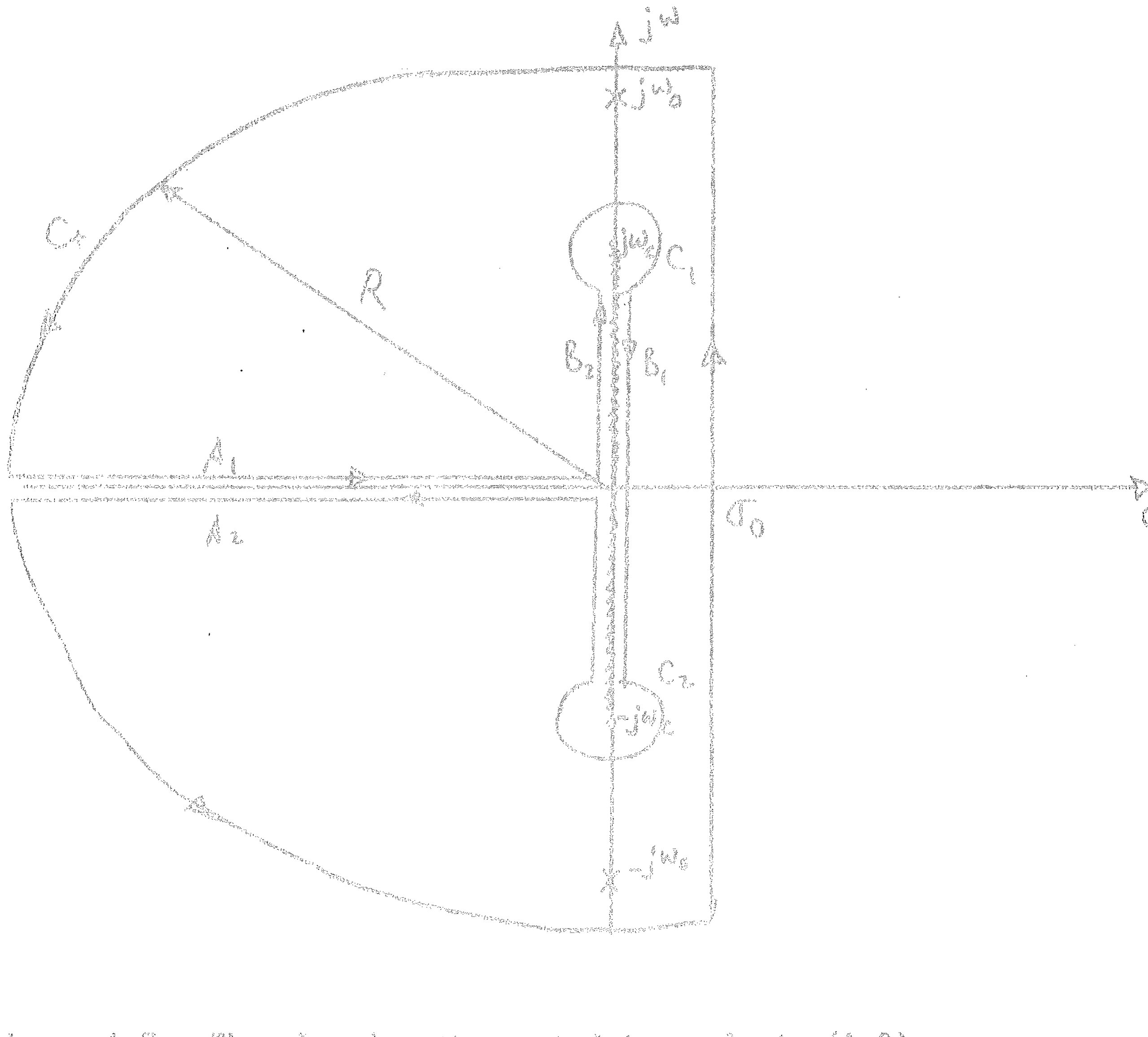


Figure A-7. The closed contour used to evaluate (A-9).

Using the same arguments as in Example 1 of this Appendix one can readily show that as $R \to \infty$ the integral over C+ vanishes when $t - x/vp > 0$. Because the integrand is single valued along the real (σ) axis, it follows that the integrals along A_1 and A_2 cancel each other because they are taken in the opposite direction and the paths A_1 and A_2 approach each other in the limit. In addition, the integrals along the arcs C_1 and C_2 vanish in the limit as the radius of each arc vanishes. Thus, that leaves only the branch cut integrals along B_1 and B_2 with which to contend. Summarizing to this point we have

(A-12)

$$\frac{1}{2\pi j}\int_{\sigma_0-j\infty}^{\sigma_0+j\infty}\left(\frac{V_0\omega_0}{s^2+\omega_0^2}\right)\exp\left[st-\frac{x}{v_p}(s^2+\omega_c^2)^{1/2}\right]ds$$

$$+\frac{1}{2\pi j}\int_{B_1}\left(\frac{V_0\omega_0}{s^2+\omega_0^2}\right)\exp\left[st-\frac{x}{v_p}(s^2+\omega_c^2)^{1/2}\right]ds$$

$$+\frac{1}{2\pi j}\int_{B_2}\left(\frac{V_0\omega_0}{s^2+\omega_0^2}\right)\exp\left[st-\frac{x}{v_p}(s^2+\omega_c^2)^{1/2}\right]ds$$

$$= \text{Residue at } j\omega_0 + \text{Residue at } (-j\omega_0).$$

It is a simple matter to calculate the residues at $\pm j\omega_0$ by expanding $\frac{V_0\omega_0}{s^2+\omega_0^2}$ into partial fractions. The result is

$$\text{Residue}\,(j\omega_0) = -j\frac{V_0}{2}\exp\left[j\omega_0 t - j\frac{x}{v_p}(\omega_0^2-\omega_c^2)^{1/2}\right]$$

$$\text{Residue}\,(-j\omega_0) = j\frac{V_0}{2}\exp\left[-j\omega_0 t + j\frac{x}{v_p}(\omega_0^2-\omega_c^2)^{1/2}\right].$$

Thus, upon using these results and recalling that the first integral of (A-12) is the response v(x,t), we rewrite (A-12) as

$$v(x,t) + \frac{1}{2\pi j}\int_{B_1}\left(\frac{V_0\omega_0}{s^2+\omega_0^2}\right)\exp\left[st - \frac{x}{v_p}(s^2+\omega_c^2)^{1/2}\right]ds$$

(A-13)

$$+ \frac{1}{2\pi j}\int_{B_2}\left(\frac{V_0\omega_0}{s^2+\omega_0^2}\right)\exp\left[st - \frac{x}{v_p}(s^2+\omega_c^2)^{1/2}\right]ds$$

$$= V_0 \sin\omega_0\left[t - \frac{x}{v_p}\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}\right].$$

As the path B_1 approaches the branch cut, the function $(s^2+\omega_c^2)^{1/2}$ approaches the pure real function $(\omega_c^2-\omega^2)^{1/2}$, while the same function approaches $-(\omega_c^2-\omega^2)^{1/2}$ along path B_2, as that path approaches the branch cut (see Figure A-6). Keeping in mind the direction of integration along B_1 and B_2 (Figure A-7), (A-13) becomes

$$v(x,t) + \frac{1}{2\pi j}\int_{j\omega_c}^{-j\omega_c}\frac{V_0\omega_0}{(s^2+\omega_0^2)}\exp\left[st - \frac{x}{v_p}(\omega_c^2-\omega^2)^{1/2}\right]ds$$

$$+ \frac{1}{2\pi j}\int_{-j\omega_c}^{j\omega_c}\frac{V_0\omega_0}{(s^2+\omega_0^2)}\exp\left[st - \frac{x}{v_p}(\omega_c^2-\omega^2)^{1/2}\right]ds$$

(A-14)

$$= V_0 \sin\omega_0\left[t - \frac{x}{v_p}\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}\right],$$

or, upon letting $s = j\omega$, and combining the two integrals

(A-15) $$v(x,t) = V_0\sin\omega_0\left(t - \frac{x}{v_p}\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}\right) - \frac{V_0\omega_0}{\pi}\int_{-\omega_c}^{\omega_c}\frac{e^{j\omega t}\sinh\frac{x}{v_p}(\omega_c^2-\omega^2)^{1/2}}{\omega_0^2-\omega^2}d\omega$$

The first term in (A-15) is clearly the steady-state or forced response because it has the same shape as the forcing function $V_0 \sin \omega_0 t$. The forced response is simply a phase-shifted (by amount $\frac{\omega_0 x}{v_p}\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}$) version of the input voltage.

The second term in (A-15) is the transient, or free-response, and consists of the superposition of the phasors

$$-\frac{V_0\omega_0}{\pi}\;\frac{\sinh(x/v_p)(\omega_c^2-\omega^2)^{1/2}}{\omega_0^2-\omega^2}$$

corresponding to all frequencies lying between $-\omega_c$ and ω_c. This term contributed to the tail in (1-61). In that equation the total response was expressed in the time-domain, i.e., it involved a convolution integral. In (A-15), we have written the total response in terms of an integral in the frequency domain.

There is another fundamental difference between (1-61) and (A-15). The first term in (1-61), $v_0(t - x/vp)$, travels with velocity v_p, whereas the first term (or steady-state term) in (A-15), $V_0 \sin \omega_0\left[t - \frac{x}{v_p}\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}\right]$ travels with velocity $\frac{v_p}{\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}} > v_p$. This latter velocity, because it affects only the phase of the sinusoidal (steady-state) term, is called the <u>phase velocity</u> at frequency ω_0.

We can gain further insight into the nature of the transient term of (A-15) by replacing the branch-cuts and contour of Fig. A-7 by those of Fig. A-8, wherein the branch-cuts lie in the left-half plane instead of along the $j\omega$-axis.

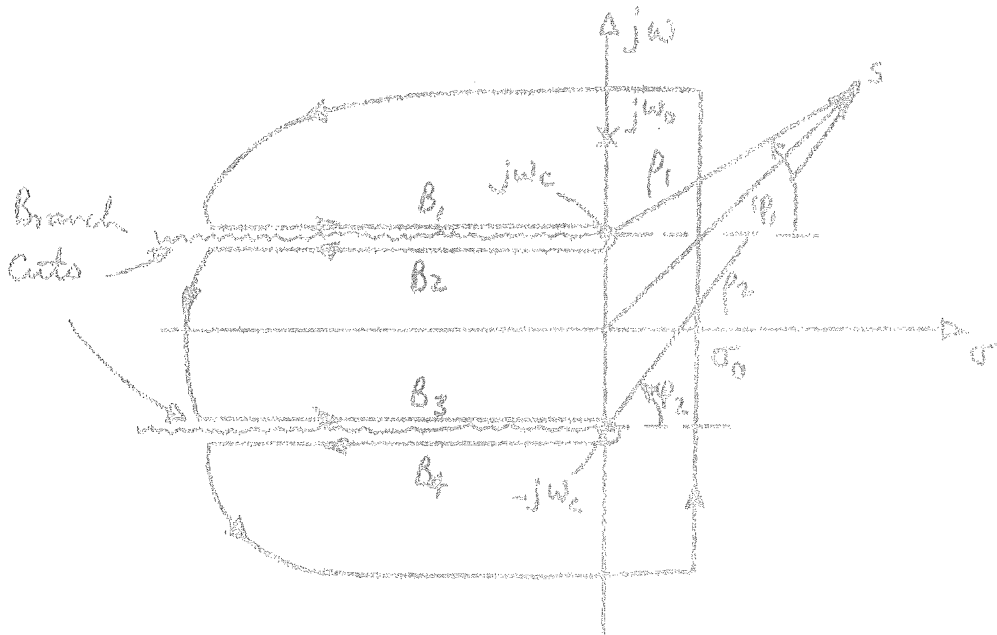


Fig. A-8. Showing the contour and branch-cuts lying in the left-half plane.

The upper sheet of the two sheeted Riemann surface is now defined by

$$(s - j\omega_c) = \rho_1 e^{j\varphi_1}, \qquad -\pi \le \varphi_1 < \pi$$

(A-16)

$$(s + j\omega_c) = \rho_2 e^{j\varphi_2}, \qquad -\pi \le \varphi_2 < \pi$$

$$(s^2 + \omega_c^2)^{1/2} = (\rho_1\rho_2)^{1/2} e^{j(\varphi_1+\varphi_2)/2} .$$

A study of (A-16), together with Fig. A-8, leads to the conclusion that the values of the function $(s^2+\omega_c^2)^{1/2}$ on the banks B_1, B_2, B_3 and B_4 of the two branch-cuts are

(A-17)

$$(s^2 + \omega_c^2)^{1/2} = \sqrt{\frac{\rho_1\rho_2}{2}}\,(-a + jb) \qquad [B_1].$$

$$= \sqrt{\frac{\rho_1\rho_2}{2}}\,(a - jb) \qquad [B_2].$$

$$= \sqrt{\frac{\rho_1\rho_2}{2}}\,(a + jb) \qquad [B_3]$$

$$= \sqrt{\frac{\rho_1\rho_2}{2}}\cdot(-a - jb) \qquad [B_4].$$

where $a = \cos\psi/2 + \sin\psi/2$, $b = \cos\psi/2 - \sin\psi/2$, $\psi = \tan^{-1}\left(\frac{|\sigma|}{2\omega_c}\right)$,

and $\sqrt{\frac{\rho_1\rho_2}{2}} = \sqrt{\frac{|\sigma|(\sigma^2 + 4\omega_c^2)^{1/2}}{2}}$.

Let us assume that the input is a switched-on complex sinusoid $V_0 e^{j\omega_0 t}$, whose Laplace transform is $V_0/(s - j\omega_0)$. The actual response is given by the imaginary part of the response to $V_0 e^{j\omega_0 t}$;

(A-18)

$$v(x,t) = \mathrm{Im}\left[\frac{V_0}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{\exp\left[st - \frac{x}{v_g}(s^2+\omega_c^2)^{1/2}\right]}{s - j\omega_0}\,ds\right].$$

Note, now, that there is a single pole located at $S = j\omega_0$. As usual, the integration implied in (A-18) is extended over the closed contour of Fig. (A-8), with the large semi-circle contributing nil in the limit. Hence, (A-18) becomes

$$v(x,t) + \operatorname{Im}\left\{\int_{B_1} + \int_{B_2} + \int_{B_3} + \int_{B_4}\right\} = \operatorname{Im}\{\text{Residue at } S = j\omega_0\} \tag{A-19}$$

$$= \operatorname{Im}\left\{V_0 \exp\left(j\omega_0 t - j\frac{x}{v_p}(\omega_0^2 - \omega_c^2)^{1/2}\right)\right\},$$

where $\int_{B_1}$, etc., are the integrals along the banks of the branch-cuts.

As an example of the evaluation of these integrals, consider $\int_{B_1}$. Along B_1, $S = j\omega_c + \sigma$, and $(S^2 + \omega_c^2)^{1/2} = \left(\frac{\rho_1\rho_2}{2}\right)^{1/2}(-a + jb)$

with ρ_1, ρ_2, a and b defined in terms of σ as above.

Therefore, the appropriate integral becomes

$$\int_{B_1} = \frac{V_0}{2\pi j}\int_{-\infty}^{0} \frac{\exp\left[(\sigma + j\omega_c)t - \frac{x}{v_p}\sqrt{\frac{\rho_1\rho_2}{2}}(-a + jb)\right]}{\sigma + j(\omega_c - \omega_0)}\, d\sigma$$

$$= \frac{V_0 e^{j\omega_c t}}{2\pi j}\int_{0}^{\infty} \frac{\exp[A - jB - \sigma t]}{-\sigma + j(\omega_c - \omega_0)}\, d\sigma,$$

where $A(\sigma) = a\frac{x}{v_p}\left(\frac{\rho_1\rho_2}{2}\right)^{1/2}$, $B(\sigma) = b\frac{x}{v_p}\left(\frac{\rho_1\rho_2}{2}\right)^{1/2}$.

Expressing the integrals along B_2, B_3 and B_4 similarly, it follows that

(A-20) $$\mathrm{Im}\left\{\int_{B_1}+\int_{B_2}+\int_{B_3}+\int_{B_4}\right\} = -\frac{V_0\cos\omega_c t}{\pi}\left[I_1-I_2\right]-\frac{V_0\sin\omega_c t}{\pi}\left[I_3-I_4\right],$$

where

$$I_1=\int_0^\infty \frac{\sigma\alpha+\beta(\omega_c+\omega_0)}{\sigma^2+(\omega_c+\omega_0)^2}\,e^{-\sigma t}\,d\sigma,$$

$$I_2=\int_0^\infty \frac{\sigma\alpha+\beta(\omega_c-\omega_0)}{\sigma^2+(\omega_c-\omega_0)^2}\,e^{-\sigma t}\,d\sigma$$

(A-21) $$I_3=\int_0^\infty \frac{\sigma\beta-\alpha(\omega_c+\omega_0)}{\sigma^2+(\omega_c+\omega_0)^2}\,e^{-\sigma t}\,d\sigma$$

$$I_4=\int_0^\infty \frac{\sigma\beta-\alpha(\omega_c-\omega_0)}{\sigma^2+(\omega_c-\omega_0)^2}\,e^{-\sigma t}\,d\sigma,$$

and $\quad \alpha = \cos B\sinh A, \quad \beta=\sin\beta\cosh A.$

Upon substituting these results back into (A-19) and using the fact that for real V_0, $\mathrm{Im}\{V_0e^{i\theta}\}=V_0\sin\theta$, we get

(A-22) $$v(x,t)=V_0\left\{\sin\omega_0\left(t-\frac{x}{v_p}\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}\right)+\left(M^2+N^2\right)^{1/2}\sin\left(\omega_c t+\theta(t)\right)\right\}$$

where $\quad M=\frac{1}{\pi}\left(I_1-I_2\right), \quad N=\frac{1}{\pi}\left(I_3-I_4\right)$, and $\theta(t)=\tan^{-1}\frac{M}{N}$.

In obtaining this final form for the second (transient) term, we have used the identity
$$A\cos\theta+B\sin\theta=\left(A^2+B^2\right)^{1/2}\sin\left(\theta+\tan^{-1}\frac{A}{B}\right).$$

The first term, of course, agrees with the steady-state expression in (A-15); the second term is, therefore, an alternate expression for the transient term in (A-15). Note that, because the integrals in (A-21) decay with time, the transient part of (A-22) decays with time. Also note that the transient is a phase modulated (and amplitude modulated as well) sine wave with carrier frequency ω_c, which is a characteristic of the transmission-line. This shows quite clearly the role of ω_c in the free response of the transmission-line. Of course (A-22) is valid only for $t > x/vp$. For $t < x/vp$, the solution $v(x,t)$ is nil.

In the last chapter we were introduced to non-sinusoidal, responses of transmission-lines and other distributed systems. Principally, our interest was in initial-value problems, and we found the Laplace transform to be well suited for their analysis. The reason for the utility of the Laplace transform is that it operates on functions that have a finite starting time (which is called the initial time) and the initial conditions or initial data, such as voltage and current, voltage and time-rate-of-change of voltage, etc., are automatically taken into account when transforming a differential equation from the time-domain into the complex frequency (s) domain.

We found that, contrary to sinusoidal, steady-state situations, the voltage at any point x of a transmission-line and time t did not at all resemble the input voltage unless the transmission-line was non-dispersive, or distortionless, i.e., unless RC=GL. The lossless line, R=G=0, is a special case of non-dispersive line.

In this chapter we will pursue the study of dispersion using, for the most part, the transmission-line model of an electromagnetic waveguide introduced in the last chapter. This transmission-line models well the most important features of dispersion. In addition it has been the subject of a great deal of recent research.

Our interest now is no longer in initial value problems, but we are still interested in determining the response to non-sinusoidal inputs. A useful tool that supplements the Laplace transform for these studies is the Fourier transform.

1. The Fourier Transform.

The Fourier transform, $V(\omega)$, of the time function, $v(t)$, is defined by

$$(2\text{-}1)\ (a) \quad V(\omega) = \int_{-\infty}^{\infty} v(t)\, e^{-j\omega t}\, dt.$$

Note the similarity between the Fourier and Laplace transforms; the latter is defined by $V(s) = \int_0^{\infty} v(t)\, e^{-st}\, dt$. The difference is in the exchange of the complex frequency variable, $s = \sigma + j\omega$, with the real frequency ω preceded by $+j$. Secondly, the Fourier transform applies to functions defined over the entire time axis $-\infty < t < \infty$, whereas the Laplace transform is taken only of time functions defined over the half-line $0 \le t < \infty$.

The inverse relation between $V(\omega)$ and $v(t)$ is given by

$$(2\text{-}1)\ (b) \quad v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)\, e^{j\omega t}\, d\omega = \int_{-\infty}^{\infty} V(f)\, e^{j2\pi f t}\, df,$$

where $\omega = 2\pi f$.

It is (2-1)(b) that justifies the statement that "any function can be thought of as the summation of complex sine waves, $e^{j\omega t}$ of all frequencies magnitudes, and phases". The factor that determines the magnitude and phase at the frequency ω is precisely $V(\omega)$, the Fourier transform of $v(t)$.

<u>Example 1.</u> Calculate the Fourier transform of the Gaussian pulse

$$v(t) = V_0 e^{-t^2/2T^2},$$

shown in Figure 2-1

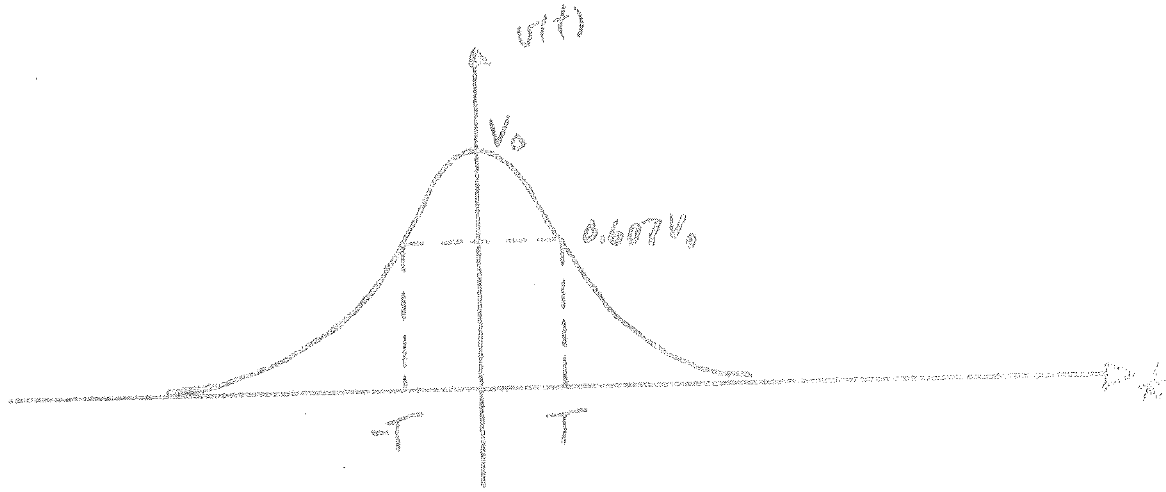


Figure 2-1. A Gaussian pulse .2T is the nominal "width" of the pulse.

<u>Solution:</u>

$$V(\omega) = V_0 \int_{-\infty}^{\infty} e^{-t^2/2T^2} e^{-j\omega t}\, dt$$

$$= V_0 \int_{-\infty}^{\infty} e^{\left[-(j\omega t) - \left(\frac{1}{2T^2}\right)t^2\right]} dt$$

Upon completing the square in the exponent, this becomes

$$V(\omega) = V_0 \int_{-\infty}^{\infty} e^{-\omega^2T^2/2} \cdot e^{-\left(t/\sqrt{2}T + Tj\omega/\sqrt{2}\right)^2} dt.$$

$$= V_0\, e^{-\omega^2T^2/2} \int_{-\infty}^{\infty} e^{-\left(t/\sqrt{2}T + j\omega T/\sqrt{2}\right)^2} dt.$$

Next, make the change of variable $\xi = \dfrac{t}{\sqrt{2}T} + \dfrac{j\omega T}{\sqrt{2}}$ for which $dt = \sqrt{2}T\, d\xi$ so that the last integral becomes

$$V(\omega) = V_0 e^{-\omega^2T^2/2} \cdot \sqrt{2}T \int_{-\infty}^{\infty} e^{-\xi^2} d\xi$$

$$= \sqrt{2\pi}\, V_0 T e^{-\omega^2T^2/2}$$

The spectrum, i.e., $V(\omega)$, is plotted in Figure 2-2. Note that it, too, is a Gaussian function of frequency with a bandwidth of $\frac{2}{T}$.

A Gaussian pulse is said to be "self-reciprocal" because it is its own transform.

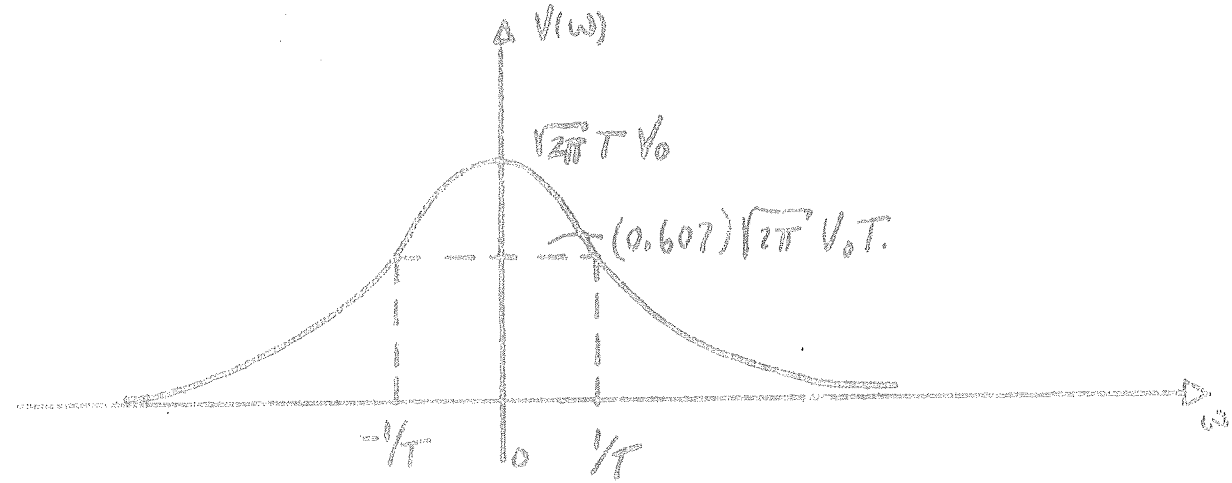


Fig. 2-2. $V(\omega)$ for a Gaussian pulse, $v(t)$.

An interesting observation is that if we reduce T to zero, keeping the product $V_0T = 1$ (which means that $V_0 \to \infty$ as $\frac{1}{T}$ when $T \to 0$), then the Gaussian pulse of Figure 2-1 would approach the delta function $\sqrt{2\pi}\,\delta(t)$, while the spectrum $V(\omega)$ approaches the constant $\sqrt{2\pi}$, thereby demonstrating that the more narrowly confined a function is in time, the broader its frequency spectrum, and the broader a function is in time, the narrower its frequency spectrum. The reason for stating that the Gaussian pulse in the example approaches $\sqrt{2\pi}\,\delta(t)$ is because the area under the curve of Figure 2-1 is not unity, but rather $\sqrt{2\pi}$.

Example 2. Calculate the Fourier transform of the modulated wave $v(t) = V_0 e^{-t^2/2T^2} \cdot e^{j\omega t}$ which consists of a (complex) sinusoidal carrier, $e^{j\omega_0 t}$, of frequency ω_0, amplitude modulated by the same Gaussian pulse of Example 1. The imaginary part of $v(t)$, which is equal to $V_0 e^{-t^2/2T^2} \sin\omega_0 t$ is sketched in Figure 2-3.

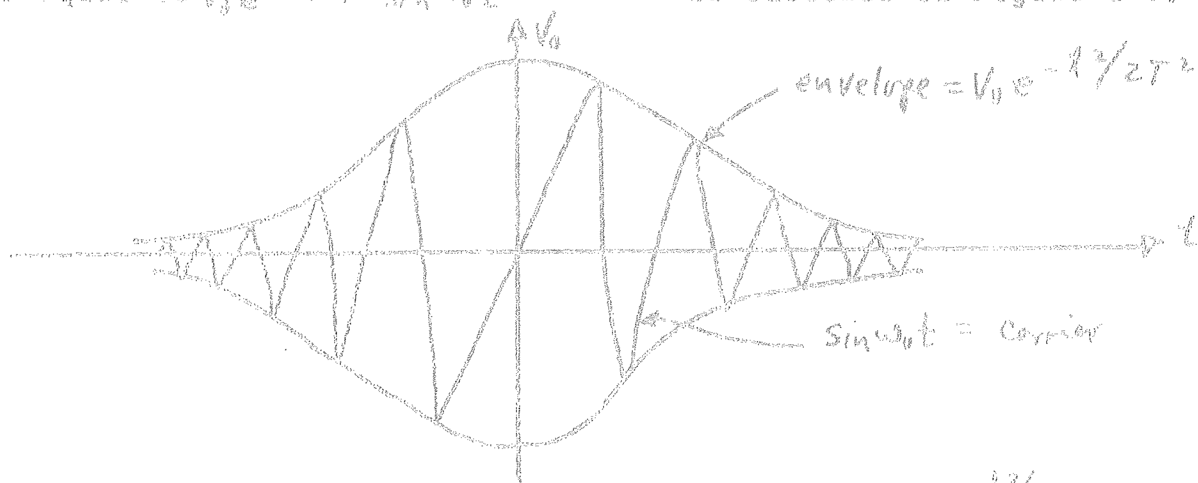


Figure 2-3. Illustrating the modulated sine wave $V_0 e^{-t^2/2T^2} \cdot \sin\omega_0 t$.

SOLUTION:

$$V(\omega) = V_0 \int_{-\infty}^{\infty} e^{-t^2/2T^2} e^{-j(\omega-\omega_0)t}\, dt.$$

At this point we need not make any more calculations after observing that this form for $V(\omega)$ is precisely the same as that for the preceding example <u>except for the replacement of ω by $\omega-\omega_0$</u>. Thus, the result is

$$V(\omega) = \sqrt{2\pi}\, V_0 T\, e^{-(\omega-\omega_0)^2 T^2/2}.$$

A sketch of the spectrum, $V(\omega)$, for the modulated carrier is, therefore, identical to the spectrum of the envelope, $e^{-t^2/2T^2}$, ~~above~~ alone, <u>except that it is shifted to the point $\omega=\omega_0$</u>. Thus, the peak of Figure 2-2 would be located at $\omega=\omega_0$ instead of at $\omega=0$.

This result is typical of all amplitude-modulated sinusoidal carriers.

2. <u>Fourier Transform of the Wave Equation</u>.

In the absence of initial conditions, the Laplace transformed wave equation becomes the Fourier transformed ~~transmission-line~~ wave equations ~~for the waveguide~~

(2-2) (a) $$\frac{dI(x,\omega)}{dx} = -\left(j\omega C + \frac{k_c^2}{j\omega L}\right) V(x,\omega)$$

(b) $$\frac{dV(x,\omega)}{dx} = -j\omega L\, I(x,\omega).$$

$V(x,\omega)$ and $I(x,\omega)$ are, respectively, the Fourier transformed voltage and current at point x. These two equations combine to yield

(2-3) (a) $$\frac{d^2V(x,\omega)}{dx^2} = -\omega^2 LC\left(1-\frac{k_c^2}{\omega^2 LC}\right)V(x,\omega) = -\left(\frac{\omega}{v_g}\right)^2\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)V(x,\omega)$$

(b) $$\frac{d^2I(x,\omega)}{dx^2} = -\left(\frac{\omega}{v_g}\right)^2\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)I(x,\omega).$$

Here $v_p = 1/\sqrt{LC}$, $\omega_c = k_c\, v_p = k_c/\sqrt{LC}$ (recall (1-57)).

simply upon letting $s=j\omega$. Thus, in (1-56)(a),(b) if we replace s by $j\omega$ we get the Fourier transformed transmission-line equations for the waveguide (2-2)(a),(b):

Of course, (2-3) could have been obtained directly from (1-57) by replacing s^2 by $-\omega^2$. The solution of (2-3)(a) corresponding to a wave propagating in the (+)x direction of the transmission-line (or waveguide) is

$$(2\text{-}4)\qquad V(x,\omega) = V(0,\omega)\, e^{-j\left(\frac{\omega}{v_p}\right)\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2} x} = V(0,\omega)\, e^{-j\beta(\omega)x}$$

where $V(0,\omega)$ is the Fourier transform of the input voltage $v_o(t)$ . . .

The function

$$(2\text{-}5)\qquad \beta(\omega) \equiv \left(\frac{\omega}{v_p}\right)\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2}$$

is called the <u>phase-shift-per-unit length</u>, or <u>propagation constant</u>. It plays an important role in the dispersive properties of the transmission-line and is plotted in Figure 2-4.

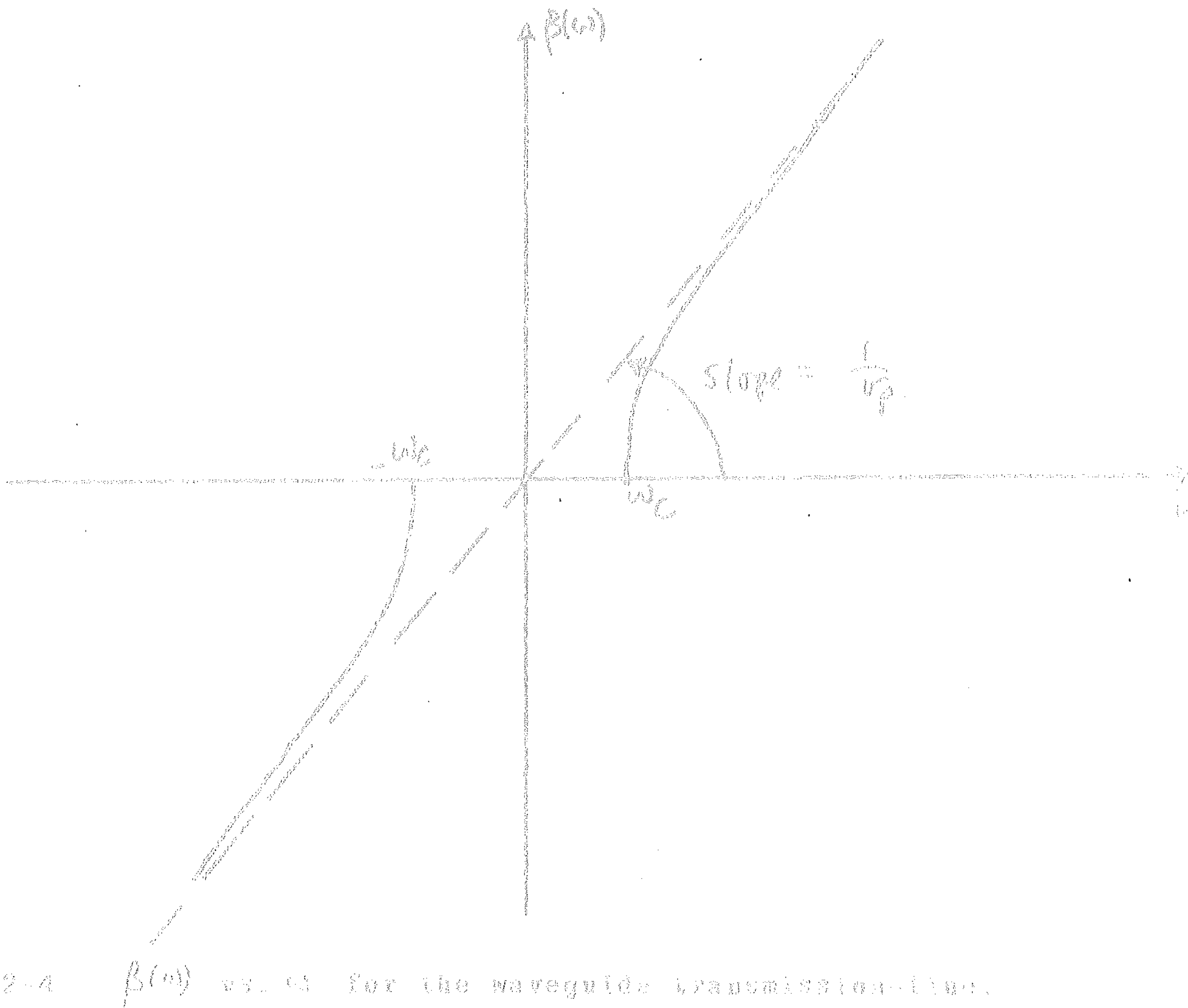


Figure 2-4 $\beta(\omega)$ vs. ω for the waveguide transmission-line.

$\beta(\omega)$, of course, is imaginary for $-\omega_c < \omega < \omega_c$, so we have left that interval blank. ω_c is called the cut-off frequency because $\beta(\omega)$ is imaginary in the above interval. Note that $\beta(\omega)$ is asymptotic to ω/v_p at high frequencies. Because ω/v_p is the propagation constant for a lossless, non-dispersive transmission-line, we say that the waveguide is non-dispersive at high frequencies. Thus, in a sense it acts as a high pass filter whose low frequency cut-off is ω_c. This will be borne out later when we calculate the transient response of the waveguide to stepped sine waves.

The instantaneous voltage at point x and time t is given by the inverse transform of (2-4)

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(0,\omega)\, e^{-j\left(\frac{\omega}{v_p}\right)\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2} x} \cdot e^{j\omega t}\, d\omega. \tag{2-6}$$

We will use (2-6) to discuss the propagation of a Gaussian modulated pulse. First let us describe the rotation of wave pockets and group velocity.

3. Wave Pockets and Group Velocity. (Narrow-Band Spectrum)

If T in the Gaussian envelope is "large" so that $2/T$, the width of the spectrum, is small compared to ω_0, the center frequency of the spectrum, then the Gaussian modulated pulse is said to be narrowband, or a wave packet. Thus, a wave packet is a signal which contains a narrow band of frequencies centered about a carrier frequency and with a pronounced peak there.

Let $V(0,\omega)$ be a wave packet in (2-6). Because $V(0,\omega)$ is highly peaked about some center (or carrier) frequency, ω_0, and is relatively small in regions well away from ω_0, it makes sense to consider expanding the phase factor $\beta(\omega)x$ in a Taylor series about ω_0:

$$\beta(\omega)x \cong \beta(\omega_0)x + x\left.\frac{\partial\beta(\omega)}{\partial\omega}\right|_{\omega=\omega_0}(\omega-\omega_0) + x\left.\frac{\partial^2\beta(\omega)}{\partial\omega^2}\right|_{\omega=\omega_0}\frac{(\omega-\omega_0)^2}{2} + \cdots$$

Now if we keep only terms up to $(\omega-\omega_0)^2$, (2-6) becomes

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(0,\omega)\, e^{j(\omega t-\beta(\omega)x)}\, d\omega$$

$$\approx \frac{1}{2\pi}\int_{-\infty}^{\infty} V(0,\omega)\, e^{j\omega t} \cdot e^{-j\beta(\omega_0)x} \cdot e^{-jx\beta'(\omega_0)(\omega-\omega_0)} \cdot e^{-jx\beta''(\omega_0)\frac{(\omega-\omega_0)^2}{2}}\, d\omega \tag{2-7}$$

$$= \frac{1}{2\pi} e^{-j\beta(\omega_0)x}\int_{-\infty}^{\infty} V(0,\omega)\, e^{j\omega t} \cdot e^{-jx\beta'(\omega_0)(\omega-\omega_0)} \cdot e^{-jx\beta''(\omega_0)\frac{(\omega-\omega_0)^2}{2}}$$

where the primes denote differentiation with respect to ω.

Upon making the change of variable $\xi = \omega - \omega_0$, (2-7) becomes

$$v(x,t) = \frac{e^{j(\omega_0 t - \beta(\omega_0)x)}}{2\pi} \int_{-\infty}^{\infty} V(0, \xi+\omega_0)\, e^{j\xi(t - x\beta'(\omega_0))} \cdot e^{-jx\beta''(\omega_0)\frac{\xi^2}{2}}\, d\xi . \tag{2-8}$$

From (2-8) it is apparent that the product $x\beta'(\omega_0)$ has dimensions of time, so that $(\beta'(\omega_0))^{-1} \equiv \left(\partial\beta/\partial\omega\big|_{\omega=\omega_0}\right)^{-1}$ has dimensions of velocity. This velocity, v_g, is called the group velocity of the wave, for reasons which will be made clear shortly. Thus

$$v_g(\omega) = \left(\frac{\partial\beta}{\partial\omega}\bigg|_{\omega=\omega_0}\right)^{-1} = \frac{\partial\omega}{\partial\beta}\bigg|_{\omega=\omega_0} . \tag{2-9}$$

Let us not forget that group velocity is associated with a wave packet, i.e., a signal with a narrow frequency spectrum centered about ω_0.

In order to appreciate the significance of the group velocity, let us assume that $\beta''(\omega_0) = 0$, so that (2-8) becomes

$$v(x,t) = \frac{e^{j(\omega_0 t - \beta(\omega_0)x)}}{2\pi} \int_{-\infty}^{\infty} V(0, \xi+\omega_0)\, e^{j\xi(t - x\beta'(\omega_0))}\, d\xi . \tag{2-10}$$

The spectrum $V(0,\omega)$ was centered about ω_0 according to the hypothesis. Thus, $V(0,\omega)$ is the spectrum of a bandpass signal with carrier frequency ω_0. $V(0,\xi+\omega_0)$, on the other hand, is obtained from the spectrum $V(0,\omega)$ by shifting to the left along the ω-axis an amount ω_0, thereby placing the spectrum $V(0,\xi+\omega_0)$ so that it peaks at $\xi=0$. This signal, because it is centered at the origin (d.c.), is called a baseband signal. It is precisely this baseband signal which served as the envelope of the modulated carrier wave. Figure 2-5 illustrates some of these points.

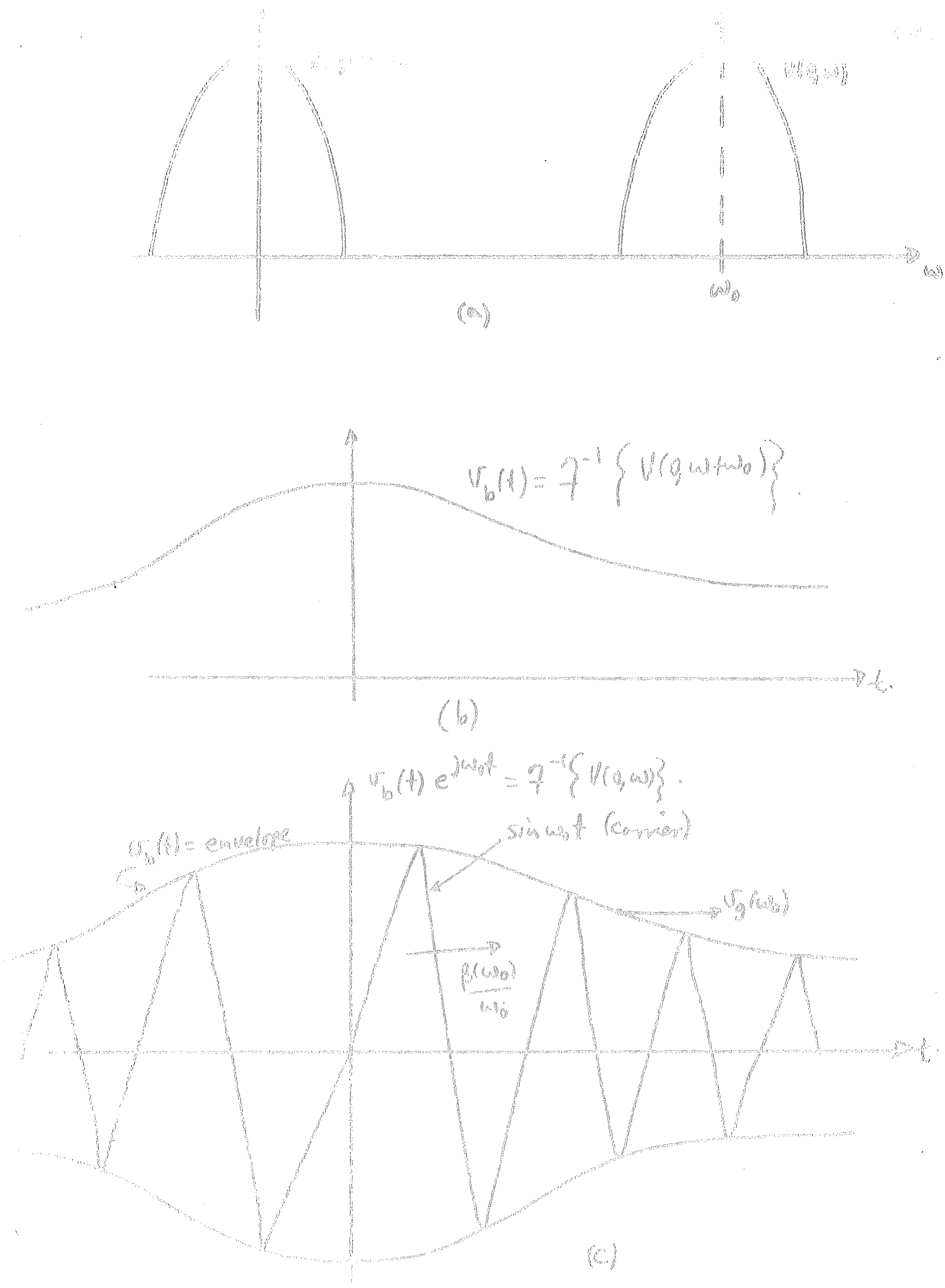


Figure 2-5. (a) Translated wave packet spectra. (b) The time function corresponding to $V(0,\omega+\omega_0)$. (c) The time function corresponding to $V(0,\omega)$.

be the baseband signal, which by our discussion is also the envelope of the frequency modulated wave at the input. Then the integral of (2-10) is precisely v_b evaluated at $t - x\beta'(\omega_0) = t - x/v_g(\omega_0)$, so that (2-10) becomes

$$v(x,t) = v_b\left(t - x/v_g(\omega_0)\right) e^{j(\omega_0 t - \beta(\omega_0)x)} \tag{2-11}$$

This result has a very interesting interpretation. First, note that it is in the form of a modulated carrier, again. Now, however, <u>the modulating envelope is propagating in the (+)x direction with a velocity, $v_g(\omega_0)$, the group velocity at frequency ω_0</u>, while the <u>carrier $e^{j(\omega_0 t - \beta(\omega_0)x)}$ propagates with velocity $\omega_0/\beta(\omega_0)$, the guide phase velocity, v_{gp}</u>. Hence, the group velocity is the velocity with which a "narrow group of waves" or, wave packet, propagates.

We remind the reader that the concepts derived here hold only for narrow spectra or "quasi-harmonic" signals. For the waveguide transmission-line, the group and phase velocities are plotted in Fig. 2-6. Expressions for these velocities are easily derived from the expression, (2-5), for $\beta(\omega)$. Thus,

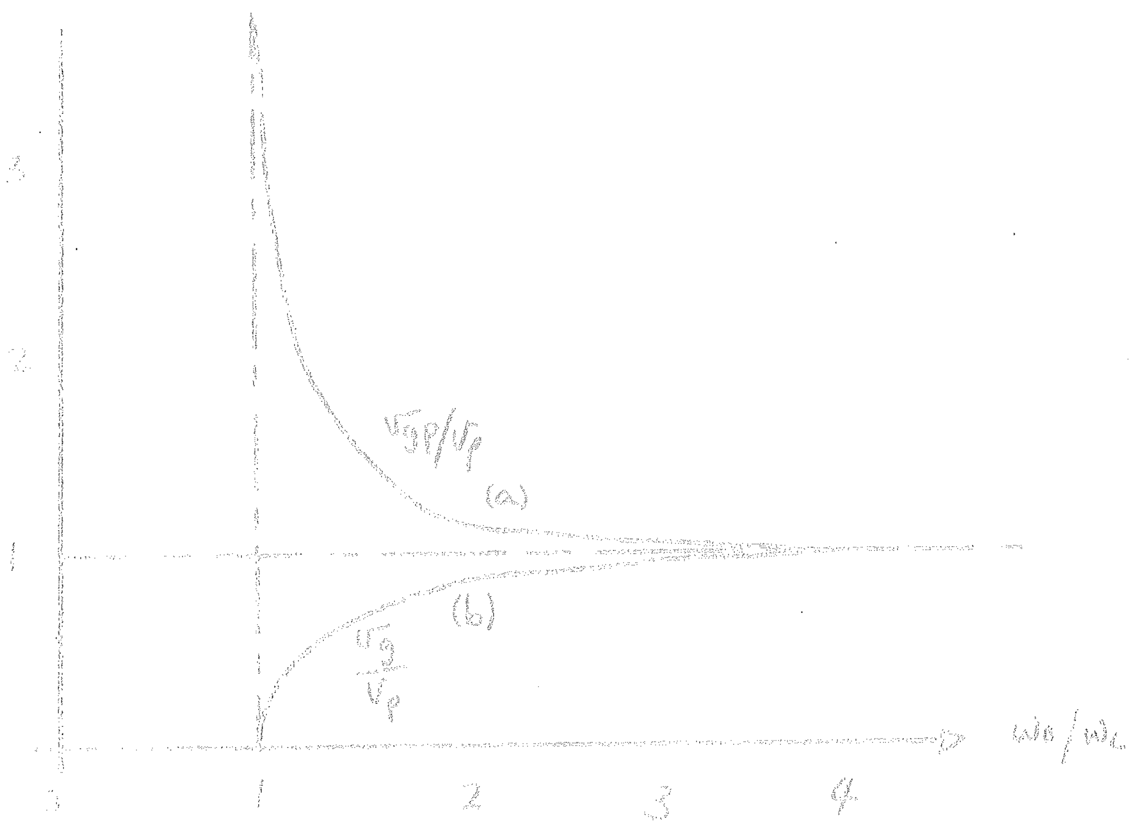


Figure 2-6. (a) Phase velocity and (b) group velocity for the waveguide.

$$\text{(a)}\quad v_{gp} = \frac{\omega_0}{\beta(\omega_0)} = \frac{v_p}{\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}}$$

(2-12)

$$\text{(b)}\quad v_g = \left(\frac{\partial\beta}{\partial\omega}\bigg|_{\omega=\omega_0}\right)^{-1} = v_p\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}.$$

Both velocities are imaginary, indicating no wave propagation, in the cut-off region $-\omega_c < \omega < \omega_c$. Finally, note that v_{gp} is always greater than v_p, the phase velocity for a distortionless line, whereas v_g is always less than v_p. We have already seen that the signal envelope travels with velocity v_g. Later we shall show that v_g is also the velocity of energy transport down the guide (See Appendix A).

4. Propagation of a Gaussian-Envelope Modulated Pulse.

Let $V(0,\omega)$ in (2-6) be the signal of Example 2 with T large enough to satisfy the hypotheses of Section 3. Then (2-8) holds.

Also, from Example 2 we have that $V(0,\xi+\omega_0) = \sqrt{2\pi}\, V_0 T e^{-\xi^2 T^2/2}$

so that (2-8) becomes

(2-13)
$$v(X,t) = \frac{V_0 T}{\sqrt{2\pi}} e^{j(\omega_0 t - \beta_0 X)} \int_{-\infty}^{\infty} \exp\left\{ j(t - X\beta_0')\xi - \left(\frac{T^2}{2} + jX\frac{\beta_0''}{2}\right)\xi^2 \right\} d\xi ,$$

where $\beta_0 = \beta(\omega_0)$, $\beta_0' = \beta'(\omega_0)$, $\beta_0'' = \beta''(\omega_0)$.

Upon completing the square in the exponent, as we did in Example 1, and carrying out the subsequent integration, there results

(2-14)
$$v(X,t) = V_0 T e^{j(\omega_0 t - \beta_0 X)} \sqrt{\frac{1}{T^2 + jX\beta_0''}}\; e^{-\frac{(t - X\beta_0')^2}{2(T^2 + jX\beta_0'')}} ,$$

The exponent $\dfrac{(t - X\beta_0')^2}{2(T^2 + jX\beta_0'')}$ can be rationalized to give

$$\frac{(t - X\beta_0')^2\left(1 - j\left(\frac{X\beta_0''}{T^2}\right)\right)}{2T^2\left(1 + \left(\frac{X\beta_0''}{T^2}\right)^2\right)} = \frac{(t - X\beta_0')^2}{2T^2\left(1 + \left(\frac{X\beta_0''}{T^2}\right)^2\right)} - j\frac{\left(\frac{X\beta_0''}{T^2}\right)(t - X\beta_0')^2}{2T^2\left(1 + \left(\frac{X\beta_0''}{T^2}\right)^2\right)} .$$

In addition

$$\frac{1}{(T^2 + jX\beta_0'')^{1/2}} = \frac{1}{T\left(1 + \left(\frac{X\beta_0''}{T^2}\right)^2\right)^{1/4}} e^{-j\frac{1}{2}\tan^{-1}\left(\frac{X\beta_0''}{T^2}\right)}$$

Therefore, our final form of (2-14) is

$$v(x,t) = \frac{V_0}{\left[1+\left(\frac{x\beta_0''}{T^2}\right)^2\right]^{1/4}} e^{-\frac{1}{2T^2}\left\{\frac{(t-x\beta_0')^2}{1+\left(\frac{x\beta_0''}{T^2}\right)^2}\right\}} \cdot \exp j\left\{\omega_0 t - \beta_0 x + \left(\frac{x\beta_0''}{T^2}\right)\left[\frac{(t-x\beta_0')^2}{1+\left(\frac{x\beta_0''}{T^2}\right)^2}\right] - \frac{1}{2}\tan^{-1}\left(\frac{x\beta_0''}{T^2}\right)\right\} \qquad (2\text{-}15)$$

This expression, believe it or not, is still in the form of a modulated carrier, only now there are more complications. First, the envelope remains Gaussian

$$\frac{V_0}{\left[1+\left(\frac{x\beta_0''}{T^2}\right)^2\right]^{1/4}} \exp\left\{-\frac{1}{2T^2}\left[\frac{(t-x\beta_0')^2}{1+\left(\frac{x\beta_0''}{T^2}\right)^2}\right]\right\},$$

but it has decreased in amplitude with increasing x and it spreads out, or disperses, with x because of the term $1+\left(\frac{x\beta_0''}{T^2}\right)^2$ in the denominator of the exponent. The peak of the envelope moves with the (group) velocity

$$v_g = 1/\beta_0' .$$

The wave packet has become <u>frequency modulated</u> about the carrier ω_0. The frequency modulating term is the quadratic

$$\left(\frac{x\beta_0''}{T^2}\right)\left\{\frac{(t-x\beta_0')^2}{1+\left(\frac{x\beta_0''}{T^2}\right)^2}\right\}.$$

The time derivative of this expression added to ω_0 gives the instantaneous frequency,

$$\omega_I(t) = \omega_0 + \left(\frac{2x\beta_0''}{T^2}\right)\frac{(t-x\beta_0')}{1+\left(\frac{x\beta_0''}{T^2}\right)^2}.$$

An additional interesting observation is that the amount of distortion of the incident pulse decreases as ω_0, the carrier frequency, increases. This can be seen by referring to (2-15) and Figure 2-4 and realizing that

$$\beta_0'' \equiv \left.\frac{\partial^2\beta}{\partial\omega^2}\right|_{\omega=\omega_0}$$

decreases at higher frequencies because the curve of $\beta(\omega)$ vs. ω approaches the straight-line asymptote $\beta \sim \omega/v_p$. Thus, for carrier frequencies well away from the waveguide cut-off, $\omega_0 >> \omega_c$, so that $\beta_0'' << T^2/x$ where x is some typical propagation distance, then

$$v(x,t) \simeq V_0 e^{-\frac{(t-x\beta_0')^2}{2T^2}} e^{j(\omega_0 t - \beta_0 x)},$$

which is simply a time-shifted-envelope and phase-shifted-carrier version of the input. This is another manifestation of the high-pass filtering nature of the waveguide.

5. Application to Communications.

In digital communications one encodes data into a time sequence of pulses, perhaps of the Gaussian form just considered. Obviously pulse overlap is not desired because the position of the pulses, or their number in a given time interval, carries the information, and overlap would result in ambiguity or error.

Because such communication systems often include waveguides or other dispersive channels, it is important to the system engineer to know the maximum rate at which digital data can be transmitted through such a channel without incurring any ambiguity.

Consider the double Gaussian pulse sequence of Figure 2-7(a). After propagating through the waveguide a certain distance the two pulses will overlap due to dispersion, as shown in Figure 2-7(b). When the exponent in the expression for the envelope in (2-15) is equal to $-\frac{1}{2}$ for each pulse there is said to exist no pulse definition. From (2-15) and Figure 2-7(b) we see that

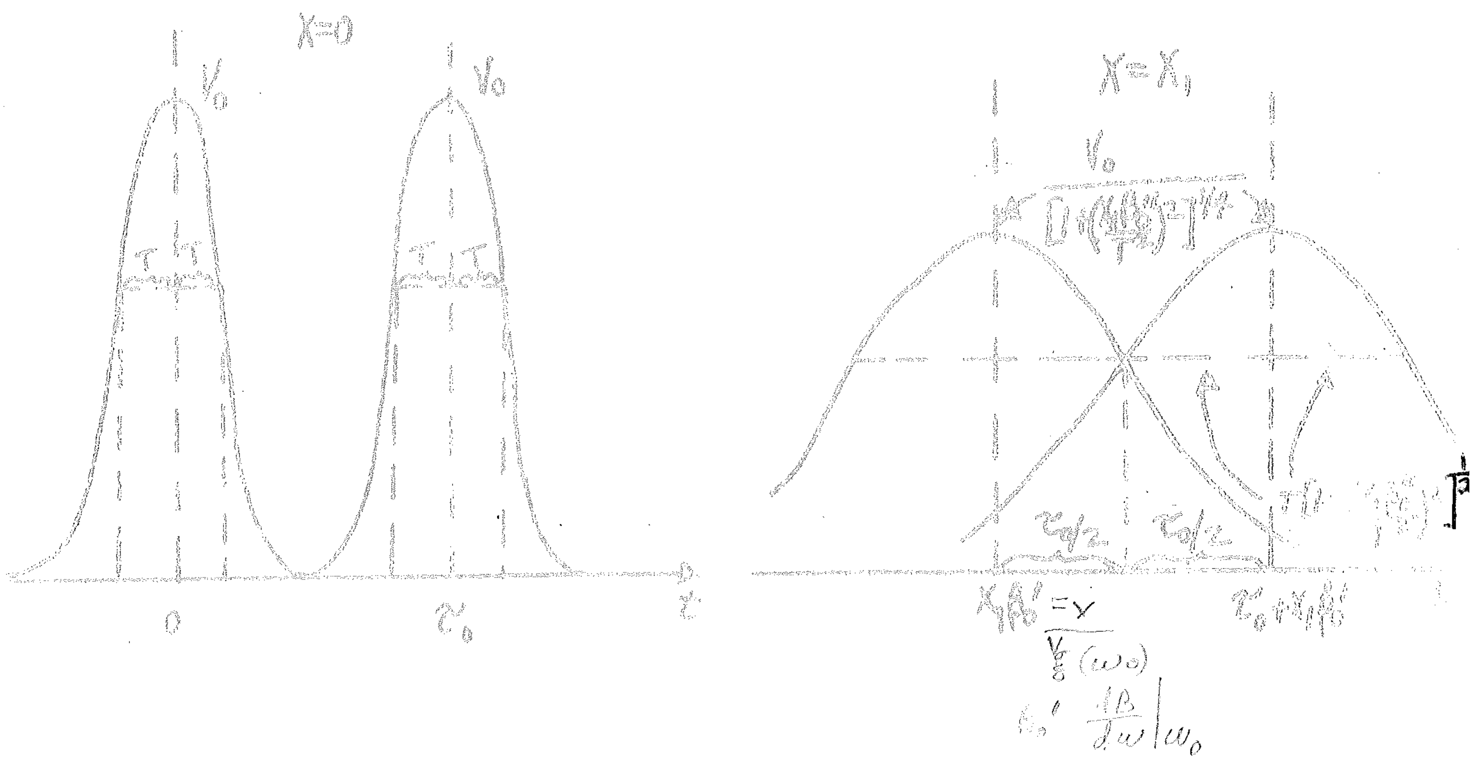


Figure 2-7. (a) Double Gaussian pulse sequence, (x=0).
(b) Definition of no pulse resolution (ambiguity). ($x=x_1$).

This condition exists when

$$\text{(2-16)}\qquad \frac{\tau_0}{2\tau} = \left[1 + \left(\frac{x_1\beta_0''}{\tau^2}\right)^2\right]^{1/2}$$

. . . gives us the maximum value of $x_1\beta_0''/\tau^2$ at which we can expect good pulse definition. For a receiver located at a distance x_2 greater than the above computed value of x_1, ambiguity in the received pulses would result because of the poor pulse resolution.

Conversely, given a fixed value for x_1, ω_0 and τ, (2-16) gives us the minimum value of the pulse position-pulse width ratio for unambiguous (good resolution) reception. These considerations are important in choosing a suitable "bit rate", $1/\tau_0$, for a digital communication system employing a dispersive channel.

5. Propagation of Broad-Band Signals: Stationary Phase.

Thus far the analysis of Sections 3 and 4 have been concerned with narrow-band, or "quasi-monochromatic", signals. Equation (2-6), however, is general and can be applied to broad-band signals as well. An example of a broad-band signal, $v_0(t)$, is a delta function, $\delta(t)$, for then $V(0,\omega) = 1$ for all ω----a decidely broad-band spectrum.

Let us generalize (2-6) and write

$$\text{(2-17)} \qquad v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(0,\omega)\, e^{-jxg(\omega)}\, d\omega,$$

where $g(\omega) = \beta(\omega) - \omega t/x$. $\beta(\omega)$ here stands for the propagation constant for a general system; (2-6) applies only to the waveguide. We shall take $\beta(\omega)$, and hence, $g(\omega)$, to be real, which implies propagation and not cut-off conditions exist.

Let us suppose that x in (2-17), is "large", that is the receiver is many wavelengths away from the transmitter. What "wavelength" we're talking about depends on the problem. Usually there exists a critical or characteristic wavelength determined by a characteristic frequency. In the quasi-monochromatic example of Sections 3 and 4, ω_0, the carrier frequency of the narrow-band pulse would be appropriate, and the characteristic wavelength would be $\lambda_0 = v_p/f_0 = 2\pi v_p/\omega_0$.

In the waveguide example the characteristic frequency would be ω_c, the cut-off frequency, and the corresponding wavelength would be $\lambda_c = v_p/f_c = 2\pi v_p/\omega_c$. Thus, we suppose that

$$x \gg \lambda_c.$$

The resulting expansion for v(x,t), because x is "large" (compared to λ_c) will be called an "asymptotic expansion" and is of great importance in wave propagation problems. Now that we have decided on an asymptotic expansion for v(x,t), let us return to (2-17) and write it as a summation

$$v(x,t) \approx \frac{1}{2\pi}\Big[\cdots V(0,\omega_{-2})\Delta\omega\, e^{-jxg(\omega_{-2})} + V(0,\omega_{-1})\Delta\omega\, e^{-jxg(\omega_{-1})}$$
$$+ V(0,\omega_0)\Delta\omega\, e^{-jxg(\omega_0)} + V(0,\omega_1)\Delta\omega\, e^{-jxg(\omega_1)} + V(0,\omega_2)\Delta\omega\, e^{-jxg(\omega_2)}$$
$$+ \cdots\Big].$$

Each term within the braces is a phasor whose magnitude (assuming $V(0,\omega)$ is real and positive, for simplicity) is $V(0,\omega_i)\,\partial\omega/2\pi$,

and whose phase angle is $-Xg(\omega_i)$. Thus, we may plot them on the phasor

diagram of Figure 2-8. In this Figure we are assuming that $V(0,\omega)$ is real, positive and relatively constant compared to the rate-of-change of $q(\omega)$ with ω .

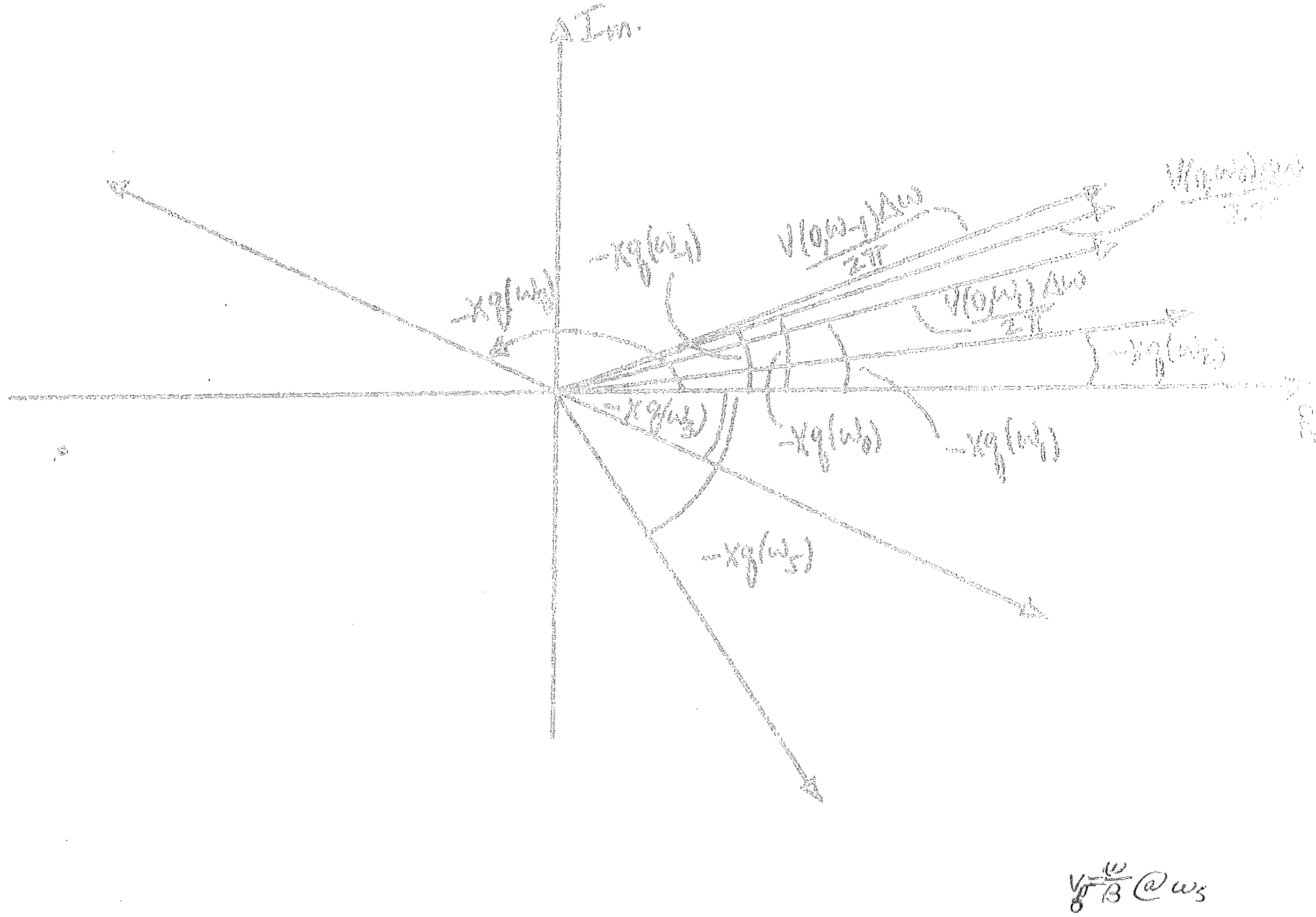


Figure 2-8, The phasor representation of (2-17). We are assuming that $V(0,\omega)$ is real, positive, and relatively constant.

Note in Figure 2-8, that due to the large value of x, even a relatively small change in the value of ω will cause a large phase shift, as for example $-Xg(\omega_4)$ and $-Xg(\omega_5)$, <u>except for a group of waves such</u> as $-Xg(\omega_{-1})$, $-Xg(\omega_0)$, $-Xg(\omega_1)$, <u>which are practically in phase.</u> Obviously, it is this group of waves which will contribute the majority value to (2-17) because the phasors in this group are in phase (approximately) with each other while phasors whose phase angles are similar to $-Xg(\omega_4)$ and $-Xg(\omega_5)$ will cancel.

[illegible] ... in the neighborhood, [illegible] the rate of change of phase angle, $\partial g/\partial\omega$ with respect to ω is zero at frequency ω_s; otherwise, a small change in ω produces a large change in phase angle, again at ω_s, because of the presence of the large factor, x. These latter terms cancel.

Thus, on the basis of the above heuristic reasoning (which can be made precise, though not here) we conclude that the principal contribution to the asymptotic expansion of (2-17) comes from the frequency at which $g(\omega)$ is stationary with respect to ω, i.e., where $\partial g(\omega)/\partial\omega = 0$. The method, therefore, is referred to as the method of stationary phase ($-x\,g(\omega)$ is the phase of the integrand of (2-17)).

The frequency ω_s at which $\partial g/\partial\omega = 0$ is called the stationary point. Let us expand $g(\omega)$ about such a stationary point, retaining only terms up to the second derivative

$$g(\omega) \simeq g(\omega_s) + \frac{\partial g}{\partial\omega}\Big|_{\omega=\omega_s}(\omega-\omega_s) + \frac{1}{2}\frac{\partial^2 g}{\partial\omega^2}\Big|_{\omega=\omega_s}(\omega-\omega_s)^2 \tag{2-18}$$

$$= g(\omega_s) + \frac{1}{2}\frac{\partial^2 g}{\partial\omega^2}\Big|_{\omega=\omega_s}(\omega-\omega_s)^2,$$

where the first derivative vanishes by definition of ω_s.

Upon substituting (2-18) back into (2-17), we obtain

$$s(r,t) \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} V(0,\omega_s)\, e^{-jxg(\omega_s)}\, e^{-jx\,g''(\omega_s)\frac{(\omega-\omega_s)^2}{2}}\,d\omega \tag{2-19}$$

$$= \frac{V(0,\omega_s)\,e^{-jxg(\omega_s)}}{2\pi}\int_{-\infty}^{\infty} e^{-jx\,g''(\omega_s)\frac{(\omega-\omega_s)^2}{2}}\,d\omega.$$

We reduce the integral by making the change of variable

$$\xi^2 = x\,|g''(\omega_s)|\,\frac{(\omega-\omega_s)^2}{2} \quad \text{so that} \quad d\omega = \sqrt{2}\,d\xi/(x\,|g''(\omega_s)|)^{1/2}.$$

Equation (2-19) becomes, under this transformation

$$s(r,t) = \frac{V(0,\omega_s)\,e^{-jxg(\omega_s)}\sqrt{2}}{2\pi\,(x\,|g''(\omega_s)|)^{1/2}}\int_{-\infty}^{\infty} e^{\mp j\xi^2}\,d\xi, \tag{2-20}$$

where the (-) argument goes with $g''(\omega_s) > 0$ and the (+) with $g''(\omega_s) < 0$. The integral in (2-20) is a well known definite integral and has the value $\sqrt{\frac{\pi}{\pm j}} = \sqrt{\pi}\,e^{\mp j\pi/4}$.

Thus (2-20) becomes

$$s(r,t) = \frac{V(0,\omega_s)}{\sqrt{2\pi}}\,\frac{e^{-j\left(xg(\omega_s) \pm \pi/4\right)}}{(x\,|g''(\omega_s)|)^{1/2}}, \qquad \begin{matrix}(+) \; g''(\omega_s) > 0 \\ (-) \; g''(\omega_s) < 0.\end{matrix} \tag{2-21}$$

Finally, if $V(0, \omega_s)$ is complex, $V(0, \omega_s) = |V(0,\omega_s)| e^{j\varphi(\omega_s)}$, then (2-21) is slightly modified

$$(2\text{-}22) \qquad v(x,t) = \frac{|V(0,\omega_s)|\, e^{-j\left(x g(\omega_s) - \varphi(\omega_s) \pm \pi/4\right)}}{\sqrt{2\pi}\,\left(x\,|g''(\omega_s)|\right)^{1/2}}.$$

A similar contribution must be added for every stationary point ω_s.

Let us go back and re-examine, the meaning of ω_s. Recall that $g(\omega) = \beta(\omega) - \omega t/x$, so that ω_s satisfies

$$(2\text{-}23) \qquad g'(\omega_s) = 0 = \beta'(\omega_s) - t/x$$

or

$$x = t/\beta'(\omega_s) = v_g(\omega_s)\, t.$$

The meaning of this result is that the dominant effects at point x and time t are carried by those wave components (or group of waves) whose group velocity, $v_g(\omega_s)$, is x/t. For a fixed x, as time varies so does $v_g(\omega_s)$ (according to (2-23)) and, hence, ω_s. Thus, a different group of waves, traveling at its own group velocity, will carry the effects at different times. Another way of stating this is that for fixed x, the stationary point, ω_s, will vary with t.

Application of (2-22) to the waveguide problem starts upon recognizing that from (2-6)

$$g(\omega) = \frac{\omega}{v_p}\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2} - \omega t/x .$$

Hence, ω_s is given by $v_g(\omega_s) = v_p\left(1-\left(\frac{\omega_c}{\omega_s}\right)^2\right)^{1/2} = x/t$, which when explicitly solved for ω_s yields

$$(2\text{-}24) \qquad \omega_s = \frac{\pm\,\omega_c}{\left[1-\left(\frac{x}{v_p t}\right)^2\right]^{1/2}}, \qquad v_p t > x,$$

Note that as $t \to \infty$, $\omega_s \to \omega_c$; also $g''(\omega_s) = \beta''(\omega_s) =$

$$= \frac{1}{v_p \omega_s\left[\left(\frac{\omega_s}{\omega_c}\right)^2 - 1\right]}.$$

Upon substitution of (2-24) into this expression there results

$$(2\text{-}25) \qquad g''(\omega_s) = \mp \frac{\left[\left(\frac{v_p t}{x}\right)^2 - 1\right]^{3/2}}{\omega_c v_p}, \qquad v_p t > x.$$

We note that there are two stationary points given by (2-24). For the first, with the (+) sign, $g''(\omega_s)$ is negative, so that (2-22) holds with the $-\pi/4$ in the exponent

$$(2\text{-}26) \qquad v_+(x,t) = \frac{|V(0,\omega_s)|}{[2\pi x|g''(\omega_s)|]^{1/2}} e^{-j(xg(\omega_s)-\varphi(\omega_s)-\pi/4)},$$

whereas for the second, with the (−) sign, $g''(\omega_s)$ is positive, so that (2-22) holds with the $+\pi/4$ in the exponent

$$(2\text{-}27) \qquad v_-(x,t) = \frac{|V(0,\omega_s)|}{[2\pi x|g''(\omega_s)|]^{1/2}} e^{-j(xg(-\omega_s)-\varphi(-\omega_s)+\pi/4)}.$$

But $g(-\omega_s) = -g(\omega_s)$, $\varphi(-\omega_s) = -\varphi(\omega_s)$ and $|g''(\omega_s)| = |g''(-\omega_s)|$, so that (2-27) becomes

$$(2\text{-}28) \qquad v_-(x,t) = \frac{|V(0,\omega_s)|}{[2\pi x|g''(\omega_s)|]^{1/2}} e^{j(xg(\omega_s)-\varphi(\omega_s)-\pi/4)}.$$

Now adding (2-26) and (2-28) yields

$$(2\text{-}29) \qquad v(x,t) = \frac{2|V(0,\omega_s)|}{[2\pi x|g''(\omega_s)|]^{1/2}} \cos(xg(\omega_s)-\varphi(\omega_s)-\pi/4)$$

$$= \frac{2|V(0,\omega_s)|}{[2\pi x|g''(\omega_s)|]^{1/2}} \cos(\omega_s t-\beta(\omega_s)x+\varphi(\omega_s)+\pi/4).$$

Finally, let us assume that the input signal $v_0(t)$ is an impulse so that $|V(0,\omega_s)| \equiv 1$, $\varphi(\omega_s) \equiv 0$. Then upon substituting the appropriate expressions for ω_s, $\beta(\omega_s)$ and $g''(\omega_s)$ into (2-29) we obtain the asymptotic impulse response of the waveguide

$$(2\text{-}30) \qquad v(x,t) = \left[\frac{2}{\pi} \cdot \frac{\omega_c v_p}{x} \frac{1}{[(\frac{v_p t}{x})^2-1]^{3/2}}\right]^{1/2} \cos\left[\frac{\omega_c t}{(1-(\frac{x}{v_p t})^2)^{1/2}} - \frac{\omega_c x/v_p}{[(\frac{v_p t}{x})^2-1]^{1/2}} + \frac{\pi}{4}\right]$$

$$v(x,t) = \left[\frac{2}{\pi} \frac{\omega_c v_p}{x} \frac{1}{[(\frac{v_p t}{x})^2-1]^{3/2}}\right]^{1/2} \cos\left[\omega_c\left(t^2-\left(\frac{x}{v_p}\right)^2\right)^{\frac{1}{2}}+\frac{\pi}{4}\right]$$

[illegible] function. The asymptotic frequency is ω_c. In fact, for large t, the entire expression for v(x,t) becomes

$$(2\text{-}31)\qquad v(x,t) \underset{t\to\infty}{\longrightarrow} \sqrt{\frac{2\omega_c x^2}{\pi v_p^2 t^3}}\,\cos(\omega_c t + \pi/4)$$

This result shows the decay with time, which decay is due to destructive interference. Equation (2-31) should be compared with the expression, (1-58), previously obtained for the impulse response of the waveguide:

$$(2\text{-}32)\qquad h(x,t) = \delta(t - x/v_p) - \frac{x\omega_c}{v_p}\frac{J_1\left[\omega_c(t^2 - x^2/v_p^2)^{1/2}\right]}{(t^2 - x^2/v_p^2)^{1/2}}\,u_{-1}(t - x/v_p).$$

As $t \to \infty$ in (2-32), h(x,t) approaches $-\dfrac{x\omega_c}{v_p}\,\dfrac{J_1(\omega_c t)}{t}$.

But (1-60) gives the asymptotic form of the Bessel function $J_1(\omega_c t)$ to be

$$J_1(\omega_c t) \longrightarrow \sqrt{2/\pi\omega_c t}\,\cos(\omega_c t - 3\pi/4).$$

Hence, the asymptotic form for h(x,t) is

$$(2\text{-}33)\qquad h(x,t) \longrightarrow \frac{-x\omega_c}{v_p t}\sqrt{\frac{2}{\pi\omega_c t}}\cos\left(\omega_c t - \frac{3\pi}{4}\right) = \frac{x\omega_c}{v_p t}\sqrt{\frac{2}{\pi\omega_c t}}\cos(\omega_c t + \pi/4),$$

which agrees exactly with (2-31). Thus, the method of stationary phase gives us an asymptotic result entirely consistent with that obtained using the Laplace transform. The beauty of the stationary phase method is that it enables one to quickly calculate the asymptotic waves at any point x and time t. There are many pitfalls in using the method, however, and one should always be aware of the limitations. As an example, we terminated the expansion of $q(\omega)$, with the second derivative $q''(\omega_s)$. When this is zero, obviously we must go to the third derivative term $q'''(\omega_s)(\omega-\omega_s)^3/3!$, and so on. A more thorough discussion of the method can be found in J. R. Wait, "Propagation of Pulses in Dispersive Media" Radio Science, Vol. 69D, pp. 1387-1401, Nov. 1965, and L. B. Felsen, "Transients in Dispersive Media, Part I: Theory", IEEE Trans. on Antennas and Propagation, Vol. AP-17, No. 2, pp. 191-200, March, 1969.

~~In an appendix we will present a more rigorous formulation of the material of this Section and Sections 3 and 4 based on Saddle point integration. By its use, we will be able to gain great insight into transient wave phenomena in dispersive media.~~

We were able to determine the asymptotic solutions (2-30) and (2-31), explicitly as a function of time for the waveguide because we had a relatively easy $\beta(\omega)$ with which to work and we were able to explicitly determine $\omega_s(x,t)$ and $q''[\omega_s(x,t)]$ as functions of x and t (see (2-24), (2-25)).

For more complicated situations we may resort to graphical means in order to ascertain at least some qualitative, if not quantitative, results. This usually requires plotting $\beta(\omega)$, $\partial\beta/\partial\omega$, $v_g(\omega)$ and the amplitude factor in (2-22) or (2-29) vs. ω. Then, as is evident from (2-23), we may determine the nature of the stationary points, ω_s, as a function of t (for a fixed x) from the intersection of the $v_g(\omega)$ curve with the horizontal line whose ordinate is x/t. Once we know the behavior of ω_s and the amplitude, as a function of t, it is a simple matter to qualitatively estimate the time response using (2-29). An example of such a graphical analysis is contained in Figure 2-9.

We have already pointed out the frequency modulation within the envelope. This is a general result, not restricted to the waveguide. We need not "track" continuously the instantaneous stationary-point frequency $\omega_s(t)$ however, in order to construct the oscillations within the envelope. Once a stationary-point ω_s has been found for a particular time, t_0, (and fixed observation point x), it provides the correct oscillation frequency for one or more cycles subsequent to t_0. This follows from (2-23), according to the following arguments:

$$\frac{d}{dt} g(\omega_s(t)) = \frac{\partial \beta}{\partial \omega_s} \cdot \frac{d\omega_s}{dt} - \frac{t}{x}\frac{d\omega_s}{dt} - \frac{\omega_s(t)}{x}$$

$$= -\frac{\omega_s(t)}{x},$$

where we have used the fundamental result $\frac{\partial \beta}{\partial \omega_s} = t/x$. Thus, the second derivative of $g(\omega_s(t))$ with respect to time is given by

$$\frac{d^2}{dt^2} g(\omega_s(t)) = -\frac{1}{x}\frac{d\omega_s}{dt}.$$ Near $t = t_0$, therefore, we have

$$g[\omega_s(t)] \approx g(\omega_{s_0}) - (\omega_{s_0}/x)(t-t_0) - \frac{1}{x}\frac{d\omega_s}{dt}\Big|_{t=t_0}(t-t_0)^2/2,$$

where $\omega_{s_0} = \omega_s(t_0)$. The second term will dominate the third if

$$\omega_{s_0} \gg \frac{1}{2}\left|\frac{d\omega_s}{dt}\right|_{t=t_0}(t-t_0).$$ Over how many periods (at the frequency ω_{s_0}) is this true? If we let $t - t_0 = nT = 2\pi n/\omega_{s_0}$, where T is the period of oscillation at the frequency ω_{s_0} and n is an integer, then the above inequality requires that $n\pi\left|\frac{d\omega_s}{dt}\right|_{t=t_0} \ll \omega_{s_0}^2$. We can get a useful expression for $\frac{d\omega_s}{dt}$ by referring back to (2-23) and differentiating with respect to t:

$$\beta''(\omega_s)\frac{d\omega_s}{dt} - \frac{1}{x} = 0 \quad \text{or} \quad \frac{d\omega_s}{dt}\Big|_{t=t_0} = \frac{1}{x\,\beta''(\omega_{s_0})}.$$

Thus, we require that

(2-34)
$$\frac{n\pi}{x\left|\beta''(\omega_{s_0})\right|} \ll \omega_{s_0}^2 .$$

[illegible]
[illegible] with frequency ω_{s_0} . Because
$x\,|\beta''(\omega_{s_0})|$ is assumed large in order for the stationary-phase method to work, it follows that ω_{s_0} is the approximate oscillation frequency for n cycles, thereby rendering unnecessary the continuous tracking of the stationary point $\omega_s(t)$ within these periods. For example, as $t \to \infty$, $\beta''(\omega_s) \to (v_p t/x)^3 \to \infty$ in (2-25). Hence, the left-hand side of (2-34) approaches zero for any value of n, no matter how large. At the same time $\omega_s \to \omega_c$ as $t \to \infty$. Therefore, we conclude that ω_c is the frequency at which (2-29) oscillates <u>for arbitrarily large periods</u> as $t \to \infty$. This is another manifestation of our earlier result, (2-31), for the asymptotic frequency of the impulse response of the waveguide.

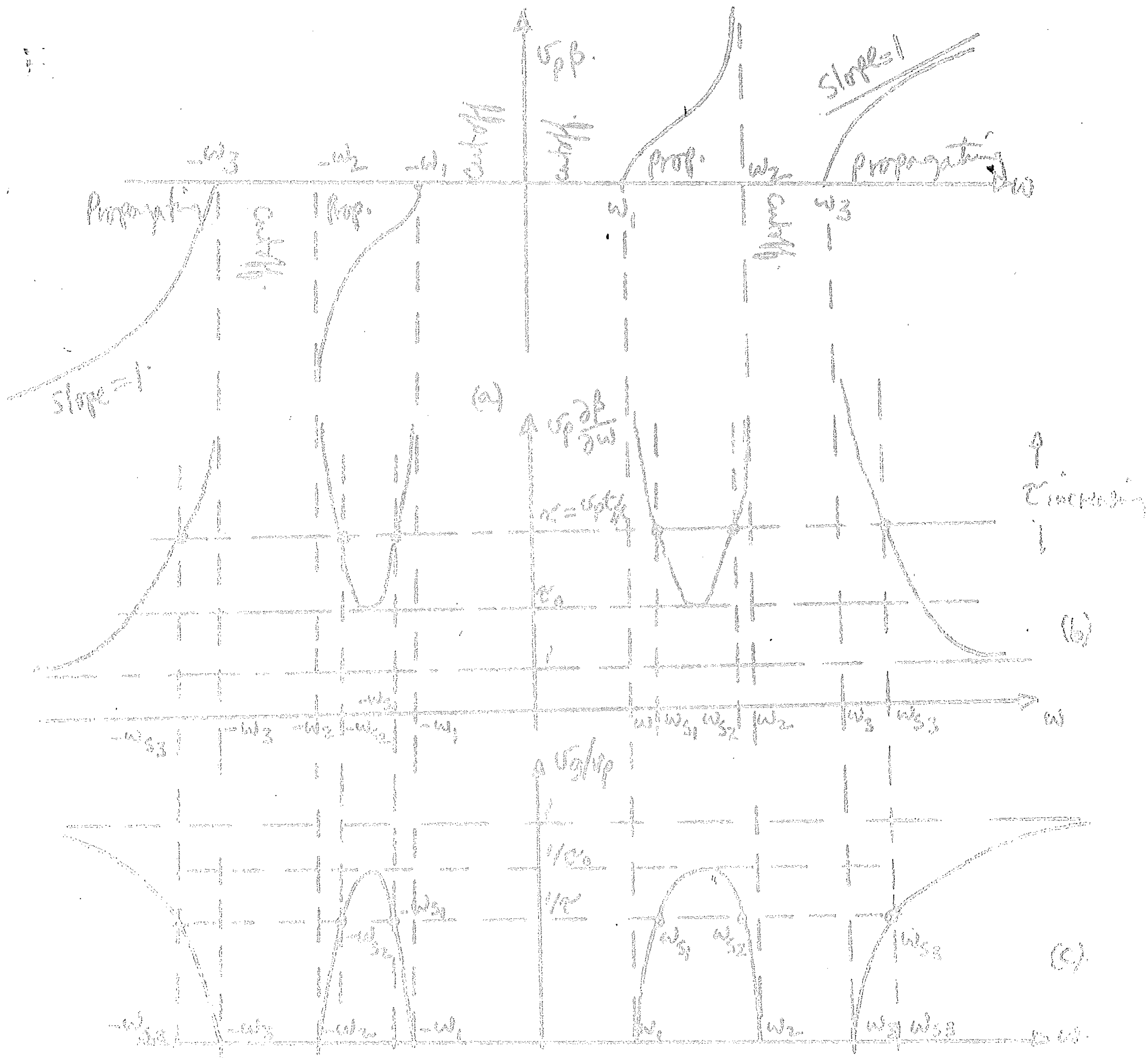


Figure 2-9. Graphical constructions for multi branched dispersion curves. (a) $v_p\beta(\omega)$ vs. ω . Note the existence of propagating and cut-off regions; (b) $v_p\partial\beta/\partial\omega$ vs ω ; (c) normalized group velocity v_g/v_p vs. ω. At the normalized time τ_0 the two stationary points $\omega_{s_1}, \omega_{s_2}$, coalesce.

that separate for $\tau \to \tau_0$. For $1 < \tau < \tau_0$, only the right-hand branch contributes a stationary point, ω_{s_3}, and any propagating disturbances. Note that at the earliest normalized time for which anything occurs ($\tau = 1$) the highest frequencies, ω_{s_3}, are present; $\omega_{s_3} \to \omega_3$, a cut-off frequency as $\tau \to \infty$. Frequencies at which $\beta(\omega) = 0$, such as ω_1, ω_3 are "cut-off" frequencies, while those at which $\beta(\omega) = \infty$, such as ω_2, are "resonance" frequencies. Note that the transient oscillations at long observation times take place at these frequencies.

There are other useful graphical schemes for depicting transient wave propagation; one such is "space-time rays." A ray is a straight line in space-time ($x - v_p t$ space) which satisfies (2-23). There is a ray for each center frequency, ω_s of a wave packet, or wave-group. The slope of the line is $v_p / v_g(\omega_s)$; to each ω_s there exists a $v_g(\omega_s)$, and, hence, a different line. Because $v_g(\omega_s) \leq v_p$ it follows that the rays lie in the sector between the $v_p t$-axis and a line of 45°; the smaller the group velocity, the more nearly is a ray parallel to the $v_p t$-axis. Rays corresponding to two different frequencies ω_s and $\omega_s + \Delta\omega_s$ are sketched in Figure 2-10. In two dimensional space-time, two rays form a ray tube.

$$q'(\omega_s) = 0 = \beta'(\omega_s) - t/x$$
$$\therefore x = v_g(\omega_s) t$$

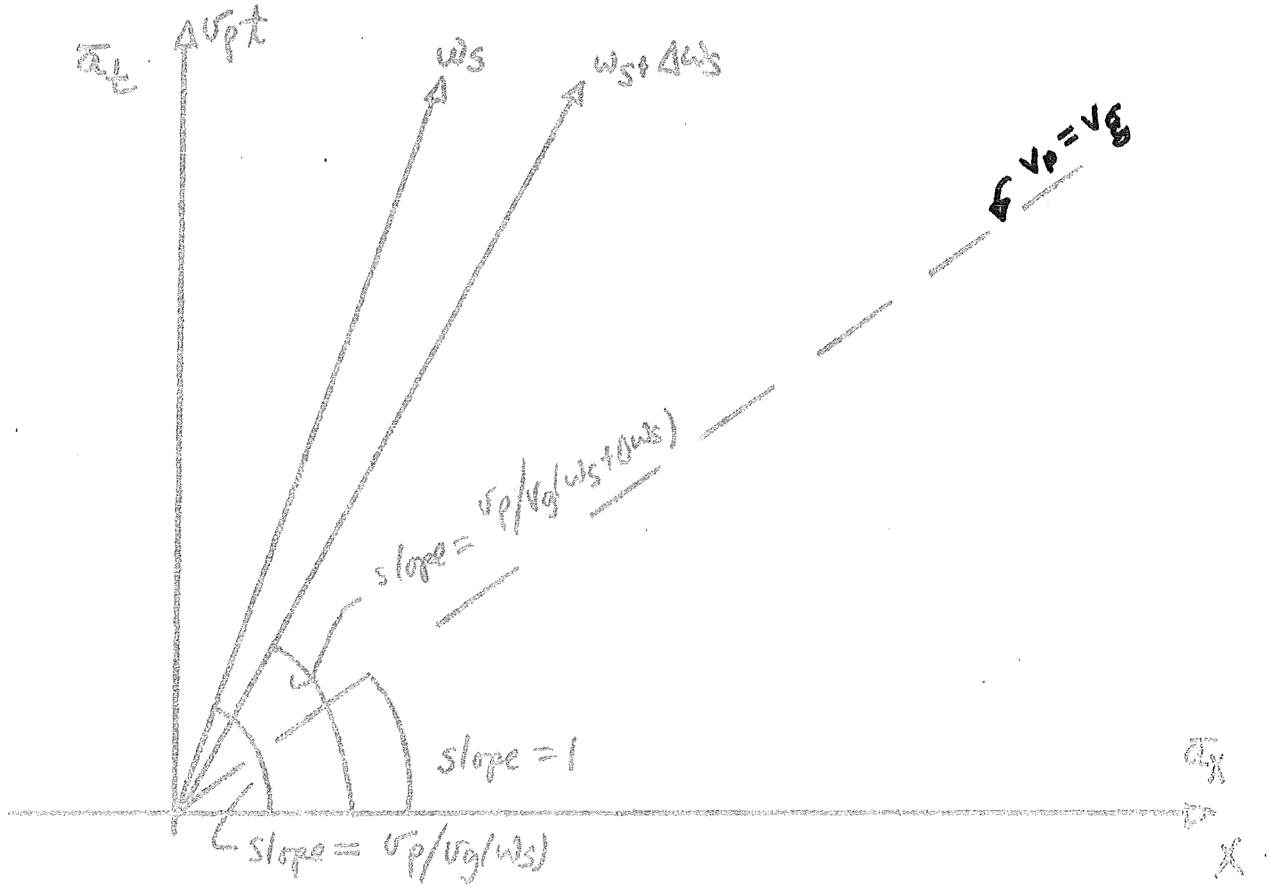


Figure 2-10. Rays corresponding to frequencies ω_s and $\omega_s + \Delta\omega_s$. Together they form a ray tube. The unit vectors $\bar{a}_x$ and $\bar{a}_t$ are used to make the $x - v_p t$ space a vector space, as described in the text.

EVANESENT-WAVE WHOSE ENERGY IS STORED (BELOW CUT-OFF FREQUENCY) [illegible] of frequency ω_g at [illegible] time, by moving parallel to the $v_p t$-axis ([illegible] keeping x fixed) one determines the frequency variation in the wave [illegible] reaching a fixed observation point x at times $t > x/v_g$. Finally, by moving parallel to the x-axis (keeping t fixed), one finds the frequency distribution of wave packets in space at any instant of time.

We can even obtain information about the variation of the amplitude of the transient oscillation from the ray diagram. In the physical coordinate space (x-space), the group of waves having center frequencies between ω_g and $\omega_g + \Delta\omega_g$, at time t_0, occupy the length interval Δx_0 (Figure 2-10), and the corresponding stored energy is proportional to $|U_{\omega_g}(x_0,t_0)|^2 \Delta x_0$. Since different wave constituents of the original group travel with different group velocities, the same packet (frequencies between ω_g and $\omega_g + \Delta\omega_g$), will spread out to occupy a length Δx at time t. The stored energy now becomes $|U_{\omega_g}(x,t)|^2 \Delta x$, where we use the same subscript as before to indicate that the wave packet consists of the same frequency range, ω_g to $\omega_g + \Delta\omega_g$. Upon invoking conservation of energy, it follows that

$$(2\text{-}35) \qquad |U_{\omega_g}(x,t)| = |U_{\omega_g}(x_0,t_0)| \sqrt{\frac{\Delta x_0}{\Delta x}} \,,$$

which states that the amplitude along a ray varies inversely as the square root of the segment, t = constant, lying between two rays. As this segment increases with time, the amplitude decreases (see Figure 2-10).

As we might have suspected, we can closely link the ray diagram of Figure 2-10 to the ω-β diagram of the dispersive transmission-line. In the two dimensional x-$v_p t$ space define a generalized group velocity vector $\bar{V}_g = \bar{a}_x v_g + \bar{a}_t v_p = \bar{a}_x \partial\omega/\partial\beta + \bar{a}_t v_p$,

where $\bar{a}_x$ and $\bar{a}_t$ are unit vectors along the x- and $v_p t$-axes, respectively; thus, the x-$v_p t$ space is a vector space. The slope of the generalized group velocity vector is the ratio of the $v_p t$- to x-components $v_p/v_g = v_p/\partial\omega/\partial\beta$, which indicates that $\bar{V}_g$ is parallel to the space-time ray (recall that the rays of Figure 2-10 satisfy $\frac{d(v_p t)}{dx} = v_p/v_g$).

Next let the ω-β space be a vector space with the negative [illegible]-axis along the $v_p t$-axis, having unit vector $\bar{a}_t$, and the positive β-axis along the x-axis, having unit vector $\bar{a}_x$. In the ω-β space we plot the dispersion curve $F(\omega, v_p\beta) = 0$. For example, for the waveguide $F(\omega, v_p\beta) = \omega^2 - v_p^2\beta^2 - \omega_c^2 = 0$, as is easily seen from (2-[illegible]). The normal vector to the dispersion curve is $\frac{1}{v_p}\frac{\partial F}{\partial\beta}\bar{a}_x - \frac{\partial F}{\partial\omega}\bar{a}_t$,

where the minus sign arises because of the assumed opposition of the [illegible] and [illegible] directions. The normal vector points in a direction whose slope is, again, the ratio of the $\bar{a}_t$ to $\bar{a}_x$ components $-\frac{\partial F}{\partial\omega} \Big/ \frac{\partial F}{v_p\partial\beta}$

$$= v_p \left(\frac{-\partial F/\partial\omega}{\partial F/\partial\beta} \right).$$

of implicit [illegible] form of calculus. Thus, we start with the implicit relation [illegible] $F(\omega, \beta)$ and from it can determine the explicit relation $\omega = \omega(\beta)$. Without actually doing this, however, we can calculate the derivative $d\omega/\partial\beta$:

$$\frac{\partial F}{\partial \omega}\cdot\frac{d\omega}{\partial\beta} + \frac{\partial F}{\partial\beta} = 0$$

or

$$\frac{d\omega}{d\beta} = v_g = -\frac{\partial F/\partial\beta}{\partial F/\partial\omega}.$$

Upon substituting this into the expression for the slope of the normal vector to the dispersion curve, we get v_p/v_g. Thus, we have shown that the normal to the dispersion curve is parallel to the generalized group velocity $\vec{v}_g$, which in turn is parallel to the space-time ray. Figure 2-11 illustrates these ideas for the waveguide. It should be evident from Figure 2-11 that the more weakly curved the dispersion curve, the narrower the corresponding ray tubes, and, hence, from (2-25) the stronger the amplitude of the transient waveform at a fixed point x as time varies. The curvature of a graph, of course, depends on the second derivative of the functional relationship between the two variables involved (here, ω and β). Thus, the greater the second derivative, the smaller the intensity. This accounts for the presence of the $[g''(\omega_s)]^{1/2}$ term in the denominator of (2-29). Obviously, the curvature is greatest at ω_c in Figure 2-11, whereas it approaches zero at high frequencies. Thus, the waveguide behaves as a high-pass filter, as we have said several times before.

We can use the ray concepts just developed, together with the $\omega-\beta$ diagram, to describe the reflection and transmission of waves from a non-dispersive (ideal) transmission-line connected to a dispersive one, or vice-versa. The important consideration is that the frequencies of the incident, reflected and transmitted wave packets must be identical; wavelengths, of course, may differ, depending on the parameters making up the transmission-lines.

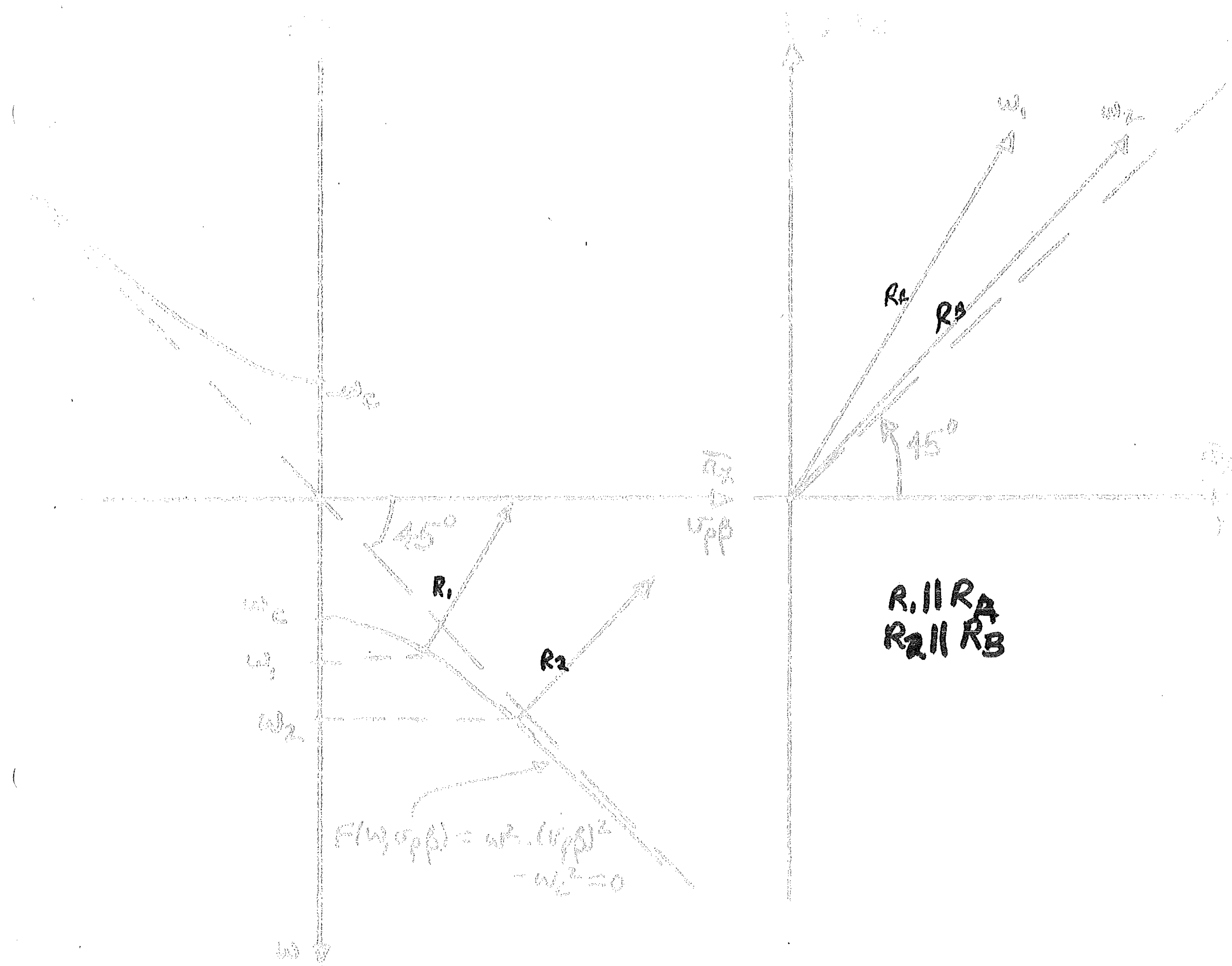


Figure 2-11. Construction of normals to dispersion curve $F(\omega, v_p\beta) = 0$ and corresponding space-time rays at two frequencies ω_1, ω_2. The rays are parallel to the normals. Only the (+z) propagating branch of the dispersion curve is shown.

Thus, Figure 2-12(a) illustrates the construction of the triplet of incident, reflected and transmitted space-time rays, all of which are characterized by having the same frequency. The non-dispersive curve is, of course, the straight line of 45°.

The case in which the incident wave, which has been excited by an impulse in time at z=0, is along the waveguide (or other dispersive, lossless line) is examined in Figure 2-12(b). Note that all of the frequency components in the incident wave propagate with the same speed in the non-dispersive line but that these different frequency components arrive at the interface $z = z_0$ at different observation times due to the dispersive effects of the waveguide.

... diverge for both the incident and reflected waves, it follows that the amplitude of the reflected waves decrease. This, again, is due to the dispersion present in the waveguide. On the other hand, the ray tubes in the non-dispersive guide maintain a constant cross section, meaning that there is no decrease in amplitude of the transmitted wave.

The converse situation of an incident wave in the non-dispersive line impinging upon the waveguide (Figure 2-12(c)), indicates that all frequencies in the incident pulse reach the interface $x=x_0$ simultaneously, ... the transmitted rays, which are excited at x_0 by the delayed impulse $\delta(t - x_0/v_p)$, exhibit dispersion, i.e., the rays are not parallel in space-time.

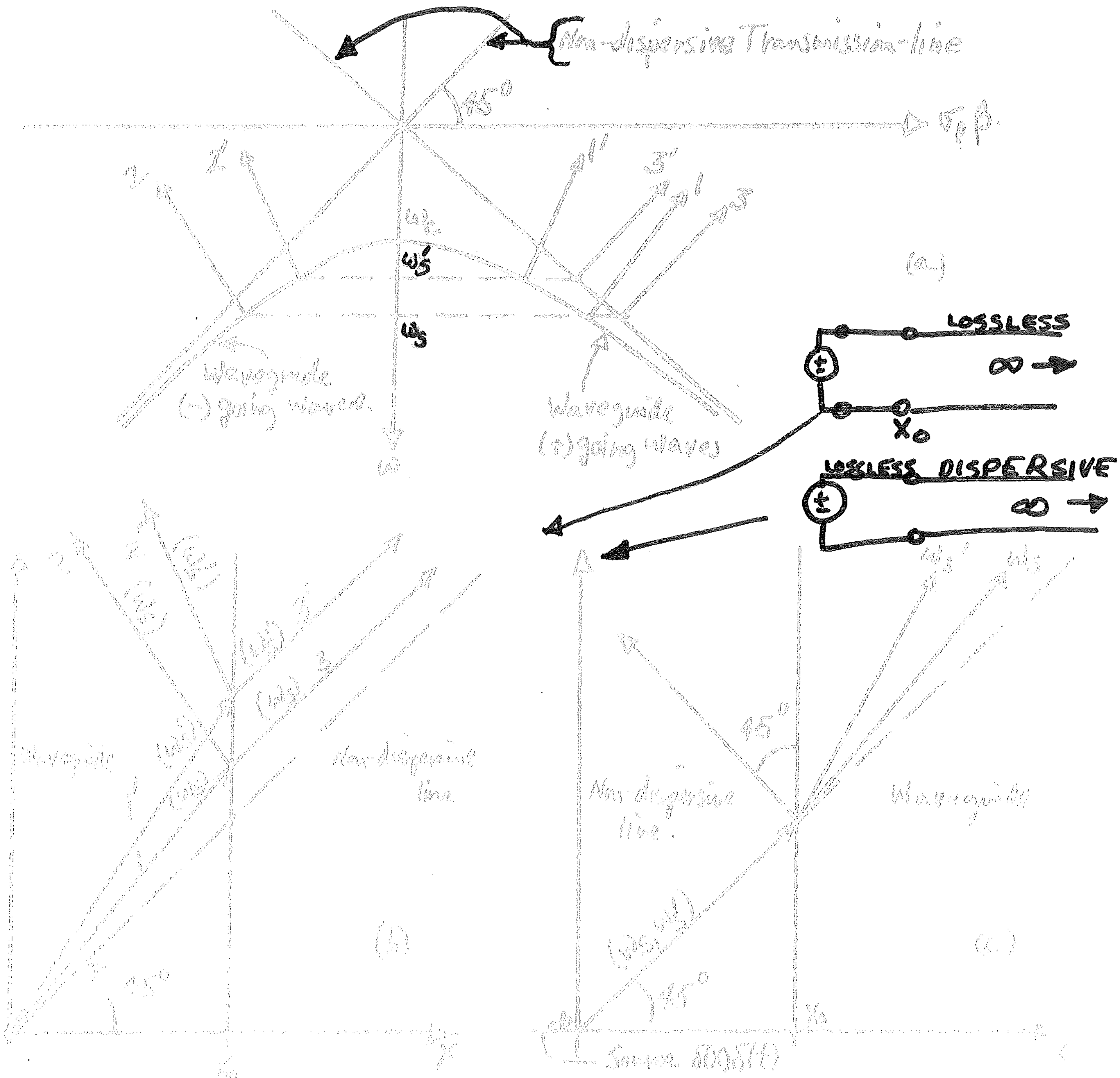
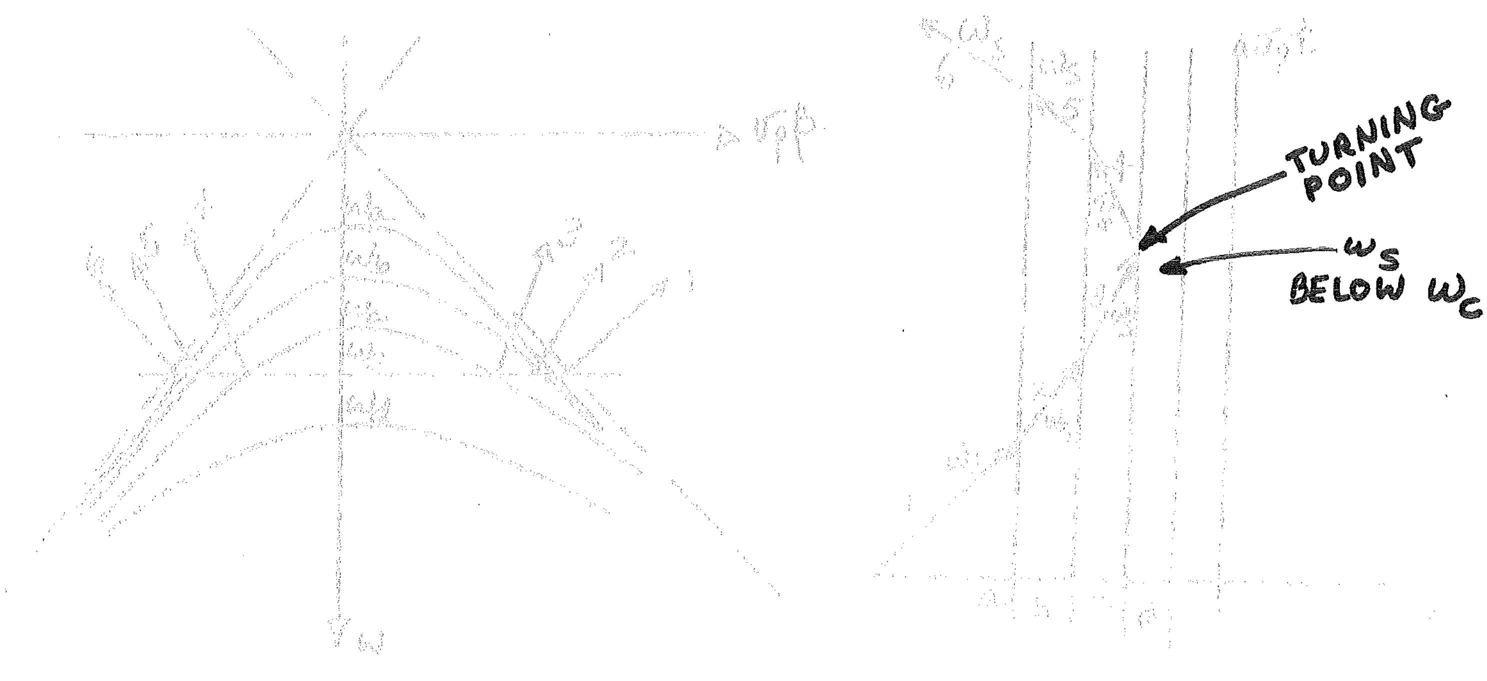


Figure 2-12. Reflection and transmission of wave packets. (a) Waveguide and ideal line dispersion curves; (b) ray plot for incidence from waveguide; (c) ray plot for incidence from ideal line

[illegible] the ideas of Figure 2-12 to [illegible] the case of [illegible] propagation in a waveguide for which the cut-off frequency, ω_c, is a slowly varying function of x. This could be the situation, perhaps, when the dimensions of the guide, or some other physical parameter, change slowly with x. Such wave propagating systems are called "spatially inhomogeneous". The "temporally inhomogeneous" case in which ω_c, for example, changes with time will be described later.

A family of dispersion curves for a spatially inhomogeneous waveguide and the corresponding space-time rays are shown in Figure 2-13. If the frequency of interest is below the cut-off frequency for curve 6, the 6 region of Figure 2-13(b) is a "turning point" for the wave; that is, there is no propagation beyond x=d. The actual situation is more complicated than that shown in Figure 2-13(b) because we have neglected the effects of multiple reflections from the various interfaces. If the system is weakly spatially inhomogeneous (as, for example, in a [illegible] instrument) the multiple reflections are not severe.



(a) (b)

Figure 2-13. Space-time rays in a spatially inhomogeneous waveguide. (a) Family of dispersion curves. (b) Space-time ray [illegible] trajectory. [illegible] corresponding [illegible]

[illegible]

In Sections 5 and 6 we described two approximate techniques which are useful, in their ranges of validity, for understanding transient pulse propagation in a dispersive media. We will now discuss a more general, and accurate, method which at once subsumes the two previous cases. It involves the use of complex variables and the inversion of the Laplace transform. We will apply it to the problem of finding the transient response of a waveguide to a sine wave switched on at $t = 0$.

Equation A-9 of Appendix A to the last chapter is the starting point:

$$v(x,t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} V(0,s)\exp\left[st - \frac{x}{v_p}\left(s^2+\omega_c^2\right)^{1/2}\right]ds, \qquad [illegible]$$

where

$$V(0,s) = \frac{\omega_0}{s^2+\omega_0^2} \qquad (2\text{-}37)$$

is the Laplace transform of the switched-on sine wave, $u_{-1}(t)\sin\omega_0 t$, of unit amplitude and angular frequency ω_0. The path of integration lies to the right of all singularities of the integrand. This requirement is satisfied by taking $\sigma_0 > 0$.

We will normalize all quantities to ω_0, thus:

$$\begin{aligned} \tau &= \omega_0 t \\ \eta &= \omega_0 x/v_p \\ \Omega &= \omega_c/\omega_0 \\ z &= s/\omega_0 = u + jv \end{aligned} \qquad (2\text{-}38)$$

Equation (2-36), in these variables, reads

$$v(\xi,\eta) = \frac{1}{2\pi j}\int_{u_0 - j\infty}^{u_0 + j\infty} \frac{\exp\left\{z\left[\xi\eta - \eta\left(1+\frac{\Omega^2}{z^2}\right)^{1/2}\right]\right\}}{z^2+1}\,dz, \qquad (2\text{-}39)$$

where we have used (2-37) and defined $\xi = \tau/\eta$. Our interest is in the asymptotic response $v(\xi,\eta)$ for large η.

This section requires a knowledge of complex function theory. [illegible] development in R. E. Haskell and C. T. Case, "Transient Signal Propagation in Lossless, Isotropic Plasmas," Volume [illegible], AFCRL 66-234([illegible]), June 1966, Air Force Cambridge Research Labs. [illegible] Hanscom Field, Bedford, Mass.

[illegible]

We have already shown in Example 3, Appendix A of the last chapter, that the response is null for $t < x/v_p$ (or $\tau < \eta$ or $\xi < 1$). This is the "causality condition" which states that no disturbance propagates faster than v_p. The conditions relating to the arrival of the wavefront, however, are now of interest. According to the initial-value theorem of Laplace transforms, the initial effects are governed by conditions as $z \to \infty$ (or $s \to \infty$). Put another way, it is the high frequencies that govern the initial step-response of a system. Accordingly, we expand the radical in the exponent in (2-39) in powers of Ω^2/z^2 keeping only terms of the first order. When this has been done, the exponent can be written as

$$z\left[\tau_\eta - \eta\left(1+\frac{\Omega^2}{z^2}\right)^{1/2}\right] = z(\tau-\eta) - \eta\frac{\Omega^2}{2z} = zr - \beta/z, \tag{2-40}$$

where $r = \tau - \eta$, $\beta = \eta\frac{\Omega^2}{2}$. Because we are going to use a large contour on which $|z^2| \gg 1$, we can neglect the unity in the denominator of (2-39), thereby rewriting (2-39), upon using (2-40), as

$$v(\xi,\eta) = \frac{1}{2\pi j}\int_{u_0-j\infty}^{u_0+j\infty} \frac{\exp(zr-\beta/z)}{z^2}\,dz, \tag{2-41}$$

For $\tau > \eta$ ($t > x/v_p$), or equivalently in (2-41), $r > 0$, the contour may be closed by an infinitely large semi-circle in the left-half plane. We have already argued that the contribution to the total closed contour integral by this semi-circle vanishes. In addition, because there are no singularities to the right of the straight line $u = u_0$, this line may be deformed into an infinitely large semi-circle lying in the right-half plane; the value of the integral along the ~~line $u = u_0$ which represents~~ (2-41). Hence, (2-41) will be replaced by the integral over the infinitely large circle $z = Re^{j\theta}$ [illegible] as $R \to \infty$, shown in Figure 2-14.

[illegible] could have gone around to the value of the integral along the [illegible] which is [illegible] (2-41).

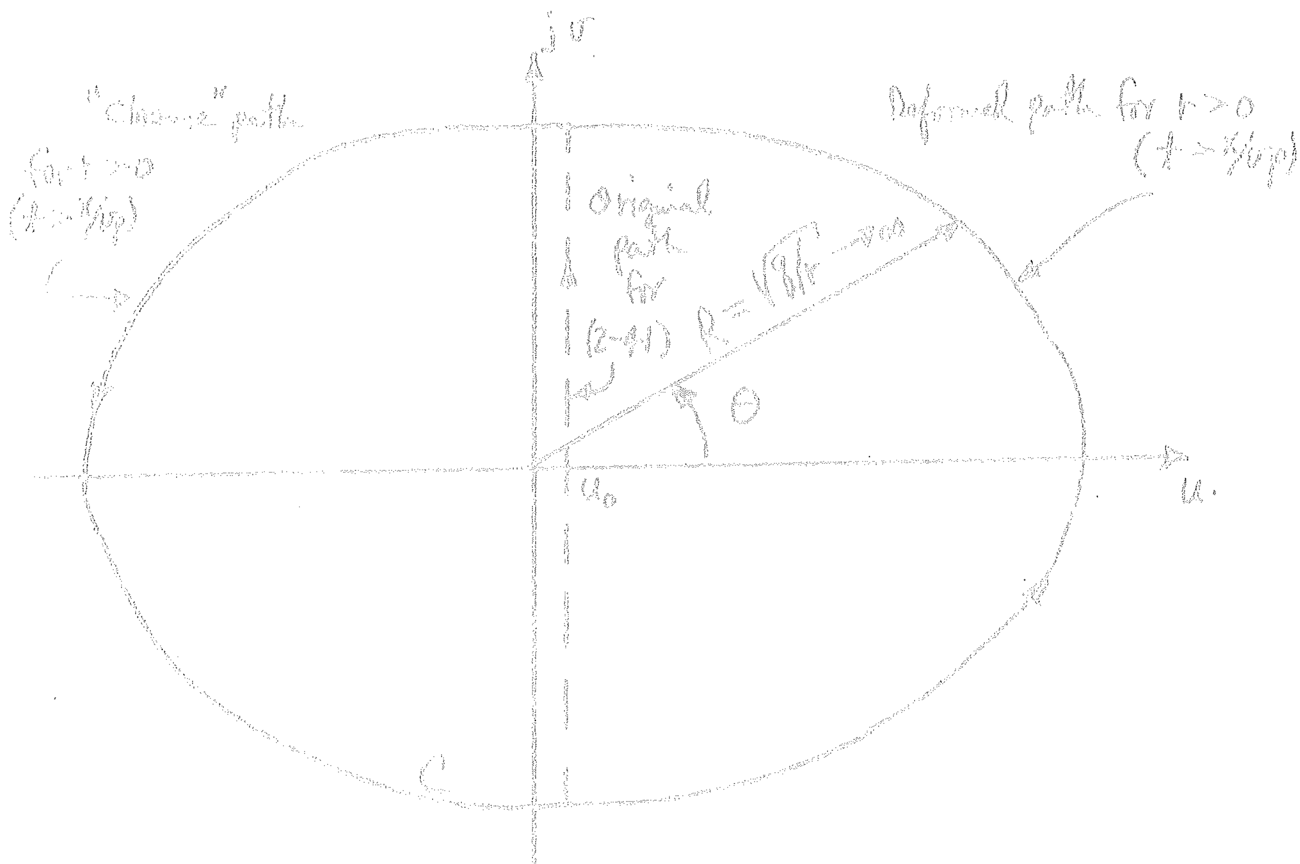


Figure 2-14. The infinite circular contour for evaluating the arrival conditions of the wavefront (i.e., $t \geq x/v_p$). The "closure" path contributes zero for $t > x/v_p$.

Over the circular path C we have $dz = jRe^{j\theta}d\theta$ and $\Omega t - \xi/z = j2\sqrt{\xi\tau}\sin\theta$. Hence (2-41), taken over C becomes

$$v(\xi,\eta) = \frac{1}{2\pi}\sqrt{\frac{\xi}{\eta}}\int_{\pi/2}^{5\pi/2} e^{j(2\sqrt{\xi\tau}\sin\theta - \theta)}\,d\theta, \qquad (2\text{-}42)$$

where we start the contour at $\theta = \pi/2$ instead of at $\theta = 0$. Upon making the further transformation $\theta = \varphi - \pi/2$, (2-42) is transformed to

$$v(\xi,\eta) = \frac{1}{2\pi}\sqrt{\frac{\xi}{\eta}}\int_{0}^{2\pi} e^{j(2\sqrt{\xi\tau}\cos\varphi + \varphi)}\,d\varphi \qquad (2\text{-}43)$$

This integral is one form of the Bessel function of the first kind and first order, $J_1(2\sqrt{\xi r})$. Hence,

$$(2\text{-}44)\qquad u(\xi,\eta) = \sqrt{\frac{r}{\xi}}\, J_1(2\sqrt{\xi r}) = \frac{\sqrt{2(\zeta-1)}}{\Omega}\, J_1\left(\sqrt{2\eta^2\Omega^2(\zeta-1)}\right), \quad \zeta \geq 1$$

This is the so-called Sommerfeld solution for the arrival of the wavefront. In order to determine its range of validity we recall that on the contour of integration $|z| = \sqrt{\xi/r}$. In addition, in the expansion of the exponent we assumed that $|z|^2 \gg \Omega^2$; also we required $|z|^2 \gg 1$. If the waveguide is not cut-off, then $\omega_0 \gg \omega_c$, so that $\Omega < 1$. Hence, the latter condition on $|z|^2$ is the more stringent and is equivalent to $\xi \gg r$. But $\xi = \eta\Omega^2/2$ and $r = \zeta - \eta$, so that the end result is the requirement $\eta(1+\Omega^2/2) \gg \zeta$, or in terms of the original variables

$$(2\text{-}45)\qquad x \gg \frac{v_p t}{\left(1+\frac{\omega_c^2}{2\omega_0^2}\right)}.$$

Thus, we must have x large; as t increases to the point where (2-45) is no longer valid the Sommerfeld solution, (2-44) breaks down. At this point we look for an approximate method for obtaining asymptotic solutions.

Saddle Points and Steepest Descent Paths

Asymptotic solutions of

$$(2\text{-}46)\qquad u(\zeta,\eta) = \frac{1}{2\pi j}\int_{a_0 - j\infty}^{a_0 + j\infty} F(z)\exp\{\eta\, w(z)\}\, dz,$$

of which (2-39) is the special case $F(z) = \frac{1}{z^2+1}$, $w(z) = z\left\{\zeta - \left(1+\frac{\Omega^2}{z^2}\right)^{1/2}\right\}$,

can be obtained by using the saddle-point method of integration. By "asymptotic" is meant that the solution will be valid for large η. Since η is the normalized distance $\frac{\omega_0 x}{v_p} = 2\pi x/\lambda_0$, this is equivalent to saying that the fixed observation point, x, is many wavelengths, $\lambda_0 = v_p/f_0$, away from the input signal.

The integral, $u(\zeta,\eta)$, is approximated most simply on a new contour, SDP (Steepest Descent Path), which passes through a saddle point in the z-plane, a saddle point being defined as that point (or points) at which $w'(z)$ the derivative of w, vanishes.

The SDP is chosen not only to pass through the saddle point but also to be such that $|\exp\{\eta w(z)\}|$ decreases most rapidly away from the saddle point. For such an SDP, then if F(z) is analytic on SDP, the principal contribution to $u(\zeta,\eta)$ for large values of η will come from the vicinity of the saddle point. If F(z) has a pole on the SDP there will be a residue contribution also.

We have $w(z) = R(u,v) + j\,I(u,v)$, $z = u + jv$,

where R, I, u and v are real. Then if we plot R and I versus u and v we get a "level map" showing the various mountains (poles) and valleys (zeros) and interconnecting passes. The stationary point, z_s, of $w(z)$ is determined by

$$\frac{dw}{dz} = \frac{\partial R}{\partial u} + j\frac{\partial I}{\partial u} = \frac{\partial I}{\partial v} - j\frac{\partial R}{\partial v} = 0, \quad z = z_s.$$

Thus $\frac{\partial R}{\partial u} = \frac{\partial I}{\partial v} = \frac{\partial I}{\partial u} = \frac{\partial R}{\partial v} = 0$ for $z = z_s$, and we see that, indeed, R and I, hence w, are stationary at z_s. It is not true, however, that the extremum of R or I is either a relative maximum or minimum, for we shall now show that the second derivative of the functions R(u,v), I(u,v) change sign as the extremum is approached along the u- and v-axes. The proof is easily effected by appealing to the Cauchy-Riemann conditions which are satisfied by the analytic function $w(z)$:

$$\frac{\partial R}{\partial u} = \frac{\partial I}{\partial v}, \quad \frac{\partial I}{\partial u} = -\frac{\partial R}{\partial v},$$

from which it follows that $\frac{\partial^2 R}{\partial u^2} = -\frac{\partial^2 R}{\partial v^2}$, $\frac{\partial^2 I}{\partial u^2} = -\frac{\partial^2 I}{\partial v^2}$.

Thus, if $\frac{\partial^2 R}{\partial u^2} < 0$, indicating a relative <u>maximum</u> with respect to <u>u</u>, then $\frac{\partial^2 R}{\partial v^2} > 0$, indicating a relative <u>minimum</u> with respect to <u>v</u>. Thus, z_s is neither a relative maximum or minimum for either R or I; it is a <u>saddle point</u> (Figure 2-15).

In order to determine the SDP through z_s we must choose that path along which R(u,v) changes most rapidly. This follows because $|\exp(mw(z))| = \exp(mR)$. Let dS denote an element of length along a path of integration. The rate-of-change of R(u,v) along this path is

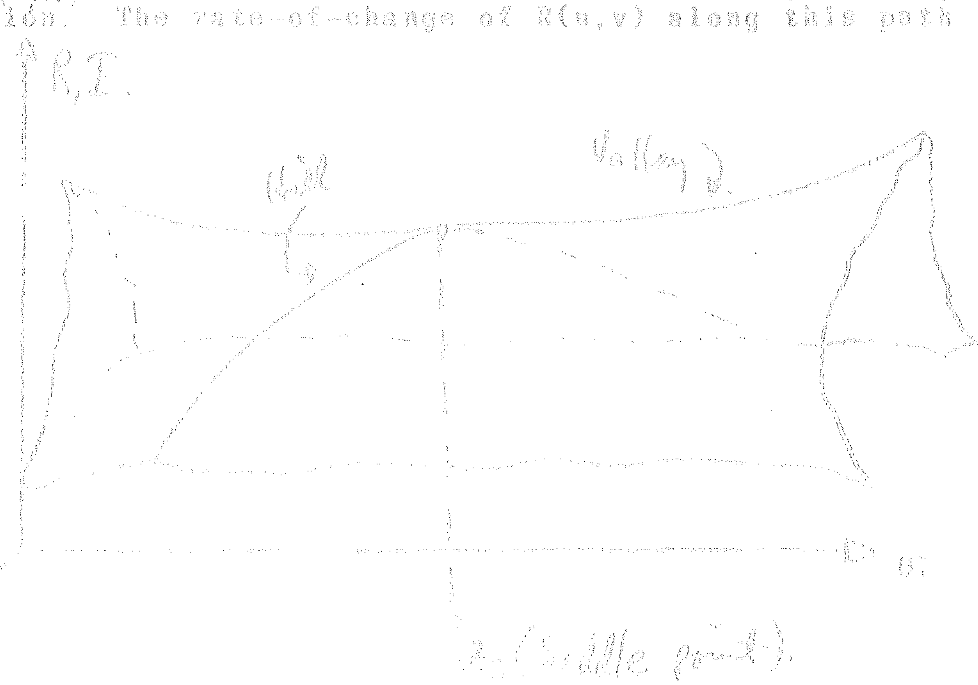


Fig. 2-15

Illustrating saddl[illegible] nature of the stationary point, z_s.

[illegible] $\dfrac{\partial v}{\partial s}$ [illegible]

[illegible] γ is the angle between the path element dS and the u-axis. [illegible] values of γ for which dR/dS are a maximum are given by

$$\frac{\partial}{\partial \gamma}\left(\frac{dR}{dS}\right) = 0 = -\frac{\partial R}{\partial u}\sin\gamma + \frac{\partial R}{\partial v}\cos\gamma = \frac{\partial R}{\partial v}\cdot\frac{du}{ds} - \frac{\partial R}{\partial u}\cdot\frac{dv}{ds}.$$

[illegible] the Cauchy-Riemann conditions this latter expression becomes

$$\frac{\partial I}{\partial v}\cdot\frac{dv}{ds} + \frac{\partial I}{\partial u}\cdot\frac{du}{ds} = \frac{dI}{ds},$$

which implies that the steepest [illegible] path, i.e., the path along which $R(u,v)$ changes most rapidly, is simply given by $I(u,v)$ = constant.

[illegible] us expand $w(z)$ in a Taylor series about z_s and keep only the second order derivative (the first derivative vanishes at z_s)

$$w(z) - w(z_s) = (R - R_s) + j(I - I_s) = \frac{1}{2}(z - z_s)^2\, w''(z_s),$$

where $R_s = R(z_s)$, $I_s = I(z_s)$. Making the substitutions $w''(z_s) = A e^{j\alpha}$, $(z - z_s) = r e^{j\theta}$, leads to

$$(R - R_s) + j(I - I_s) = \frac{1}{2} A r^2 e^{j(2\theta + \alpha)} = \frac{1}{2} A r^2 \left[\cos(2\theta + \alpha) + j\sin(2\theta + \alpha)\right].$$

[illegible], the steepest descent path given by $I = I_s$ implies that $2\theta + \alpha = n\pi$, $n = 0, 1, 2, \ldots$. Along this path(s)

$$R - R_s = \frac{1}{2} A r^2 \cos n\pi, \quad n = 0, 1, 2, \ldots.$$

Now if $n = 0, 2$ then $R > R_s$, which [illegible] that the path is one of steepest ascent away from the saddle [illegible]. Similarly $n = 1, 3$ correspond to paths of steepest descent.

In the case we are considering $w(z) = z\left[\xi - \left(1 + \left(\frac{a}{z}\right)^2\right)^{1/2}\right]$. [illegible] saddle points are obtained by setting $w'(z) = 0$

$$(9) \qquad z_s = \pm \frac{j a \xi}{(\xi^2 - 1)^{1/2}}.$$

[illegible], therefore, two complex-conjugate saddle points lying on the [illegible] and moving with time for fixed x (recall [illegible]).

[illegible] saddle points start out at [illegible] for $\xi = 1$ [illegible] [illegible] as $\xi \to \infty$ [illegible]. Thus, because [illegible] there will be [illegible] of time (or ξ) at which the poles located at [illegible] will be [illegible]. The value of ξ corresponding to capture is

$$[illegible] \quad \xi_c = \frac{1}{(1 - [illegible]^2)^{1/2}}.$$

$$w''(z_s) = -\Omega^2 z_s \left[1+(\Omega/z_s)^2\right]^{-3/2},$$

At the saddle point

$$z_s^+ = j\frac{\Omega\xi}{(\xi^2-1)^{1/2}}, \quad w''(z_s^+) = \frac{(\xi^2-1)^{3/2}}{\Omega} e^{-j\pi/2},$$

and at

$$z_s^- = -j\frac{\Omega\xi}{(\xi^2-1)^{1/2}}, \quad w''(z_s^-) = \frac{(\xi^2-1)^{3/2}}{\Omega} e^{j\pi/2}.$$

Therefore, the lines of steepest descent through z_s^+ satisfy $\theta_s = \frac{3\pi}{4}$ $(n=1)$ and $\frac{7\pi}{4}$ $(n=3)$ and through z_s^-, $\theta_s = \frac{\pi}{4}$ $(n=1)$, $\frac{5\pi}{4}$ $(n=3)$.

The SDP, saddle-points, valleys, poles, branch-points and cuts are shown in Figure 2-15. The deformed path is SDP in the vicinity of the saddle points $\pm jz_s$. Otherwise it remains in the valley region and runs parallel to the $j\omega$ axis.

[illegible]

... the ... [illegible]

[illegible]

where

[illegible]

Therefore the ... [illegible] ... change ... [illegible]

... and ... [illegible] ... through ... [illegible]

[illegible]

The SDP, saddle-points, ... poles, ... and cuts are shown in Figure 2-1?. The ... pole ... SDP in the vicinity of ... stable pole ... Otherwise ... in the ... region ... runs parallel to the ... axis.

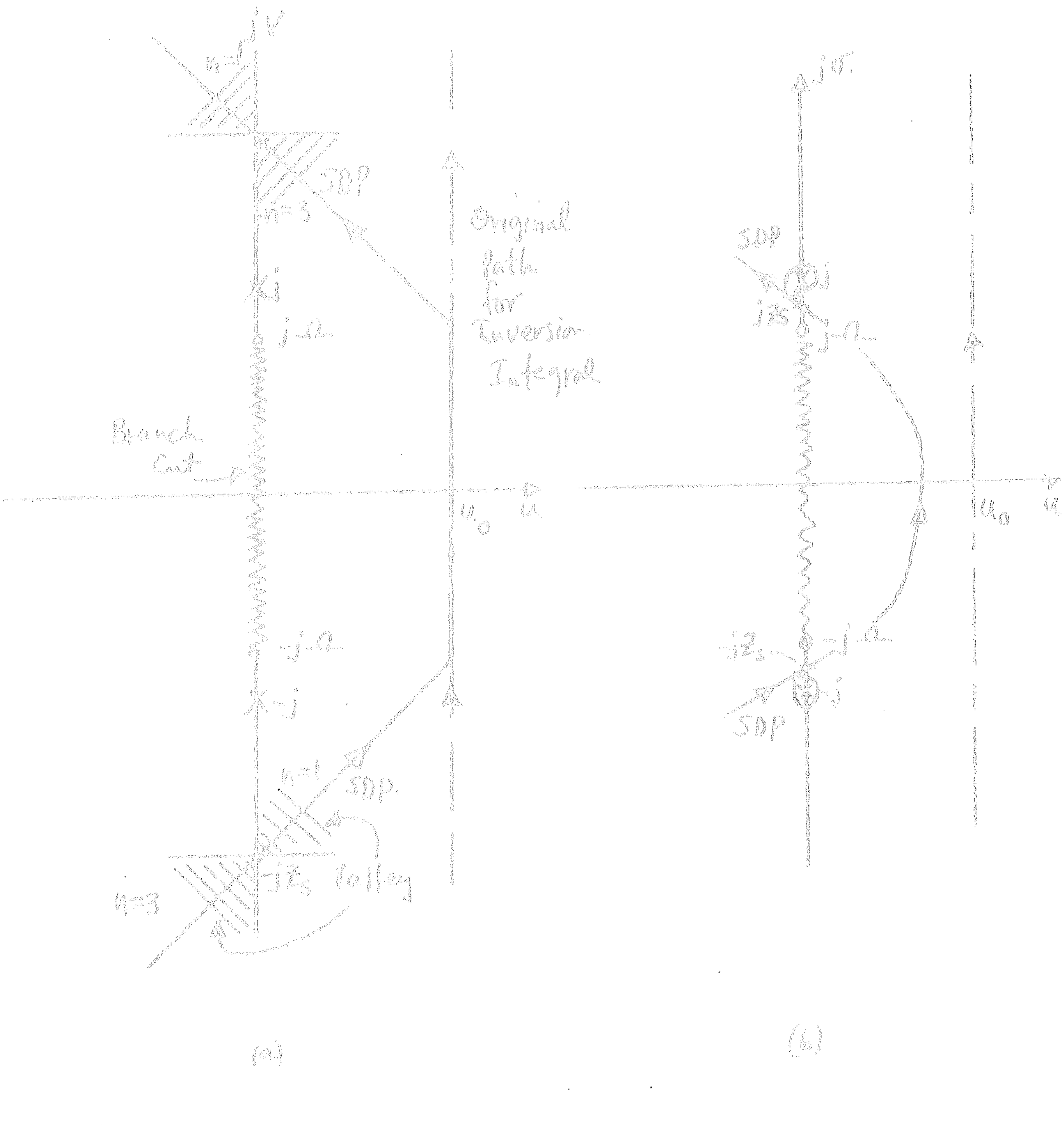


Figure 2-16. The deformed and SDP paths before (a) and after (b) capture of the poles. The deformed path remains in the valley regions away from the stationary point and runs parallel to the original path

C. Asymptotic Evaluation of $U(\xi,\eta)$ when $z_s \neq 1$.

When z_s is not in the vicinity of the poles (either above or below the poles, Figure 2-16(a) and (b), respectively), then the major contribution to the integral comes from the saddle points. It should be observed, however, that in the case of Figure 2-16(b), there will be a residue contribution because the path of integration encloses a pole.

We expand F(z) and $w(z)$ of (2-46) in a Taylor series about the stationary point z_s

$$F(z) = F(z_s) + F'(z_s)(z-z_s) + \frac{F''(z_s)}{2}(z-z_s)^2 + \cdots \tag{2-51}$$

$$w(z) = w(z_s) + w'(z_s)(z-z_s) + \frac{w''(z_s)}{2}(z-z_s)^2 + \cdots$$

$$\simeq w(z_s) + \frac{w''(z_s)}{2}(z-z_s)^2$$

where $w'(z_s) = 0$ by definition of the stationary point. Note that it is reasonable to expand F(z) in a Taylor series about z_s when z_s is away from the pole of F(z), for then F(z) is analytic.

Deforming the original contour, $u = u_0$, into the SDP and introducing (2-51) into (2-46) yields

$$U(\xi,\eta) \simeq \frac{1}{2\pi j}\int_{SDP}\left(\sum_{n=0}^{\infty}\frac{F^{(n)}(z_s)}{n!}(z-z_s)^n\right)e^{\eta w(z_s)}\cdot e^{\eta w''(z_s)\frac{(z-z_s)^2}{2}}\,dz, \tag{2-52}$$

Because the major contribution to this integral comes from the SDP in the vicinity of $z = z_s$, we introduce the transformation

$$z - z_s^+ = \ell e^{-j\pi/4}, \quad dz = e^{-j\pi/4}\,d\ell, \quad -\varepsilon \le \ell \le \varepsilon \tag{2-53}$$

$$z - z_s^- = \ell e^{j\pi/4}, \quad dz = e^{j\pi/4}\,d\ell, \quad -\varepsilon \le \ell \le \varepsilon,$$

where ε is some small number. This transformation deforms the path to a segment of a straight line passing through both saddle points.

Upon substituting (2-53) into the nth integral of (2-52), we are led to consider

$$\frac{1}{2\pi j}\int_{SDP}(z-z_s)^n e^{\eta w''(z_s)\frac{(z-z_s)^2}{2}}\,dz \simeq \tag{2-54}$$

$$\simeq \frac{e^{-j(n+1)\pi/4}}{2\pi j}\int_{+\varepsilon}^{-\varepsilon}\ell^n e^{-\eta\left\{\frac{(\xi^2-1)^{1/2}}{2a}\right\}\ell^2}\,d\ell$$

$$+ \frac{e^{j(n+1)\pi/4}}{2\pi j}\int_{-\varepsilon}^{+\varepsilon}\ell^n e^{-\eta\left\{\frac{(\xi^2-1)^{1/2}}{2a}\right\}\ell^2}\,d\ell.$$

In arriving at the final form of (2-54) we summed over both saddle points $z_s^{\pm} = \pm j\Omega\xi/(\xi^2-1)^{1/2}$ and have used the expressions

$$w''(z_s^{\pm}) = \mp j\frac{(\xi^2-1)^{3/2}}{\Omega}.$$

In addition, according to (2-53) and Figure 2-16, the integration over the saddle point z_s^{+} goes into the order of 1 from $+\xi$ to $-\xi$.

According to our assumption of large η, the limits of integration of (2-54) may be extended to $\pm\infty$. Therefore, (2-52), together with (2-54), becomes

$$v(\xi,\eta) = \sum_{n=0}^{\infty}\left\{\frac{e^{j(n+1)\pi/4}}{2\pi j}\cdot\frac{F^{(n)}(z_s^{-})}{n!}e^{\eta w(z_s^{-})}\right.$$

(2-55)

$$\left.-\frac{e^{-j(n+1)\pi/4}}{2\pi j}\cdot\frac{F^{(n)}(z_s^{+})}{n!}e^{\eta w(z_s^{+})}\right\}\int_{-\infty}^{\infty}\varrho^n e^{-\eta\left\{\frac{(\xi^2-1)^{3/2}}{2\Omega}\right\}\varrho^2}d\varrho.$$

The integral appearing on the right-hand side vanishes for n-odd because the integrand is then an odd function and the limits of integration are symmetrical. For n-even, the integral can be written in terms of the gamma function, $\Gamma(a)$, defined by

$$\Gamma(a) = \int_0^{\infty} e^{-\rho}\rho^{(a-1)}d\rho, \quad Re(a) > 0,$$

with the result that

$$\int_{-\infty}^{\infty}\varrho^n e^{-\eta\left\{\frac{(\xi^2-1)^{3/2}}{2\Omega}\right\}\varrho^2}d\varrho = \Gamma\left(\frac{n+1}{2}\right)\left(\frac{2\Omega}{\eta(\xi^2-1)^{3/2}}\right)^{\left(\frac{n+1}{2}\right)}, \quad n \text{ even}$$

$$= 0, \quad n \text{ odd}.$$

The values of $\Gamma(m+1/2)$, $m = 0, 1, 2, \cdots$ are easily inferred from the recursion formula $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Finally, upon letting $n = 2m$, $m = 0, 1, 2, \cdots$ and substituting the expressions for $w(z_s^{\pm})$, $w(z_s^{\pm}) = \pm j\Omega(\xi^2-1)^{1/2}$, into (2-55) we arrive at our asymptotic expansion

(2-56) $$v(\xi,\eta) = \sum_{m=0}^{\infty}\Gamma\left(m+\frac{1}{2}\right)\left(\frac{2\Omega}{\eta(\xi^2-1)^{3/2}}\right)^{(m+1/2)}\frac{A^{2m}(\xi)}{\pi(2m)!}\cdot\sin\left[(2m+1)\frac{\pi}{4}\right.$$

$$\left.-\eta\Omega(\xi^2-1)^{1/2}+\varphi_{2m}(\xi)\right],$$

[illegible]
and have observed that $f^{(n)}(z_s^+)$ is the complex conjugate of $F^{(n)}(z_s^-)$.

The asymptotic nature (for large η) of the result is manifested by the appearance of the decaying terms $\left(\frac{1}{\eta}\right)^{(n+1/2)}$.

Often the lowest order (n=0) term is sufficient. Assuming this, we have

(2-57)
$$\psi(\eta) \approx \frac{\Gamma\left(\frac{1}{2}\right)}{\pi}\sqrt{\frac{2a}{\eta}}\,\frac{1}{(\xi^2-1)^{1/4}}\,F(z_s)\sin\left[\frac{\pi}{4}-\eta a(\xi^2-1)^{1/2}\right]$$
$$=\sqrt{\frac{2a}{\pi\eta}}\;\frac{(\xi^2-1)^{1/4}}{\xi^2(1-a^2)-1}\,\cos\left[\eta a(\xi^2-1)^{1/2}+\frac{\pi}{4}\right],$$

where we have used $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $F(z_s^-)=\frac{1}{1+z_s^{-2}}=\frac{(\xi^2-1)}{\xi^2(1-a^2)-1}$

and $\sin\left(\frac{\pi}{4}-\theta\right)=\cos\left(-\frac{\pi}{4}-\theta\right)=\cos\left(\theta+\frac{\pi}{4}\right)$.

The solution (2-57), called the <u>anterior</u> (or "precursor") transient, is valid for times long before the saddle point moves into the vicinity of a pole. Since, by (2-50), $\xi=\xi_c=\frac{1}{(1-a^2)^{1/2}}$ is the normalized time at which the saddle point crosses the pole, the anterior solution is valid for

(2-58)
$$1<\xi<<\frac{1}{(1-a^2)^{1/2}}.$$

The solution for which $\xi\approx 1$ was the Sommerfeld solution, and that for which $\xi\approx\frac{1}{(1-a^2)^{1/2}}$ is the build-up of the main signal.

If the saddle point [illegible] as shown in Figure 2-16(b) so that the poles at $\pm i$ are enclosed within the contour, it is necessary to add residue contributions [illegible]. The residues of the integrand of (2-46) are

(2-59)
$$[illegible]$$
$$[illegible]$$

Thus, the contribution to (2-46) due to the pole singularities at $z = \pm j$ is the sum of the two terms of (2-59) and is given by $\sin[\xi-(1-\Omega^2)^{1/2}]$.

The total solution when the saddle-point is well away from the pole, but between the poles and branch points located at $\pm j\Omega$, is

(2-60)
$$v(\xi,\eta) = \sqrt{\frac{2\Omega}{\pi\eta}}\,\frac{(\xi^2-1)^{1/4}}{\xi^2(1-\Omega^2)-1}\cos\left[\eta\Omega(\xi^2-1)^{1/2}+\frac{\pi}{4}\right] + \sin\eta\left[\xi-(1-\Omega^2)^{1/2}\right], \quad \frac{1}{(1-\Omega^2)^{1/2}} \ll \xi.$$

We can express (2-60) in terms of x and t by making use of the definitions (2-38)! When this is done, we see that the amplitude of the first term decays as $1/\xi^{3/2} = (x/v_p t)^{3/2}$, and the phase varies as

$$\eta\Omega(\xi^2-1)^{1/2}+\pi/4 = \left(\omega_c^2 t^2 - \omega_c^2\frac{x^2}{v_p^2}\right)^{1/2}+\pi/4 .$$

Hence, the frequency of the first term is given by the time derivative of the phase: $f(t) = \omega_c t/(t^2 - x^2/v_p^2)^{1/2}$.

Because the frequency varies with time, this term is said to be frequency modulated, with an asymptotic frequency $f(t) \to \omega_c$ as $t \to \infty$.

The second term is the usual sinusoidal, steady-state (or forced) response $\sin\omega_0\left(t - \frac{x}{v_p}\left(1-\left(\frac{\omega_c}{\omega_0}\right)^2\right)^{1/2}\right)$. These results should be compared with Example 3, Appendix A of the last chapter.

The solution, (2-60), is called the <u>posterior</u> solution, occurring as it does for $t \to \infty$.

D. <u>Asymptotic Evaluation of $v(\xi,\eta)$ when $z_s \approx 1$</u> (Main Signal Build-up).

The remaining region of interest is when the saddle-point is in the vicinity of a pole, for then the expansion of (2-51) is no longer generally usable because F is not analytic at the saddle-point.

Let us, therefore, start by writing (2-46) as

(2-61)
$$v(\xi,\eta) = \frac{1}{4\pi}\left\{\int_{u_0-j\infty}^{u_0+j\infty}\frac{\exp[\eta(\xi z-(z^2+\Omega^2)^{1/2})]}{z+j}\,dz - \int_{u_0-j\infty}^{u_0+j\infty}\frac{\exp[\eta(\xi z-(z^2+\Omega^2)^{1/2})]}{z-j}\,dz\right\}.$$

this, in turn, can be written

$$(2\text{-}62)\quad u(\xi,\eta) = \frac{1}{2\pi}\left\{ \int_{u_0}^{u_0+j\infty} \frac{\exp[\eta w(z)]}{z+j}\,dz - \int_{u_0-j\infty}^{u_0} \frac{\exp[\eta w(z)]}{z-j}\,dz \right.$$
$$\left. + \int_{u_0-j\infty}^{u_0} \frac{\exp[\eta w(z)]}{z+j}\,dz - \int_{u_0}^{u_0+j\infty} \frac{\exp[\eta w(z)]}{z-j}\,dz \right\}.$$

Upon deformation of the paths of integration of these four integrals so that the paths pass through the saddle points (2-49), it is clear that the paths of the first two integrals will pass, respectively through $\frac{j\Omega\xi}{(\xi^2-1)^{1/2}}$, $\frac{-j\Omega\xi}{(\xi^2-1)^{1/2}}$.
Along these two deformed contours the integrands of the first two integrals are regular, there being poles located, respectively, at $z = -j$ and $z = +j$.

Because these integrands are regular, the theory of Subsection C is applicable to the first two integrals, with the result that

$$(2\text{-}63)\quad \int_{u_0}^{u_0+j\infty} \frac{\exp[\eta w(z)]}{z+j}\,dz = j\sqrt{2\pi}\,\frac{(\Omega/\eta)^{1/2}}{(\xi^2-1)^{3/4}}\left(\frac{-j}{1+\Omega\xi/(\xi^2-1)^{1/2}}\right)\exp\left\{j\left(\Omega\eta(\xi^2-1)^{1/2}+\frac{\pi}{4}\right)\right\}$$

$$\int_{u_0-j\infty}^{u_0} \frac{\exp[\eta w(z)]}{z-j}\,dz = j\,\frac{\sqrt{2\pi}\,(\Omega/\eta)^{1/2}}{(\xi^2-1)^{3/4}}\left(\frac{j}{1+\Omega\xi/(\xi^2-1)^{1/2}}\right)\exp\left\{-j\left(\Omega\eta(\xi^2-1)^{1/2}+\frac{\pi}{4}\right)\right\}.$$

Consider now an integral of the form of the fourth one in (2-62)

$$(2\text{-}64)\quad I = \int_{SDP} \frac{\exp[\eta w(z)]}{z-z_p}\,dz,$$

where SDP is the segment of steepest descent path in the vicinity of and through the saddle point. We proceed as before and expand $w(z)$ about the saddle-point, which we take to be z_s, since that will be the relevant saddle point for the fourth integral and retain only the first two terms of [illegible] in (2-51)

$$(2\text{-}65)\qquad I = \exp[\eta w(z_s^+)] \int_{S_{DP}} \frac{\exp\left[\frac{\eta}{2} w''(z_s^+)(z - z_s^+)^2\right]}{z - z_p}\, dz .$$

Upon making the first transformation of (2-53), and using the fact that $w''(z_s^+) = -\frac{i(\xi^2-1)^{3/2}}{\Omega}$, we obtain

$$(2\text{-}66)\qquad I = -\exp[\eta w(z_s^+)] \int_{-\varepsilon}^{\varepsilon} \frac{\exp\left[-\eta\left(\frac{(\xi^2-1)^{3/2}}{2\Omega}\right)\ell^2\right]}{\ell+\beta}\, d\ell$$

$$\approx -\exp[\eta w(z_s^+)] \int_{-\infty}^{\infty} \frac{\exp\left[-\eta \frac{(\xi^2-1)^{3/2}}{2\Omega}\ell^2\right]}{\ell+\beta}\, d\ell ,$$

where $\beta = (z_s^+ - z_p)e^{i\pi/4}$ and we have removed the limits of integration to $\pm\infty$ under the assumption that η is large.

We now multiply and divide the integrand of (2-66) by $\ell - \beta$

$$(2\text{-}67)\qquad I = -\exp[\eta w(z_s^+)]\left\{\int_{-\infty}^{\infty} \frac{\ell \exp\left[-\eta \frac{(\xi^2-1)^{3/2}}{2\Omega}\right]\ell^2}{\ell^2-\beta^2}\, d\ell - \int_{-\infty}^{\infty} \frac{\beta \exp\left[-\eta \frac{(\xi^2-1)^{3/2}}{2\Omega}\right]\ell^2}{\ell^2-\beta^2}\, d\ell\right\}.$$

The first integral vanishes because the integrand is odd and the limits of integration are even; the second is evaluated by writing $\frac{1}{\ell^2-\beta^2} = \int_0^\infty e^{\gamma(\ell^2-\beta^2)}\, d\gamma$. Then the expression (2-67) becomes, upon inverting the order of integration

$$I = \exp[\eta w(z_s^+)]\,\beta \int_0^\infty e^{-\gamma\beta^2}\left\{\int_{-\infty}^{\infty} \exp[-(\alpha+\gamma)\ell^2]\, d\ell\right\} d\gamma ,$$

where $\alpha = \frac{\eta(\xi^2-1)^{3/2}}{2\Omega}$. The integral over ℓ is [illegible] in any table of integrals to be [illegible]. Therefore, if we make the change of variables $(a+q)^{1/2} = y/|\beta|$, (2-68) can be rewritten

(2-69) $$I = \exp\{q\,w(z_s^t)\}\cdot 2\sqrt{\pi}\, e^{-a\beta^2} \frac{\beta}{|\beta|} \int_{|\beta|\sqrt{a}}^{\infty} \exp\left(y\frac{\beta}{|\beta|}\right)^2 dy .$$

For the problem at hand, $z_s^t = j\frac{\Omega\xi}{(\xi^2-1)^{1/2}}$, $z_p = j$ so that $\beta = j\left(\frac{\Omega\xi}{(\xi^2-1)^{1/2}}-1\right)e^{j\pi/4}$, for $\xi<\xi_c$, where $\xi_c = \frac{1}{(1-\Omega^2)^{1/2}}$ is the critical value of ξ for which the saddle point cross the poles. $\left(\frac{\Omega\xi}{(\xi^2-1)^{1/2}}-1\right) > 0$, so that $\beta/|\beta| = je^{j\pi/4}$. Now define $B = |\beta|\sqrt{a}$; then for $\xi<\xi_c$ the following hold: $B = \left(\frac{\Omega\xi}{(\xi^2-1)^{1/2}}-1\right)\left(\frac{\eta}{2a}\right)^{1/2}(\xi^2-1)^{3/4}$, $a\beta^2 = -jB^2$.

In addition, $w(z_s^t) = j\Omega(\xi^2-1)^{1/2}$, so that (2-69) becomes

(2-70) $$I = j\,2\sqrt{\pi}\exp\left\{j\left[\eta\Omega(\xi^2-1)^{1/2}+B^2+\frac{\pi}{4}\right]\right\}\int_{B}^{\infty} e^{-jy^2}dy, \quad \xi<\xi_c .$$

For $\xi>\xi_c$, $\left(\frac{\Omega\xi}{(\xi^2-1)^{1/2}}-1\right)<0$, so that $\frac{\beta}{|\beta|} = -je^{j\pi/4}$,

$a\beta^2 = -jB^2$.

Then, when we take into account the fact that the pole at [illegible] has been enclosed by the SDP, the expression for I becomes

$$\text{(2-71)} \qquad I = -j^{1/2}\sqrt{\pi}\, \exp\left\{ j\left[\eta\, \Omega (\xi^2-1)^{1/2} + \delta + \frac{\pi}{4} \right] \right\} \int_{R}^{\infty} e^{-j t^2}\, dt$$

$$+\, j\pi \exp\left\{ j\eta\left(\xi - (1+\Omega^2)^{1/2}\right) \right\}, \qquad \xi > \xi_c .$$

Here R is the negative of what it is in (2-70).

The fourth integral of (2-62) has values which are just the negative of (2-70) and (2-71), whereas the third integral is just the complex conjugate of the fourth.

When we combine (2-63), together with the results for the third and fourth integrals as stated above, (2-62) yields

$$\sigma(\xi,\eta) = \frac{1}{\sqrt{2\pi}}\left(\frac{\Omega}{\eta}\right)^{1/2} \frac{\cos\left[\Omega\eta(\xi^2-1)^{1/2} + \pi/4\right]}{(\xi^2-1)^{3/4}\left[1 + \dfrac{\Omega\xi}{(\xi^2-1)^{1/2}}\right]} + \frac{1}{2}\left\{ \left[1 + C(v) + S(v)\right]\sin\delta \right.$$

(2-72)

$$\left. + \left[C(v) - S(v)\right]\cos\delta \right\}, \quad \xi < \xi_c = \frac{1}{(1+\Omega^2)^{1/2}}$$

$$\sigma(\xi,\eta) = \frac{1}{\sqrt{2\pi}}\left(\frac{\Omega}{\eta}\right)^{1/2} \frac{\cos\left[\Omega\eta(\xi^2-1)^{1/2} + \pi/4\right]}{(\xi^2-1)^{3/4}\left[1 + \dfrac{\Omega\xi}{(\xi^2-1)^{1/2}}\right]} - \frac{1}{2}\left\{ \left[1 - C(v) - S(v)\right]\sin\delta \right.$$

$$\left. - \left[C(v) - S(v)\right]\cos\delta \right\} + \sin\left[\eta\left(\xi - (1-\Omega^2)^{1/2}\right)\right], \quad \xi > \xi_c$$

where

$$C(v) = \int_0^v \cos\left[\frac{\pi\alpha^2}{2}\right] d\alpha \qquad \text{(Cosine Fresnel Integral)}$$

$$\text{(2-73)} \qquad S(v) = \int_0^v \sin\left[\frac{\pi\alpha^2}{2}\right] d\alpha \qquad \text{(Sine Fresnel Integral).}$$

and [illegible] $\delta = \Omega\eta(\xi^2-1)^{1/2}$ [illegible] $\frac{\pi}{2}v^2$ [illegible] $\left(\frac{\eta}{[\text{illegible}]}\right)^{1/2}\left(1 + \frac{\Omega\xi}{(\xi^2-1)^{1/2}}\right)(\xi^2-1)^{3/4}$.

[illegible] and [illegible] of (2-73) as follows:

$$\int_{\beta}^{\infty} e^{-j\eta^2} d\eta = \int_0^{\infty} e^{-j\eta^2} d\eta - \int_0^{\beta} e^{-j\eta^2} d\eta = \frac{1}{2}\sqrt{\frac{\pi}{2}}(1-j) - \sqrt{\frac{\pi}{2}}\left[C(q) - jS(q)\right] \tag{2-74}$$

$$= \frac{\sqrt{\pi}}{2} e^{-j\frac{\pi}{4}} \left\{1 - (1+j)\left[C(q) - jS(q)\right]\right\},$$

where $q = \sqrt{\frac{2}{\pi}}\,\beta$.

The reader can show that the solution (2-72) is continuous at $\xi = \xi_c$, i.e., when the saddle-point crosses the pole. For times when $\xi \sim \xi_c$ the first term in (2-72) is of the order $1/\sqrt{\eta}$, which for large η is negligible during the main signal build-up. Upon expanding the arguments of the sine and cosine functions in the remaining terms of (2-72) in a Taylor series about ξ_c, it follows that if $(\xi - \xi_c)^2 \ll \frac{2\Omega^4}{(1-\Omega^2)^2}$, then $\gamma \approx \eta\left(\xi - (1-\Omega^2)^{1/2}\right)$.

Thus the dominant terms for the main signal build-up for each of the expressions in (2-72) is the same

$$v(\xi,\eta) \approx \frac{1}{2}\left\{\left[1 + C(v) + S(v)\right]\sin\gamma + \left[C(v) - S(v)\right]\cos\gamma\right\} \tag{2-75}$$

$$= A\sin\left[\eta\left(\xi - (1-\Omega^2)^{1/2}\right) + \theta_0\right],$$

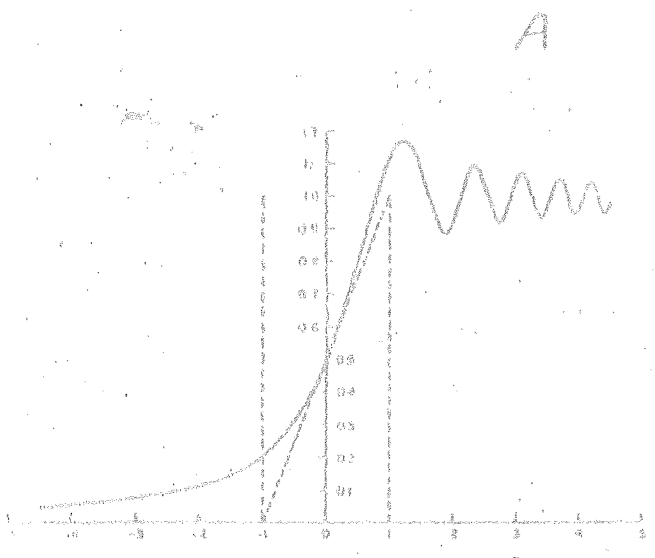
where

$$A = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{2} + C(v)\right)^2 + \left(\frac{1}{2} + S(v)\right)^2\right]^{1/2} \tag{2-76}$$

$$\theta_0 = \tan^{-1}\left[\frac{C(v) - S(v)}{1 + C(v) + S(v)}\right]$$

$$v = \left(\frac{\eta}{\pi\Omega}\right)^{1/2}\left(1 - \frac{\Omega\xi}{(\xi^2-1)^{1/2}}\right)(\xi^2-1)^{3/4}.$$

Equation (2-75) is the approximate expression for the main signal build-up for large $\mathcal{M}$. The amplitude function A is plotted in Figure 2-17. Because the ripples of Fig. 2-17 are due to the Fresnel functions, C(v), S(v), they are called "Fresnel ripples." They occur in the diffraction of light by sharp edges as well. In fact, they occur in many systems when a sharp discontinuity is involved.



B

Value of Extremum	v
1.17 max	1.22
0.88 min	1.87
1.10 max	2.35
0.93 min	2.74
1.07 max	3.06
0.93 min	3.39
1.05 max	3.67
0.94 min	3.93
1.05 max	4.18
0.95 min	4.42

Figure 2-17. (a) Amplitude, A(v), of the main signal build up [illegible]
(b) Values of v at which [illegible]

In Figure 2-18 we plot the envelope of the transient response of a unit step-carrier signal which has propagaged in a lossless waveguide with $\eta = \frac{\omega_0 x}{v_p} = 10^4$ and $\Omega = \frac{\omega_c}{\omega_0} = 0.8$. If $\omega_0 \sim 6 \cdot 10^{10} \frac{rad}{sec}$ ($f_0 \sim 10\,GHz$) and $v_p = 3 \cdot 10^8$, then $x \sim 50$ meters.

No signal arrives prior to $t = x/v_p$ ($\zeta = 1$). Immediately in the vicinity of $\zeta = 1$, the response is given by the Sommerfeld solution (A of Figure 2-18).

Following the Sommerfeld region the anterior transient, or precursor, given by (2-57) and labeled by B is shown. The main-signal build-up, labeled region C, follows the precursor and is itself followed by the posterior solution (D) oscillating about its final value of unity (2-60).

The amplitude of the precursor and Sommerfeld solutions are small and become smaller as the normalized distance η increases; the time between the arrival of the signal wavefront and the arrival of the main signal increases with increasing η, also, the main signal can be considered to arrive at $\zeta_c = \frac{1}{(1-\Omega^2)^{1/2}}$, which corresponds to the time of arrival at the group velocity $t_g = x/v_g = x \frac{\partial \beta}{\partial \omega}\Big|_{\omega = \omega_0}$, where from (2-36), $\beta(\omega) = \frac{\omega}{v_p}\left(1 - \left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2}$.

At this instant of time the saddle-point is exactly crossing the pole and the amplitude of the envelope response is equal to 1/2. The envelope response becomes oscillatory after the saddle-point has crossed the pole.

Finally, we should keep in mind that Figure 2-18 shows only the envelope response; the instantaneous frequency varies with time. The earliest arrival is made by high frequencies; the asymptotic transient frequency is ω_c as $t \to \infty$. Of course, the steady-state frequency is that of the forcing function, ω_0.

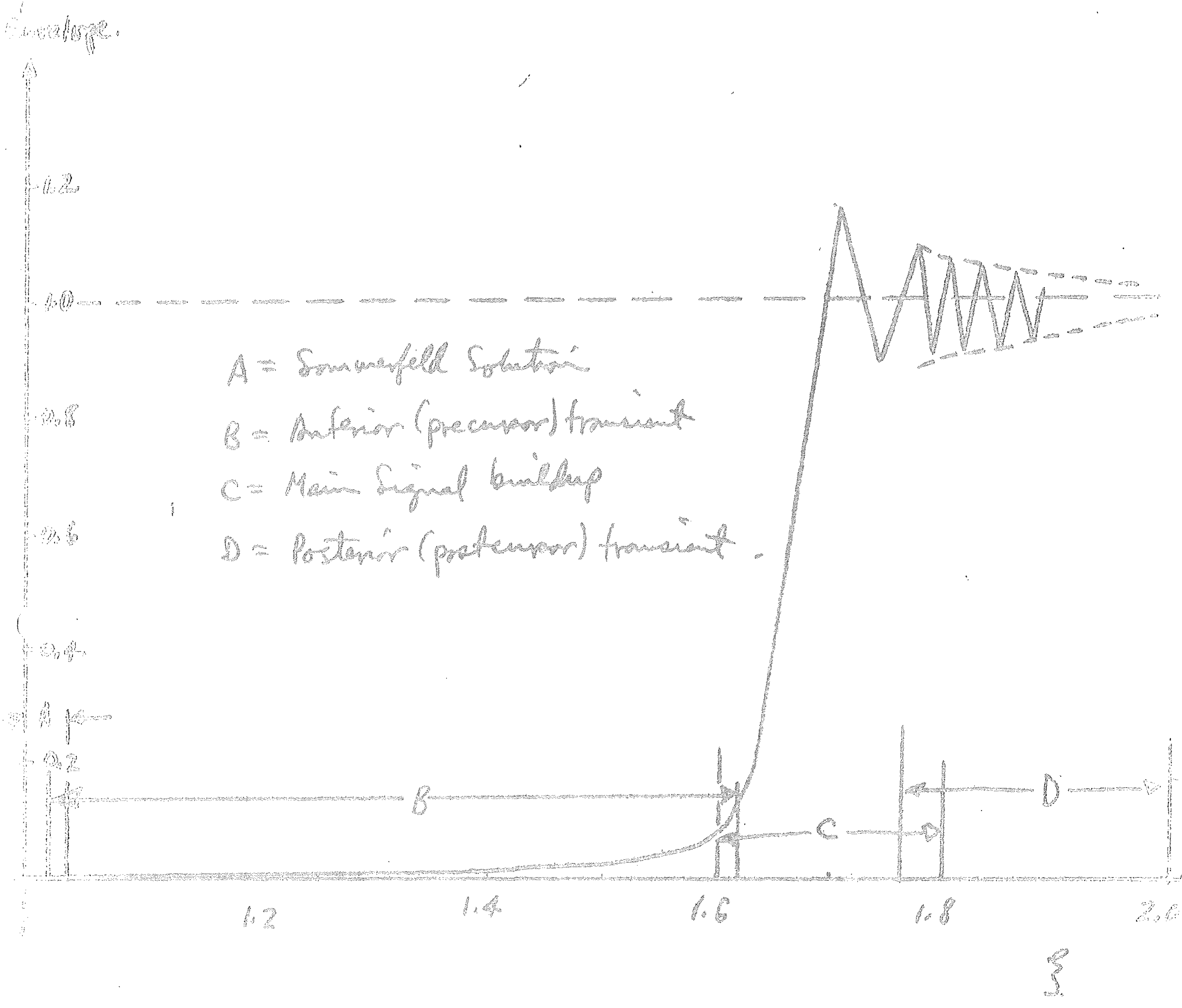


Figure 2-18. Envelope of the response of a lossless waveguide (or other equivalent dispersive medium whose $\beta(\omega) = \frac{1}{v_p}(\omega^2 - \omega_c^2)^{1/2}$) when $\eta = \frac{\omega_0 x}{v_p} = 10^4$, $\Omega = \frac{\omega_c}{\omega_0} = 0.8$

8. Return to the Convolution Integral

A. Pulsed Carrier Propagation

The analytical methods of this chapter have been centered about the transform techniques. In particular we have obtained theoretical results based on the inversion of Fourier and Laplace transforms. There have appeared recently several interesting computer studies of transients in waveguides*, or waveguide-like dispersive systems (Appendix B describes how plasmas behave similarly to a waveguide as far as dispersion is concerned), and these studies have been based on the convolution integral of the last chapter. In particular, recall the impulse response (1-58)

$$(2\text{-}77) \quad h(x,t) = \delta(t - x/v_p) - \frac{\omega_c x}{v_p} \frac{J_1\left[\omega_c(t^2 - x^2/v_p^2)^{1/2}\right]}{(t^2 - x^2/v_p^2)^{1/2}} u_{-1}(t - x/v_p),$$

and the response to an arbitrary function $V_0(t)$, obtained by convolving the impulse response with $V_0(t)$ to get (1-61)

$$(2\text{-}78) \quad v(x,t) = \left[v_0(t - x/v_p) - \frac{\omega_c x}{v_p} \int_{x/v_p}^{t} v_0(t-\tau) \frac{J_1\left[\omega_c(\tau^2 - x^2/v_p^2)^{1/2}\right]}{(\tau^2 - x^2/v_p^2)^{1/2}} d\tau\right] u_{-1}(t - x/v_p)$$

Letting the input signal be a pulsed sine wave of frequency $\omega_0 = 2\omega_c$ and lasting for 20 cycles of ω_c, propagating through a waveguide of length $x = 10\lambda_c = 10 v_p/f_c$ (the length, x, is one-half the "length" $v_p T$, of the pulse, where T is pulse duration), Schulz-DuBois computed the curves of Figure 2-19 using (2-78). The top line shows the input signal, while the second line shows the output one would expect if there were no dispersion; the envelope would propagate with the group velocity and the phase is governed by the phase velocity, v_p. Thus, the second line is the figure one would expect using (2-11), which was the response to an amplitude modulated carrier under the condition that $\beta''(\omega_0) = 0$, where ω_0 is the carrier frequency. The bottom line is the actual transient response computed from (2-78). It shows a slight "ripple" in the amplitude of the envelope. This ripple is called the "Fresnel ripple" and was analyzed in Section 7.

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* E. O. Schulz-DuBois, "Sommerfeld Pre- and Postcursor in the Context of Waveguide Transients", IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-18, No. 8, Aug. 1970, pp. 455-460. R. E. McIntosh and S. E. El-Khamy, "Compression of Transmitted Pulses in Plasmas", IEEE Transactions on Antennas and Propagation, Vol. AP-18, No. 2, March, 1970, pp. 236-241.

The meanings of the time symbols are as follows: t_b and t_e denote the times at which the input signal begins and ends, respectively. The subscripts on t_b and t_e denote input beginning and ending times corresponding to the distance traveled, x, divided by the appropriate velocity. Thus, t_{bp} and t_{ep} denote the input signal beginning and ending times delayed by the amount x/v_p, while t_{bg}, t_{eg} denote, respectively, the beginning and ending times delayed by x/v_g where v_p is the phase velocity and v_g is the group velocity evaluated at the carrier frequency ω_0. v_p should not be confused with the guide-phase velocity v_{gp} at frequency ω_0, i.e., $\frac{\omega_0}{\beta(\omega_0)}$.

Between t_{bp} and t_{bg}, the signal is called the Sommerfeld precursor, or "anterior transient", which was analyzed in Section 7 using saddlepoint integration. The precursor is much smaller than the main signal which arrives at time t_{bg}, traveling with velocity $v_g(\omega_0)$. The envelope at t_{bg} is one-half its steady-state value, in agreement with Figures 2-17 and 2-18. Note that the precursor and postcursor, that part of the signal after t_{eg}, are a small fraction of the total pulse width, $t_{eg}-t_{bg}$. This is definitely not the case in Figure 2-20, which depicts the same transient response as in Figure 2-19, but with an applied frequency of $\omega_0 = 1.2\,\omega_c$. The poorer pulse response is indicative of the fact that we are now operating at too low a frequency for a high-pass filter (the waveguide) to give a faithful reproduction of the input.

It is clear in Figure 2-20 that the group velocity v_g at this lower frequency is much less than (almost 1/2) the phase velocity, v_p. In Figure 2-19, however, we see that $v_p \approx v_g$, which agrees with the fact that the slope of the $\beta(\omega)$ vs. ω curve for a waveguide is asymptotic to $1/v_p$ at high frequencies, as in Figure 2-4.

Rather dramatic results, showing the effects of operating with a carrier frequency below cut-off, are depicted in Figures 2-21 and 2-22. The former figure illustrates the transient response when the carrier frequency, $\omega_0 = 0.8\,\omega_c$, while the latter illustrates the response to a d.c. pulse ($\omega_0 = 0$). For frequencies below cut-off, a group velocity is not defined (or if defined it will be imaginary) and the steady-state response suffers severe exponential damping. The output obviously bears no resemblance to the input (top line in each figure). This is the result one would expect when operating a high-pass filter of any kind below its cut-off frequency.

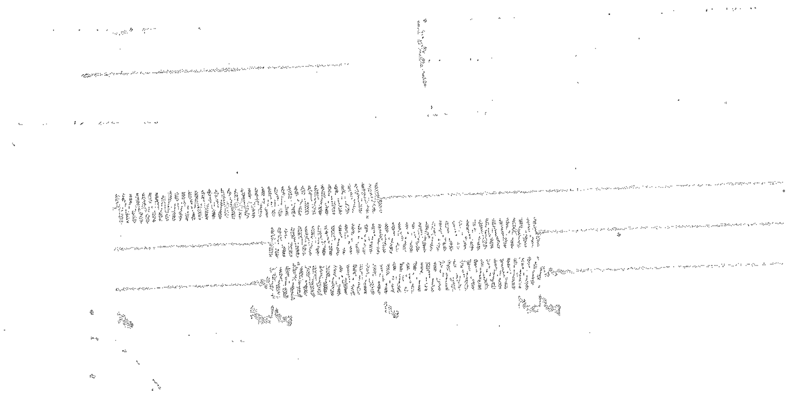


Figure 2-19. Time-domain response to a pulsed carrier for waveguide of length $x = 10\lambda_c$. The carrier frequency ω_0 of the input pulse is $2\omega_c$; the pulse contains 40 periods at ω_0

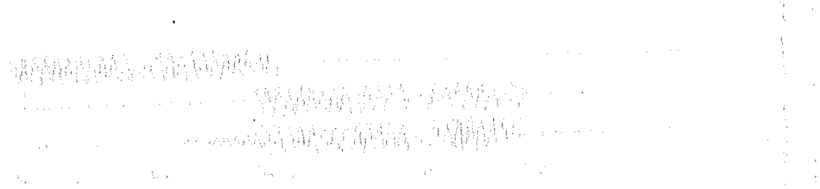


Figure 2-20. Transient response as in Figure 2-[illegible], but for $\omega_s = 1.2\,\omega_c$. [illegible]
post-cursors.

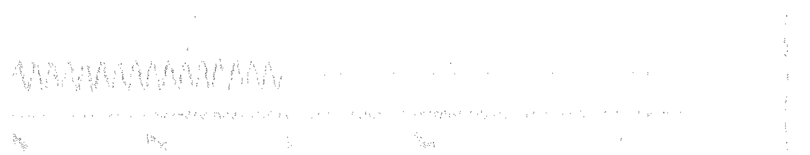


Figure 2-21. Transient response as in Figure 2-19, but for $\omega_s = 0.8\,\omega_c$. Below cut-off [illegible]
attenuation.

Let us now go back to the situation depicted in Figure 2-19 and verify the computed results (always a good practice) and at the same time gain insight into the nature of phase and group velocity. We have already pointed out that the second line was the response expected using (2-11)

$$(2\text{-}79)\qquad v(x,t) = v_b\left(t - \frac{x}{v_g(\omega_0)}\right)\sin\left(\omega_0 t - \beta(\omega_0)x\right) = v_b\left(t - \frac{x}{v_g(\omega_0)}\right)\sin\omega_0\left(t - \frac{x}{\frac{\omega_0}{\beta(\omega_0)}}\right),$$

where we have taken the imaginary part of (2-11) in order to produce the real, or physical, waveform. The baseband signal, $v_b(t)$, is the envelope of the carrier $\sin\omega_0 t$. $\beta(\omega)$ is given by the usual expression

$$(2\text{-}80)\qquad \beta(\omega) = \frac{\omega}{v_p}\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{1/2},$$

while the group velocity, $v_g(\omega_0)$, and guide-phase velocity, $\omega_0/\beta(\omega_0)$, are given, respectively, by

$$(2\text{-}81)\qquad \begin{aligned} v_g(\omega_0) &= v_p\left[1 - \left(\frac{\omega_c}{\omega_0}\right)^2\right]^{1/2} \\ v_{gp}(\omega_0) &= \omega_0/\beta(\omega_0) = \frac{v_p}{\left[1 - \left(\frac{\omega_c}{\omega_0}\right)^2\right]^{1/2}} \end{aligned}$$

From (2-79), we see that the envelope, $v_b(t)$, moves with the group velocity, $v_g(\omega_0)$, whereas the carrier, $\sin\omega_0 t$, moves with the phase velocity, $v_{gp}(\omega_0)$, of the guide. Therefore let us draw a space-time ray diagram with axes x/v_p and t (Figure 2-23). The three rays shown correspond to $v_g(\omega_0)$, v_p and $v_{gp}(\omega_0)$ for $\omega_0 = 2\omega_c$. A simple calculation, based on (2-81), shows that $v_g(\omega_0) = 0.866\, v_p$, $v_{gp}(\omega_0) = 1.15\, v_p$.

Because our observation point is located 10 wavelengths (based on ω_c) away from the source, it follows that from $x = 10\lambda_c = 10\, v_p/f_c$ we have $x/v_p = 10/f_c$. Hence, we draw a horizontal line intersecting the x/v_p-axis at $10/f_c$. The intersection of the rays with this line gives us the time instants t_{bgp}, t_{bp} and t_{bg}, where t_{bgp} is the beginning time based on travel to x with velocity v_{gp}.

Because the phase of the carrier travels with velocity v_{gp}, the equivalent situation on the ray diagram is to move a fixed starting phase point on the sine wave at $x/v_p = 0$ along the ray (or a parallel ray) labeled v_{gp} of Figure 2-23. This has been done for the point of the sine wave corresponding to zero phase shift (i.e., the start of the sine wave). Therefore the corresponding zero phase shift of the sine wave located at x would occur at $t = t_{bgp}$.

[illegible] the true start of the pulse does not arrive at x until time t_{bg}. Thus, the sine wave located between t_{bgp} and t_{bg} is drawn dotted because it really "hasn't arrived yet", i.e., it cannot be detected by physical instruments (note that if it could, then we would be able to measure distrubances propagating at velocities higher than that of light in free space, v_p).

Thus, we see that v_{gp} determines only the phase of the received signal! v_g, the group velocity gives us the time of arrival of the signal itself.

In order to calculate the relative phase shifts between the transmitted and received signals, refer to Figure 2-23 and observe that $10/f_c = 1.15\, t_{bgp}$ or $t_{bgp} = 8.66/f_c = 17.32/f_0$. But $1/f_0$ is one period of the input signal. Thus, the zero phase-shift point (or starting point) of the received sine wave carrier starts after about seventeen and one-third periods of the input carrier. Ignoring a whole number of periods, we see that the received carrier is about 1/3 of a period (or 120°) out of phase (lagging) the input. Or, to put it another say, the positive peak of the received carrier leads the negative peak of the input carrier by about 60° (1/6 cycle). This is in agreement with the computer result of Figure 2-19 (top two lines of the figure).

Finally, the starting point (or phase) of the actual received signal at t_{bg} can be determined by appealing to Figure 2-23 and realizing that $t_{bg} - t_{bgp} = 5.68/f_0$ at $x/v_p = 10/f_c$. Thus, the signal starts with a phase shift of $(0.68)(2\pi)$ after the zero phase shift (ignoring the integral (5) no. of periods). This is a phase shift of 245°, which means that the signal starts at t_{bg} on the negative part of the cycle, again, as shown in the computer result of Figure 2-19.

8. Pulse Compression in a Waveguide.

By now we realize that dispersion of a wideband pulse is due to the different propagation rates (i.e., phase and/or group velocity) of the different frequency components, or wave packets. Generally, dispersion results in a distortion of the transmitted pulse (we have seen this abundantly in this chapter) and is considered a nuisance. There are instances, however, in which dispersive filters are extremely important because they have a salutary effect on the transmitted pulse. That is, they are deliberately used to shape a transmitted pulse into a prescribed form at the output.

We have seen that when Gaussian modulated, single-frequency carrier pulses propagate through a waveguide (or any other dispersive medium with the same $\beta(\omega)$ vs ω characteristic) the envelope of the pulse is broadened to the extent that pulse overlap exists (Figure 2-7). Hence, the Gaussian pulse is not the optimum pulse to use when communicating digital data through a waveguide. Is there a better pulse, in the sense that transmission through the waveguide will not expand the pulse but will, rather, compress it into a sharp spike, thereby improving the resolution between adjacent data pulses? The answer is yes; the pulse will be taken to have a rectangular envelope (as in subsection A) but will be frequently modulated, i.e., the carrier will no longer be the single frequency ω_0, but will vary with time according to an as-yet-to-be-determined law.

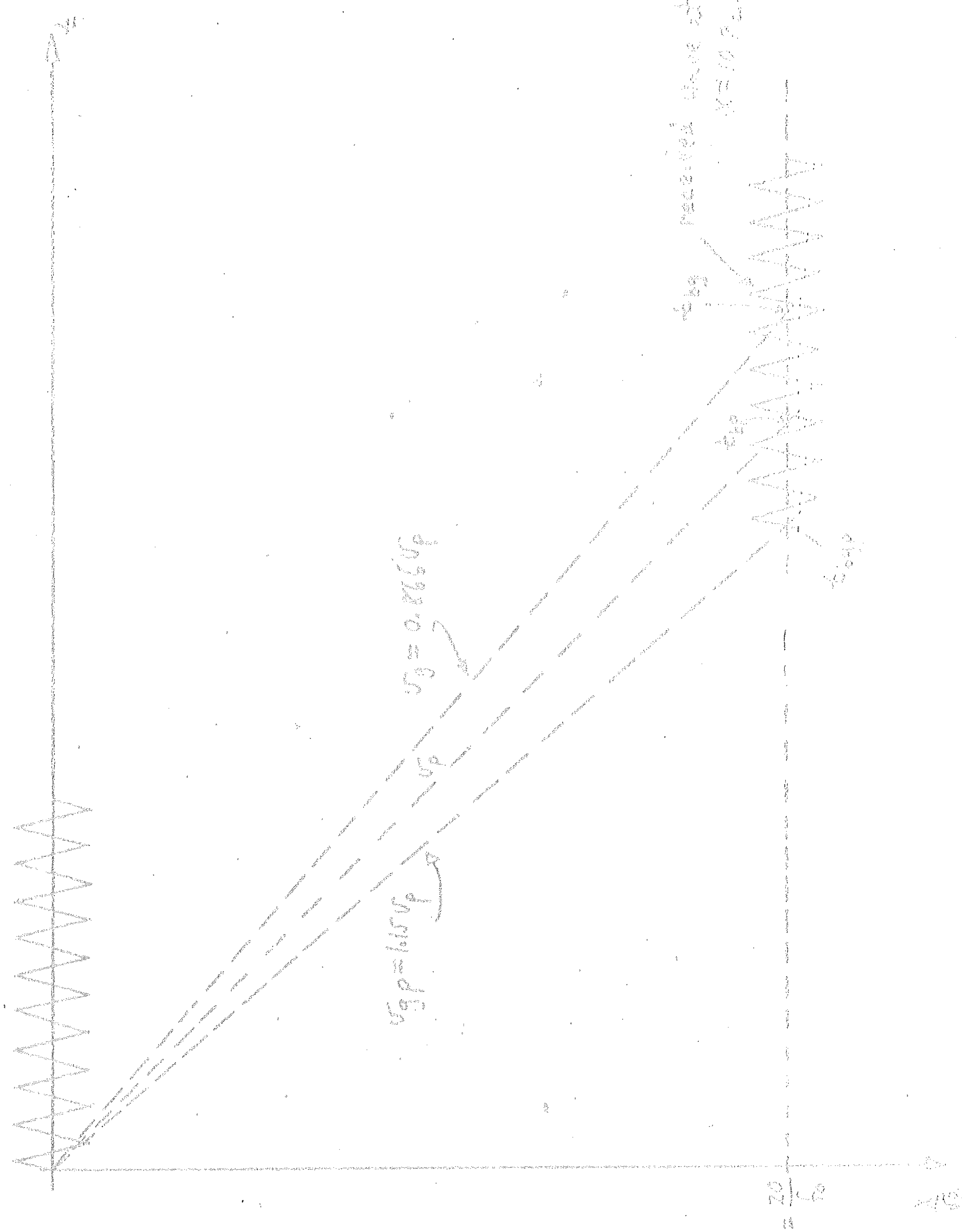


Figure 2-23. Space-time ray diagram to illustrate the effects of group and (guide) phase velocities [illegible]

Here is where dispersion enters: different parts of the signal will travel with different speeds because the frequency varies over the signal duration. By adjusting the frequency variation, we can cause the energy in the trailing portion of the pulse to travel faster than that in the leading portion so that all parts of the pulse will arrive at a fixed location virtually simultaneously, resulting in a <u>collapsed</u> (or compressed) <u>pulse</u> possessing a significant spike. Obviously the signal's frequency variation will have to be <u>matched</u> to the waveguide's dispersion characteristic. When this is achieved, the waveguide acts as a <u>matched filter</u> for the signal. The theory of matched filtering is more extensive than what we wish to pursue here. Excellent references are J. M. Wozencraft and I. M. Jacobs, <u>Principles of Communication Engineering</u>, John Wiley & Sons, New York, 1965 and J. R. Klauder, A. C. Price, S. Darlington, and W. J. Albersheim, "The Theory and Design of Chirp Radars", Bell Syst. Tech. J., Vol. 39, pp. 745-808, July, 1960.

Refer to Figure 2-24(a) in which we show the proposed frequency-modulated, rectangular envelope input signal to the waveguide. Part (b) of Figure 2-24 shows the manner in which the instantaneous frequency of the pulse varies with time. We will derive an analytic expression for this variation shortly. Finally, Figure 2-24(c) is a sketch of the usual group velocity expression for a waveguide $v_g = v_p[1-(\omega_c/\omega)^2]^{1/2}$ with $\omega = \omega(t)$ given in Figure 2-24(b). Thus, we can write

$$v_g = v_g(t) = v_p[1-(\omega_c/\omega(t))^2]^{1/2}.$$

It will again be convenient to introduce normalized (dimensionless) variables: $T = \omega_c t$, $Y = \omega_c x/v_p$, $\omega(t)/\omega_c = \Omega(T)$.

Using these variables, the input signal of normalized duration T_0 can be written

$$v_0(T) = V_0 \, I_m \{e^{j\theta(T)}\}[u_{-1}(T) - u_{-1}(T-T_0)], \quad (2\text{-}82)$$

Where the instantaneous modulated phase $\theta(T)$ is related to $\Omega(T)$ by

$$\Omega(T) = \frac{d\,\theta(T)}{dT}. \quad (2\text{-}83)$$

The unit step functions in (2-82), of course, provide the rectangular envelope.

At the normalized position Y and time T, the output voltage is given by the convolution (2-78), expressed in normalized variables

$$v(Y,T) = \left[v_0(T-Y) - Y\int_Y^T v_0(T-\rho)\frac{J_1[(\rho^2-Y^2)^{1/2}]}{(\rho^2-Y^2)^{1/2}}\,d\rho\right]\cdot u_{-1}(T-Y). \quad (2\text{-}84)$$

Upon substituting (2-82) into (2-84), there results

$$
\begin{aligned}
v(Y,T) &= 0, \qquad T < Y \\
&= V_0 \, I_m \left\{ e^{j\theta(T-Y)} - Y \int_{Y}^{T} e^{j\theta(T-\rho)} \frac{I_1\left[(\rho^2 - Y^2)^{1/2}\right]}{(\rho^2 - Y^2)^{1/2}} \, d\rho \right\}, \qquad Y \le T \le Y + T_0 \\
&= V_0 \, I_m \left\{ -Y \int_{T-T_0}^{T} e^{j\theta(T-\rho)} \frac{I_1\left[(\rho^2 - Y^2)^{1/2}\right]}{(\rho^2 - Y^2)^{1/2}} \, d\rho \right\}, \qquad T > Y + T_0,
\end{aligned}
\tag{2-85}
$$

where the last expressions follow because $v_0(T-Y) = 0$ for $T > Y + T_0$.

Now all that remains before putting (2-85) into a computer in order to determine the voltage at (Y,T) is to decide upon the correct form for $\theta(T)$. We determine $\theta(T)$ by using the following line of reasoning: the variation of group velocity as a function of normalized time, T, is $v_g(T) = v_p\left[1 - 1/\Omega^2(T)\right]^{1/2}$,

so that the normalized <u>group time delay</u> that each element of the pulse experiences in propagating a distance x in the waveguide is given by

$$
D(T) = \omega_c t_g = \omega_c \frac{x}{v_g} = \frac{\omega_c x}{v_p} \cdot \frac{1}{\left[1 - 1/\Omega^2(T)\right]^{1/2}} = \frac{Y}{\left[1 - \frac{1}{\Omega^2(T)}\right]^{1/2}}.
\tag{2-86}
$$

Remember, each part of the input pulse travels with a different group velocity because the instantaneous frequency corresponding to that part of the pulse is different than the frequency of any other part, and different frequency components travel with different velocities due to the dispersive property of the waveguide.

Each [illegible] of the pulse [illegible] at the receiver at the same time as the leading edge if $\theta(0) - \theta(T) = T$, that is the leading edge of the pulse must be delayed relative to the part of the pulse T units later by exactly T units. Utilizing this reasoning, and (2-86), there obtains

$$(2\text{-}87) \quad \frac{Y}{\left[1-\left(\frac{1}{\Omega(0)}\right)^2\right]^{1/2}} = \frac{Y}{\left[1-\left(\frac{1}{\Omega(T)}\right)^2\right]^{1/2}} + T$$

where $\Omega(0)$ is the normalized frequency at the leading edge of the pulse, $T = 0$. Hence, upon letting ω_0 be the angular frequency of the leading edge, (2-87) is easily solved to yield

$$(2\text{-}88) \quad \Omega(T) = \frac{1-\xi}{(\xi^2 - 2\xi + \rho^2)^{1/2}},$$

where $\rho = \frac{1}{\Omega(0)} = \frac{\omega_c}{\omega_0}$, $\xi = \frac{T}{Y}(1-\rho^2)^{1/2}$, and $\frac{Y}{(1-\rho^2)^{1/2}} = T_B$ represents the normalized time required for the leading edge to arrive at Y traveling at its group velocity.

Finally, using (2-88) we determine $\theta(T)$ (using (2-63)) to be

$$(2\text{-}89) \quad \theta(T) = \int_0^T \Omega(T)\,dT = \frac{Y}{(1-\rho^2)^{1/2}}\left[\rho - (\xi^2 - 2\xi + \rho^2)^{1/2}\right].$$

Upon substituting (2-89) into (2-65), we can determine the output voltage. This has been done on a computer using values of the parameters such that enhancement of the signal occurred for a receiver located at the normalized distance $Y = 15$ with $\rho = 0.95$ and a pulse length, T_0, such that the total frequency spread (or bandwidth) was 4, i.e., the ratio of Ω_{max} to Ω_0 equalled 4 (Figure 2-24 (b), which is a sketch of (2-88), shows that Ω increases monotonically with T, so that its maximum value occurs at the trailing edge of the pulse, T_0). The results are shown in Figure 2-25).

The effect of dispersion is to considerably modify the envelope so that at the "design optimum" position $Y = 15$ the envelope is maximized to about three times its original magnitude at the [illegible] and, in addition, the pulse is very narrow, thereby reducing the possibility of overlap with another signal. We remind the reader that this pulse compression is due to the dispersive nature of the transmission medium, enabling the various frequency components of the input signal to "catch up" with one another at the predetermined position $Y = 15$.

The normalized time required for the leading edge to arrive has already been [illegible] to be $Y/(1-\rho^2)^{1/2}$. For the values used in [illegible] to [illegible] the agreement with Figure 3-25(f) for the location in time of the narrow spike.

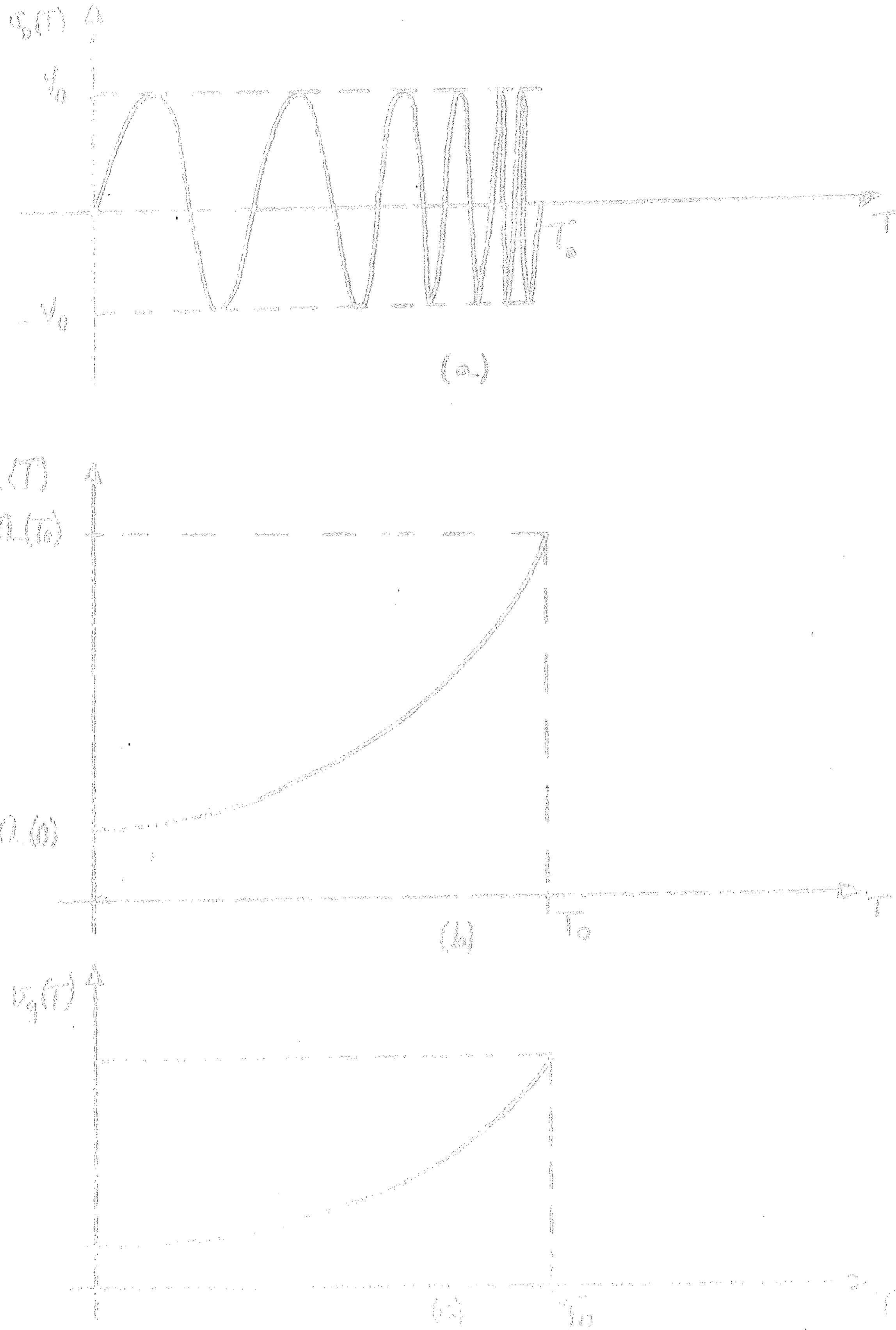
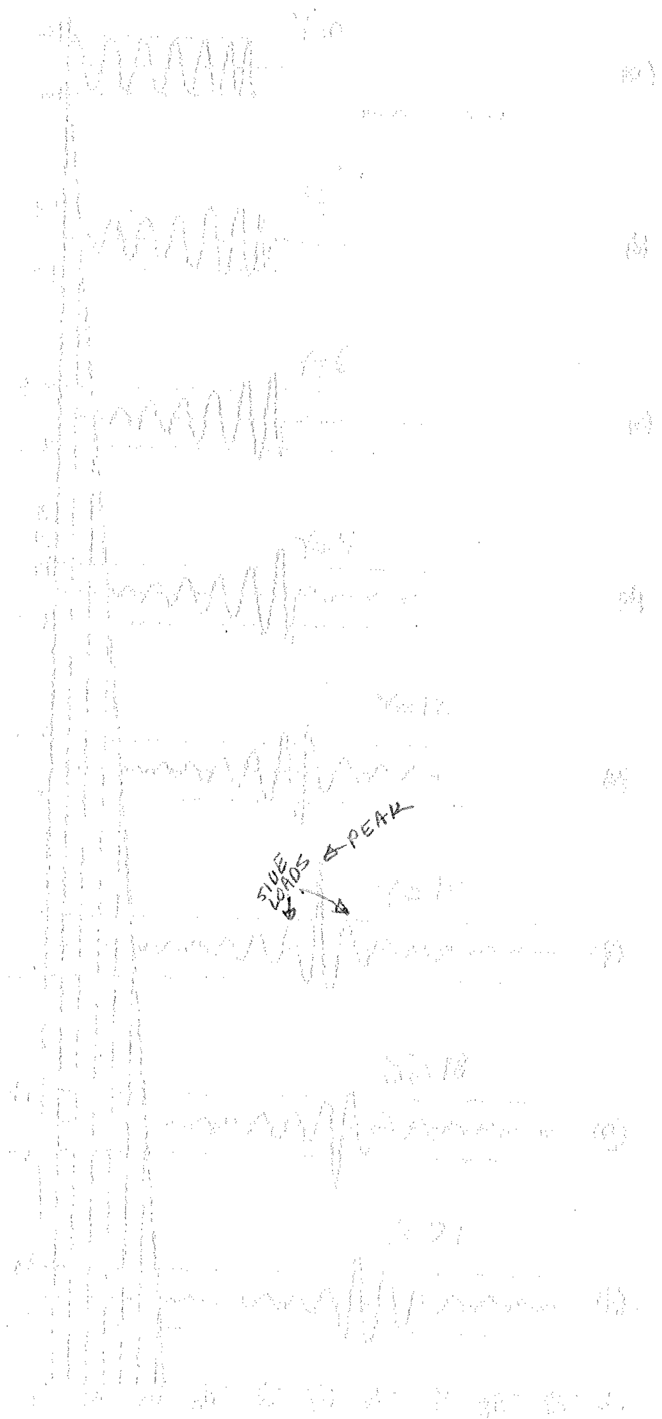


Figure 3-24 The frequency modulated input pulse (a), normalized instantaneous frequency (b) and instantaneous group velocity (c) associated with pulse compression in a waveguide.

$Y = \frac{\omega_c X}{V_P}$



[illegible]
Figure 2-25. First, as has been pointed out, the compressed spike arrives at the normalized time $T = Y/(1-p^2)^{1/2}$. This, however, is simply the group time delay (or time of arrival based on group velocity) of the main signal; there are additional pre- and post-cursors as described in Section 7 and subsection A of this Section. These latter contributions are due to the fact that the sudden rise and fall time of the envelope must contribute a response to the over-all transient. Figure 2-25 also displays the fact that no signal components arrive <u>before</u> Y/T = 1; the first response occurs at Y/T = 1 (this corresponds to the Sommerfeld "early times" solution of Section 7).

Finally, though we have not illustrated the fact here, it can be shown* that enhanced compression (that is, a larger amplitude of the compressed pulse) occurs with longer propagation distances. This is due to the fact that a longer input signal (hence, more energy) can be used and longer distances allow the signal more time to be compressed and the effects of the low-frequency carrier are better separated from the carrier which possesses a greater bandwidth due to the increased pulse length.

Though the rectangular envelope signal has been enhanced, in the sense of pulse discrimination, by the dispersive waveguide, it is still not the optimum signal to use. The optimum signal to be used with a waveguide has a non-rectangular envelope** and would probably be far more difficult to generate in practice.

It is an interesting fact that some mammals, including bats, use a pulse compression "sonar" system to locate obstacles and meals. The signals they emit are generally rather complicated, too, and apparently the characteristics of the signals, e.g., pulse length, pulse repetition rate, frequencies, etc., can be varied depending on the task at hand. When one considers that the signals are transmitted, received, and processed in as small a volume as a bat, one appreciates the engineering design of bats and other biological systems.

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* McIntosh and El-Khamy, op. cit.

** R. E. McIntosh and S.E. El-Khamy, "Optimum Pulse Compression Through a Plasma Medium", IEEE Trans. on Ant. and Prop., Vol. AP-18, No. 5, Sept. 1970, pp. 666-672.

APPENDIX A.   GROUP VELOCITY AS THE VELOCITY OF ENERGY FLOW

Consider a wave traveling in the positive x-direction of a waveguide. The voltage phasor of frequency ω, at point x is

$$V(x) = V_+ e^{-j\beta(\omega)x} \tag{A-1}$$

where $\beta(\omega) = \left(\frac{\omega}{v_p}\right)\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2}$ is the familiar dispersion relation for the waveguide. The positive-going current wave phasor is obtained from the appropriate transmission-line equation to be

$$I(x) = \frac{-1}{j\omega L}\cdot\frac{dV}{dx} = \frac{\beta V_+}{\omega L}e^{-j\beta x} = \frac{V_+}{Z_c}e^{-j\beta x}, \tag{A-2}$$

where $Z_c = \frac{\omega L}{\beta} = \frac{\sqrt{L/C}}{\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2}} = \frac{Z_0}{\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2}}$ is the characteristic impedance of the transmission-line (waveguide) and Z_0 is the characteristic impedance of a lossless, non-dispersive line with inductance-per-unit length, L, and shunt capacitance-per-unit length, C. Z_c is real if $\omega > \omega_c$ and imaginary otherwise. In general, if Z is the series impedance-per-unit length and Y the shunt admittance-per-unit length of a transmission-line, then

$$Z_c = \sqrt{\frac{Z}{Y}} \quad \text{and} \quad j\beta = \sqrt{ZY}$$

We define an energy flow velocity, v_e, as the ratio of the time-averaged power at point X, P, to the time-averaged stored energy-per-unit-length of the line, U.

Note that this ratio does have the correct dimensions of length over time.

From (A-1) and (A-2), the time-averaged power carried by a (+) going wave is

$$P = \frac{1}{2}\mathrm{Re}\left[V(x)\,I^*(x)\right] = \frac{|V_+|^2}{2Z_c}, \quad \omega > \omega_c \tag{A-3}$$
$$= 0, \quad \text{otherwise.}$$

Obviously, then, waves with $\omega > \omega_c$ propagate (deliver time-averaged power) and those with $\omega < \omega_c$ are evanescent and don't deliver time-averaged power, in agreement with our earlier analysis based on the propagation constant.

The total time-averaged energy density is the sum of the time-averaged energy stored, per-unit-length, in the inductor, U_m, and that stored as electrical energy, U_e, in the capacitance per-unit-length. In order to calculate U_m and U_e, we appeal to the incremental circuit of Figure A-1.

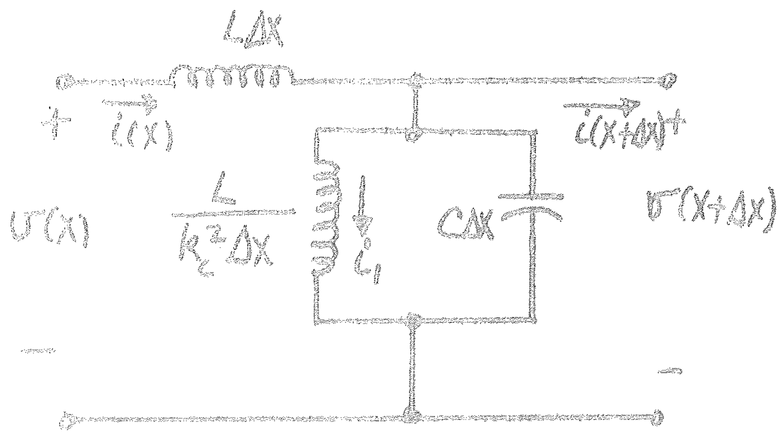

Figure A-1. Incremental section of waveguide transmission-line.

The current flowing in the shunt ind&cator is

$$\text{(A-4)} \quad I_1 = \frac{V(x+\Delta x)\, k_c^2\, \Delta x}{j\omega L},$$

so that the time-averaged energy stored within it is (recall that the time-averaged energy stored in an indicator is $\frac{1}{4} L|I|^2$)

$$\text{(A-5)} \quad \frac{L}{k_c^2 \Delta x} \cdot \frac{|I_1|^2}{4} = \frac{|V(x+\Delta x)|^2}{4\omega^2 L}\, k_c^2 \Delta x .$$

Therefore, the time-averaged energy stored per-unit-length in the distributed shunt inductance at x is

$$\text{(A-6)} \quad U_{m_1} = \frac{|V|^2 k_c^2}{4\omega^2 L} = \frac{|V_+|^2 k_c^2}{4\omega^2 L}, \quad \omega > \omega_c,$$

where the second equality follows from (A-1). To this must be added
$\frac{1}{4} L|I|^2 = \frac{1}{4} L|V_+|^2/Z_c^2 = \frac{C}{4}\left(1-\left(\frac{\omega_c^2}{\omega^2}\right)\right)|V_+|^2$,
which is the energy stored per-unit-length in the series distributed inductance. Thus, the total U_m is given by

$$U_m = \frac{|V_+|^2 k_c^2}{4\omega^2 L} + \frac{C}{4}\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)|V_+|^2$$

$$\text{(A-7)} \qquad = \frac{C}{4}|V_+|^2\left(\frac{\omega_c}{\omega}\right)^2 + \frac{C}{4}|V_+|^2\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)$$

$$= \frac{C}{4}|V_+|^2$$

The energy density stored per-unit-length in the distributed capacitance is

$$\text{(A-8)} \qquad U_e = \frac{C}{4}|V_+|^2 = \frac{C}{4}|V|^2 = U_m .$$

The result, $U_e = U_m$, is a general one for lossless waveguiding or transmission-line systems.

Upon using (A-3), (A-7) and (A-8), we finally arrive at our result

$$\text{(A-9)} \qquad v_e = \frac{P}{U_m + U_e} = \frac{|V_+|^2/2Z_c}{\frac{C}{2}|V_+|^2} = \frac{\left[1-\left(\frac{\omega_c}{\omega}\right)^2\right]^{1/2}}{\sqrt{LC}} = v_p\left(1-\left(\frac{\omega_c}{\omega}\right)^2\right)^{1/2}$$

$$= v_g , \quad \omega > \omega_c$$

$$= 0, \quad \omega < \omega_c .$$

Thus, we have our important conclusion that the group and energy flow velocities are equal in a lossless transmission-line. Hence, we can use the usual dispersion relations, space-time rays, etc., to infer conclusions concerning energy flow in a system.

B. PLASMA AND THE RELATIVISTIC ELECTRON

Plasma. For our purposes a plasma is simply a gas of free electrons which interacts with an electromagnetic wave. Thus, the appropriate one dimensional (Maxwell) equations are

$$\text{(B-1)}\quad \text{(a)}\ \frac{\partial \mathcal{E}_y}{\partial x} = -\mu_0 \frac{\partial \mathcal{H}_z}{\partial t}$$

$$\text{(b)}\ \frac{\partial \mathcal{H}_z}{\partial x} = -J - \varepsilon_0 \frac{\partial \mathcal{E}_y}{\partial t}$$

$$\text{(c)}\ J = -Nqv.$$

E_y is the y-component of the electric field vector, $\bar{\mathcal{E}}$, $\mathcal{H}_z$, the z-component of $\mathcal{H}$, the magnetic field intensity, J, the current density resulting from the motion of the elctrons whose velocity is v, number density (#/volume) N and charge − q. μ_0 is the magnetic permeability of free space (= $4\pi.10^{-7}$ henries/meter) and ε_0 is the dielectric constant of free space (=$(1/36\pi).10^{-9}$).

Newton's law for the motion of the electrons under the influence of the electric field, E_y, is

$$\text{(B-2)}\quad \frac{\partial v}{\partial t} = -\frac{q}{m}\mathcal{E}_y,$$

where m is the mass of an electron. Substituting this equation into (B-1) (c) yields

$$\text{(B-3)}\quad \frac{\partial J}{\partial t} = \frac{Nq^2}{m}\mathcal{E}_y .$$

Differentiation of (B-1)(a) with respect to x and (B-1) (b) with respect to t yield

$$\text{(B-4)}\quad \text{(a)}\ \frac{\partial^2 \mathcal{E}_y}{\partial x^2} = -\mu_0 \frac{\partial^2 \mathcal{H}_z}{\partial t\,\partial x}$$

$$\text{(b)}\ \frac{\partial^2 \mathcal{H}_z}{\partial x\,\partial t} = -\frac{\partial J}{\partial t} - \varepsilon_0 \frac{\partial^2 \mathcal{E}_y}{\partial t^2} = -\frac{Nq^2}{m}\mathcal{E}_y - \varepsilon_0 \frac{\partial^2 \mathcal{E}_y}{\partial t^2},$$

where use has been made of (B-3) in obtaining the final form of (B-4) (b). Upon substituting (B-4) (b) into (B-4) (a) we obtain the appropriate wave equation

$$\text{(B-5)}\quad \frac{\partial^2 \mathcal{E}_y}{\partial x^2} = \mu_0 \frac{Nq^2}{m}\mathcal{E}_y + \mu_0\varepsilon_0 \frac{\partial^2 \mathcal{E}_y}{\partial t^2} = \frac{1}{c^2}\left(\omega_p^2 \mathcal{E}_y + \frac{\partial^2 \mathcal{E}_y}{\partial t^2}\right),$$

where $c = \left(\frac{1}{\mu_0 \epsilon_0}\right)^{1/2}$ is the velocity of light in free-space ($= 3.10^8$ m/sec), and $\omega_p = \left(\frac{Nq^2}{\epsilon_0 m}\right)^{1/2}$ is called the <u>plasma frequency</u>. The equation for the Fourier transform, $E_y(x,\omega)$ of $\mathcal{E}_y(x,t)$ is obtained by replacing $\partial^2/\partial t^2$ by $-\omega^2$ in (B-5)

$$\text{(B-6)} \qquad \frac{d^2 E_y}{dx^2} + \left(\frac{\omega}{c}\right)^2 \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)^{1/2} E_y = 0.$$

This is precisely of the same form as the corresponding equation for the waveguide, upon calling ω_p the "cut-off frequency", ω_c, of the plasma. Hence, everything that has been said in this chapter concerning dispersion, pulse distortion, etc., holds for a plasma. Indeed, many of the references cited dealt not with waveguides, but rather with plasmas.

2. <u>Realivistic Electron.</u> In quantum (or "wave") mechanics one defines a probability wave function $\psi(x,t)$ whose salient property is that $|\psi|^2 dx$ represents the probability that at time t a particle will be located between x and x + dx. $\psi(x,t)$ satisfies Schrodinger's wave equation

$$\text{(B-7)} \qquad \mathcal{H}\psi(x,t) = j\hbar \frac{\partial \psi}{\partial t},$$

where H is called the Hamiltonian operator for the system under consideration and $\hbar = h/2$, where h is Planck's constant, a universal constant.

The operator, H, is obtained in the following manner: take the total energy of the system (kinetic plus potential) and express it in terms of momentum p and position x. Then replace p, wherever it appears, by the operator $(\hbar/j)\,\partial/\partial x$. The resulting expression is the Hamiltonian operator H.

As an example, consider a relativistic electron in an external field whose potential energy is V(x). According to the theory of relativity, the kinetic energy of a particle of rest mass m, moving with momentum p is given by

$$\text{(B-8)} \qquad \text{Kinetic Energy} = \left(c^2 p^2 + m^2 c^4\right)^{1/2},$$

with c again being the velocity of light in free space.

Applying the rules just stated, the Hamiltonian operator is given by

$$\text{(B-9)} \qquad \mathcal{H} = \left(-\hbar^2 c^2 \frac{\partial^2}{\partial x^2} + m^2 c^4\right)^{1/2} + V(x),$$

and Schrodinger's wave equation becomes

(B-10) $$\mathcal{H}\psi = \left[\left(-\hbar^2c^2\frac{\partial^2}{\partial x^2} + m^2c^4\right)^{1/2} + V(x)\right]\psi = j\hbar\frac{\partial\psi}{\partial t}.$$

Consider now a free particle, i.e., one for which V(x) = 0. Equation (B-10) becomes

(B-11) $$\left(-\hbar^2c^2\frac{\partial^2}{\partial x^2} + m^2c^4\right)^{1/2}\psi = j\hbar\frac{\partial\psi}{\partial t}.$$

If we "operate" on (B-11) by multiplying both sides by the operator $\left(-\hbar^2c^2\partial^2/\partial x^2 + m^2c^4\right)^{1/2}$ we get

(B-12) $$-\hbar^2c^2\frac{\partial^2\psi}{\partial x^2} + m^2c^4\psi = j\hbar\frac{\partial}{\partial t}\left[\left(-\hbar^2c^2\frac{\partial^2}{\partial x^2} + m^2c^4\right)^{1/2}\psi\right].$$

But by (B-11) the term within the square brackets in (B-12) is $j\hbar\frac{\partial\psi}{\partial t}$, so that (B-12) becomes

(B-13) $$-\hbar^2c^2\frac{\partial^2\psi}{\partial x^2} + m^2c^4\psi = -\hbar^2\frac{\partial^2\psi}{\partial t^2},$$

which, upon rearrangement is identical in form to the waveguide (or plasma) wave equation (B-5)

(B-14) $$\frac{\partial^2\psi}{\partial x^2} = \frac{m^2c^2}{\hbar^2}\psi + \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{1}{c^2}\left(\frac{m^2c^4}{\hbar^2}\psi + \frac{\partial^2\psi}{\partial t^2}\right).$$

Therefore, the "cut-off" frequency for the free relativistic electron is $\omega_c = mc^2/\hbar$, and the "cut-off" energy $\hbar\omega_c = mc^2$ is simply the "rest energy" mc^2 of the electron. This means that the minimum energy a free electron can possess is its rest energy, and that is achieved when the momentum of the particle vanishes. This is consistent with our notion that the group velocity of the wave (particle) is nil at cut-off. The group velocity is obtained in the usual manner

(B-15) $$v_g = \left(\frac{\partial\beta(\omega)}{\partial\omega}\right)^{-1} = c\left(1 - \left(\frac{mc^2}{\hbar\omega}\right)^2\right)^{1/2},$$

where $\beta(\omega) = \frac{\omega}{c}\left(1 - \left(\frac{mc^2}{\hbar\omega}\right)^2\right)^{1/2}$

The phase velocity of the particle is

$$(B-16) \qquad v_p = \frac{\omega}{\beta(\omega)} = \frac{c}{\left[1-\left(\frac{mc^2}{\hbar\omega}\right)^2\right]^{1/2}},$$

and is always greater than c, whereas v_g is always less than c.

Let us return to (B-15) and call $\hbar\omega$ the energy, E, of the system. Then (B-15 reads

$$(B-17) \qquad v_g = c\left(1-\left(\frac{mc^2}{E}\right)^2\right)^{1/2},$$

and this may be solved for E as a function of v_g:

$$(B-18) \qquad E = \frac{mc^2}{\sqrt{1-\left(\frac{v_g}{c}\right)^2}},$$

which is the classical relativistic expression for a free particle's energy when moving with velocity v_g. Thus, in quantum mechanics we always associate the group velocity, v_g, of the probability wave with the particle velocity.

APPENDIX C. THE WKBJ APPROXIMATION.*

Consider

$$\frac{d^2V}{dx^2} + k^2(x)\,V = 0 \qquad \text{(C-1)}$$

which is of the form (2-100) except that we have changed the independent variable from t to x and the coefficient function from $\omega^2(t)$ to $k^2(x)$. The reason for doing this is to introduce a physically motivated derivation of the WKBJ approximation through the idea of wave propagation.

The WKBJ approximation, named in honor of its 20th century discoverers Wentzel, Kramers, Brillouin and Jeffreys, though it was known in the nineteenth century by Liouville and Green, is useful when $k^2(x)$ is a "slowly-varying" function of x.

Consider a transmission-line (or other wave propagating medium) that has a continuously varying wave number k(x) and extending from 0 to x_n as in Figure C-1. Partition the interval $[0, x_n]$ into intervals such that k(x) is approximately constant over each subinterval. Thus, over the ith subinterval, $x_{i-1} < x < x_i$, $k(x)$ takes on the constant value $k_i = k(\xi_i)$, where ξ_i is some point (perhaps the midpoint) of the ith subinterval. Then, we have approximated a continuously variable parameter transmission-line by a series of constant parameter lines. The continuously variable line, and its concomitant effects, are obtained by taking the limit of the corresponding effects of the discrete lines as the length (or interval spacings) of the constant parameter lines goes to zero. (See Figure C-1).

At each interface of the piecewise constant lines an incident wave will be partially transmitted and reflected. The reflected waves will be re-reflected and re-transmitted. These higher order effects are ignored because the variation in k between adjacent transmission-lines is small, so that the original reflected wave will be small. This is the physical basis of the WKBJ approximation.

---

* We follow J. Heading, An Introduction to Phase Integral Methods, Methuen and Co., 1962. The material is also discussed in most quantum mechanics texts.

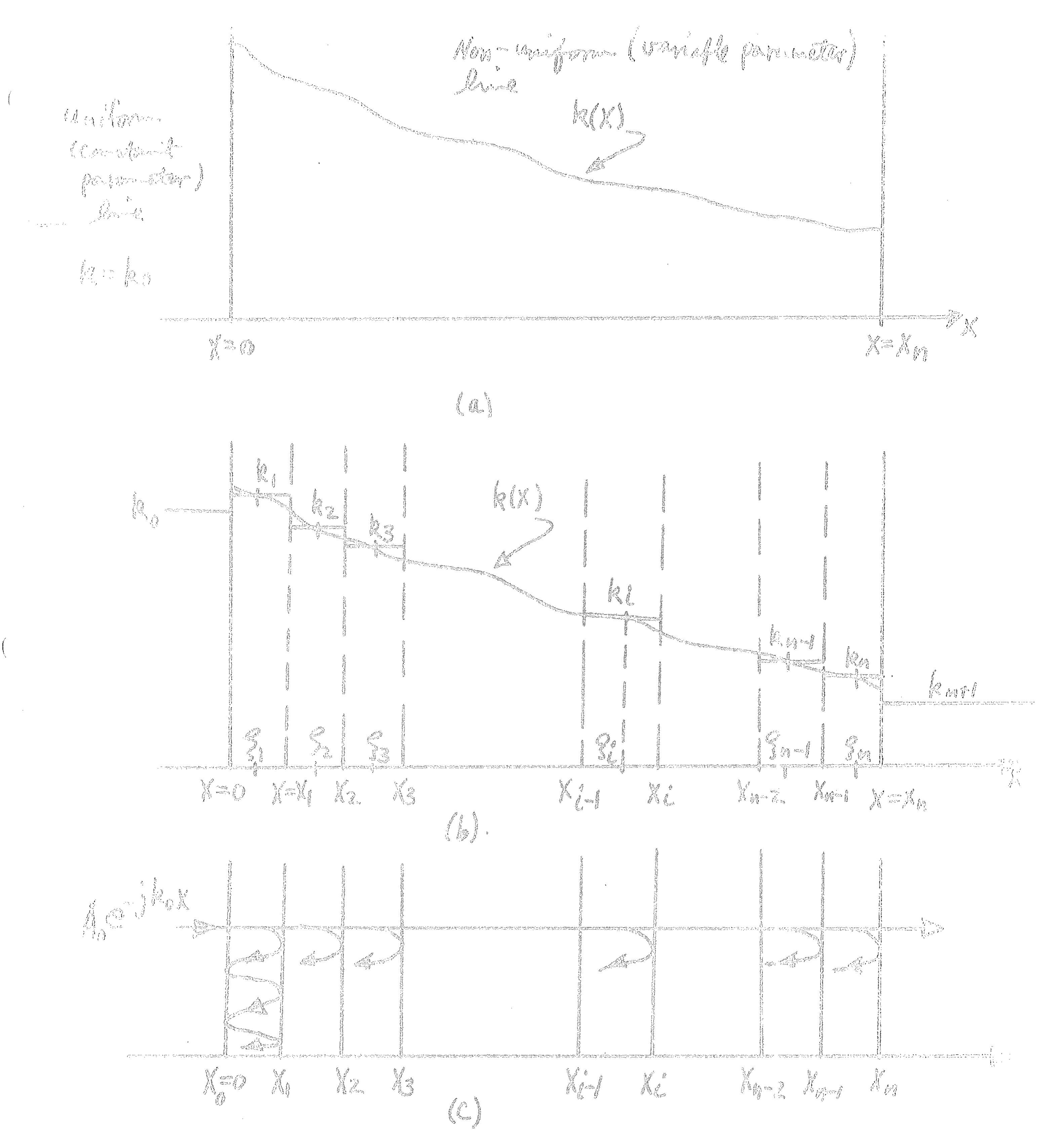

Figure C-1. Depicting available parameter (non-uniform) transmission-line extending from $x=0$ to $x=x_n$ (a) The continuously varying wave number, $k(x)$. (b) piecewise constant approximation. (c) Illustrating transmission and reflection at the boundaries of the piece-wise constant lines. In the WKBJ approximation the effects of the reflected (singly and multiply reflected) waves are ignored.

For the line just to the left of x_i, the wave is given by

(C-2) $$V = A_i e^{-jk_i x} + B_i e^{jk_i x} ,$$

and just to the right there is only a transmitted wave

(C-3) $$V = C_i e^{-jk_{i+1} x} .$$

Note that the propagation constant (or wave number) is k_{i+1} just to the right of $x = x_i$.

We wish V to be continuous, together with its first derivative, at $x = x_i$. Thus,

(C-4) $$A_i e^{-jk_i x_i} + B_i e^{jk_i x_i} = C_i e^{-jk_{i+1} x_i}$$
$$-k_i A_i e^{-jk_i x_i} + k_i B_i e^{jk_i x_i} = -k_{i+1} C_i e^{-jk_{i+1} x_i} .$$

We can eliminate the coefficient B_i and obtain

(C-5) $$C_i = A_i \cdot \frac{2k_i}{k_i + k_{i+1}} e^{-j(k_i - k_{i+1}) x_i} .$$

Notice that the magnitude of the ratio C_i/A_i is simply the transmission coefficient of the ith slab, and the exponent determines the phase shift of the transmitted wave at x_i relative to the incident wave $A_i e^{-jk_i x_i}$.

We can use this result for any index value i, if we take the incident wave number at x=0 to be k_0, and set $x_0=0$. In addition, if we assume that the magnitude, A_0, of the incident wave at x=0 is unity, then the transmitted output at $x=x_n$ is simply the product of terms like (C-5) going from i=0 to i=n (starting with $A_0=1$).

(C-6) $$V_{out} = C_{n+1} e^{-jk_{n+1} x_n} = \frac{2k_0}{k_0+k_1} \cdot \frac{2k_1}{k_1+k_2} \cdot \cdots \cdot \frac{2k_n}{k_{n-1}+k_n} \cdot$$
$$\cdot e^{-j(k_0-k_1)x_0} \cdot e^{-j(k_1-k_2)x_1} \cdot \cdots \cdot e^{-j(k_n-k_{n+1})x_n} \cdot e^{-jk_{n+1} x_n} .$$

Now consider the factor $\frac{2k_i}{k_i+k_{i+1}}$ and expand it as follows:

(C-7) $$\frac{2k_i}{k_i+k_{i+1}} = \frac{2k_i}{2k_i + (k_{i+1}-k_i)} = \frac{1}{1+\left(\frac{k_{i+1}-k_i}{2k_i}\right)}$$
$$= 1 - \left(\frac{k_{i+1}-k_i}{2k_i}\right) + \left(\frac{k_{i+1}-k_i}{2k_i}\right)^2 + \cdots$$

By assumption ($k_{i+1} - k_i$) $\ll$ 2 k_i if the wave number is slowly varying. Hence, the infinite geometric series (with alternating signs) can be terminated with the result

(C-8) $$\frac{2k_i}{k_i+k_{i+1}} \simeq 1-\left[\frac{k_{i+1}-k_i}{2k_i}\right] \simeq \exp\left[-\frac{k_{i+1}-k_i}{2k_i}\right].$$

The latter equality follows upon expanding $\exp\left[-\frac{k_{i+1}-k_i}{2k_i}\right] = 1-\left[\frac{k_{i+1}-k_i}{2k_i}\right]+\frac{1}{2!}\left[\frac{k_{i+1}-k_i}{2k_i}\right]^2+\cdots$ and keeping only the first two terms.

Finally, then, we can substitute (C-8) back into (C-6), rearrange terms and get

(C-9)
$$V_{out} = \exp\left[-\frac{k_1-k_0}{2k_1}\right]\cdot\exp\left[-\frac{k_2-k_1}{2k_2}\right]\cdot\ldots\cdot\exp\left[-\frac{k_n-k_{n-1}}{2k_n}\right]\cdot$$
$$\cdot\exp\left[-jk_0x_0-jk_1(x_1-x_0)-jk_2(x_2-x_1)-\cdots-jk_n(x_n-x_{n-1})\right]$$
$$=\exp\left\{-\frac{k_1-k_0}{2k_1}-\frac{k_2-k_1}{2k_2}-\cdots-\frac{k_n-k_{n-1}}{2k_n}\right.$$
$$\left.-j\left[k_1(x_1-x_0)+k_2(x_2-x_1)+\cdots+k_n(x_n-x_{n-1})\right]\right\},$$

where we have used the fact that $x_0=0$ in dropping the term $-jk_0x_0$ in the last exponent. In passing to the continuous limit, we obtain integrals in place of the summations:

(C-10)
$$V_{out} = \exp\left\{-\int\frac{dk}{2k}-j\int_0^{x_n}k(\eta)\,d\eta\right\}$$
$$=\exp\left\{-\frac{1}{2}\ln k-j\int_0^{x}k(\eta)\,d\eta\right\}$$
$$=\exp\left\{\ln\frac{1}{k^{1/2}}\right\}\exp\left\{-j\int_0^{x}k(\eta)\,d\eta\right\}$$
$$=\frac{1}{\left[k(x)\right]^{1/2}}\exp\left[-j\int_0^{x}k(\eta)\,d\eta\right],$$

where we have replaced x_n by x to represent the final variable point.

For a wave incident from the right and propagating to the left, we would replace the exponent's negative sign by a plus sign.

Hence, taking into account that the magnitudes of the incident waves need not have been unity, the most general WKBJ solution of (C-1) is

(C-11)
$$\psi(x) = \frac{A}{[k(x)]^{1/2}} \exp\left(-j\int_0^x k(\eta)\,d\eta\right) + \frac{B}{[k(x)]^{1/2}} \exp\left(j\int_0^x k(\eta)\,d\eta\right).$$

It appears that because we have the product of two functions and $[k(x)]^{-1/2}$ and $\exp\left(\pm j\int_0^x k(\eta)\,d\eta\right)$ in (C-11), the derivative of either term would involve two terms. In the WKBJ "slowly varying" assumption, we discard one term. Thus,

(C-12)
$$\frac{d}{dx}\left[[k(x)]^{-1/2} e^{-j\int_0^x k(\eta)\,d\eta}\right] = -\frac{1}{2}[k(x)]^{-3/2}\frac{dk}{dx}\exp\left(-j\int_0^x k(\eta)\,d\eta\right) - j[k(x)]^{1/2}\exp\left(-j\int_0^x k(\eta)\,d\eta\right) \simeq -j[k(x)]^{1/2}\exp\left(-j\int_0^x k(\eta)\,d\eta\right),$$

because the magnitude of the ratio of the first term to the second is

(C-13)
$$\left|\frac{\frac{dk}{dx}}{[k(x)]^2}\right| << 1$$

by virtue of the smallness of dk/dx. Thus the derivative of either term of the WKBJ solution (C-11) involves only the derivative of the exponential, treating $[k(x)]^{-1/2}$ as a constant.

APPENDIX D. EVALUATION OF $v(x,t)$ [illegible]

A. Evaluation by means of Inverse Fourier Transform

We start with (2-105) and substitute into (2-96):

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\cos kv_p t_0 \cos (v_p^2k^2+b^2)^{1/2}(t-t_0)$$

(D-1)
$$- v_p k \sin kv_p t_0 \frac{\sin (v_p^2k^2+b^2)^{1/2}(t-t_0)}{(v_p^2k^2+b^2)^{1/2}}\Big\}e^{-jkx}\,dk.$$

Use will be made of the Fourier transform relation

(D-2)
$$\int_{-\infty}^{\infty} e^{jxy}\frac{\sin\left(b\sqrt{a^2+x^2}\right)}{\sqrt{a^2+x^2}}\,dx = \begin{cases} 0, & |y| > b \\ \pi J_0\left(a\sqrt{b^2-y^2}\right), & |y| < b. \end{cases}$$

which can be found in W. Magnus and F. Oberhettinger, Formulas and Theorems for the Functions of Mathematical Physics, Chelsea Pub. Co., New York, 1949, p. 118.

Consider the first term of the integrand of (D-1), and write it as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos kv_p t_0 \cos (v_p^2k^2+b^2)^{1/2}(t-t_0)\, e^{-jkx}\,dk$$

(D-3)
$$= \frac{1}{4\pi}\cdot\frac{d}{dt}\int_{-\infty}^{\infty}\left(e^{jk(v_p t_0 - x)} + e^{-jk(v_p t_0 - x)}\right)\frac{\sin (v_p^2k^2+b^2)^{1/2}(t-t_0)}{(v_p^2k^2+b^2)^{1/2}}\,dk.$$

Reference to (D-2) shows that the integral on the righthand side of (D-3) can be easily evaluated, with the result that (D-3) is equal to

$$\frac{1}{4v_p}\frac{\partial}{\partial t}\Big\{J_0\Big[\frac{b}{v_p}\big(v_p^2(t-t_0)^2-(v_pt_0-x)^2\big)^{1/2}\Big]u_{-1}\big(t-t_0-|t_0-x/v_p|\big)$$

$$+J_0\Big[\frac{b}{v_p}\big(v_p^2(t-t_0)^2-(v_pt_0+x)^2\big)^{1/2}\Big]u_{-1}\big(t-t_0-|t_0+x/v_p|\big)\Big\}$$

(D-4)
$$=\frac{1}{4v_p}\Big\{\delta\big(t-t_0-|t_0-x/v_p|\big)+\delta\big(t-t_0-|t_0+x/v_p|\big)$$

$$-v_p b(t-t_0)\frac{J_1\Big[\frac{b}{v_p}\big(v_p^2(t-t_0)^2-(v_pt_0-x)^2\big)^{1/2}\Big]}{\big(v_p^2(t-t_0)^2-(v_pt_0-x)^2\big)^{1/2}}u_{-1}\big(t-t_0-|t_0-x/v_p|\big)$$

$$-v_p b(t-t_0)\frac{J_1\Big[\frac{b}{v_p}\big(v_p^2(t-t_0)^2-(v_pt_0+x)^2\big)^{1/2}\Big]}{\big(v_p^2(t-t_0)^2-(v_pt_0+x)^2\big)^{1/2}}u_{-1}\big(t-t_0-|t_0+x/v_p|\big)\Big\},$$

where use has been made of the identity

(D-5)
$$\frac{d}{dx}J_0(x)=-J_1(x).$$

Next, we consider the second term of (D-1) and write it as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-v_pk)\sin kv_pt_0\,\frac{\sin(v_p^2k^2+b^2)^{1/2}(t-t_0)}{(v_p^2k^2+b^2)^{1/2}}e^{-jkx}\,dk$$

$$=\frac{v_p}{4\pi}\int_{-\infty}^{\infty}\big(e^{jk(v_pt_0-x)}-e^{-jk(v_pt_0+x)}\big)(jk)\frac{\sin(v_p^2k^2+b^2)^{1/2}(t-t_0)}{(v_p^2k^2+b^2)^{1/2}}dk$$

(D-6)
$$=-\frac{d}{dx}\left(\frac{v_p}{4\pi}\right)\int_{-\infty}^{\infty}\big(e^{jk(v_pt_0-x)}-e^{-jk(v_pt_0+x)}\big)\frac{\sin(v_p^2k^2+b^2)^{1/2}(t-t_0)}{(v_p^2k^2+b^2)^{1/2}}dk$$

$$=-\frac{d}{dx}\left(\frac{v_p}{4\pi}\right)\Big\{\frac{\pi}{v_p}J_0\Big[b\big((t-t_0)^2-(t_0-x/v_p)^2\big)^{1/2}\Big]\cdot u_{-1}\big(t-t_0-|t_0-x/v_p|\big)$$

$$-\frac{\pi}{v_p}J_0\Big[b\big((t-t_0)^2-(t_0+x/v_p)^2\big)^{1/2}\Big]\cdot u_{-1}\big(t-t_0-|t_0+x/v_p|\big)\Big\}.$$

various intervals of the x-axis given by

$$
\text{(D-7)} \qquad
\begin{aligned}
u_{-1}\left(t - t_0 - \left|t_0 - x/v_p\right|\right) &= u_{-1}\left(t - 2t_0 + x/v_p\right), & x < v_p t_0 \\
&= u_{-1}\left(t - x/v_p\right), & x > v_p t_0 . \\
u_{-1}\left(t - t_0 - \left|t_0 + x/v_p\right|\right) &= u_{-1}\left(t + x/v_p\right), & x < -v_p t_0 \\
&= u_{-1}\left(t - 2t_0 - x/v_p\right), & x > -v_p t_0 .
\end{aligned}
$$

When (D-7) is substituted into (D-6) and the differentiation carried out, there results

$$
\text{(D-8)} \qquad
\begin{aligned}
= -\frac{1}{2}\Bigg\{ & \frac{J_1\left[\frac{b}{v_p}\left(v_p^2(t-t_0)^2 - (x-v_p t_0)^2\right)^{1/2}\right]\left(\frac{b}{v_p}\right)(x - v_p t_0)}{\left[v_p^2(t-t_0)^2 - (x - v_p t_0)^2\right]^{1/2}} \, u_{-1}\left(t - t_0 - \left|t_0 - x/v_p\right|\right) \\
& - \frac{J_1\left[\frac{b}{v_p}\left(v_p^2(t-t_0)^2 - (x+v_p t_0)^2\right)^{1/2}\right]\left(\frac{b}{v_p}\right)(x + v_p t_0)}{\left[v_p^2(t-t_0)^2 - (x + v_p t_0)^2\right]^{1/2}} \, u_{-1}\left(t - t_0 - \left|t_0 + x/v_p\right|\right) \\
& + \frac{1}{v_p}\underbrace{\left[\delta(t - 2t_0 + x/v_p) - \delta(t + x/v_p)\right]}_{x < -v_p t_0} + \frac{1}{v_p}\underbrace{\left[\delta(t - 2t_0 + x/v_p) + \delta(t - 2t_0 - x/v_p)\right]}_{-v_p t_0 < x < v_p t_0} \\
& + \frac{1}{v_p}\underbrace{\left[\delta(t - 2t_0 - x/v_p) - \delta(t - x/v_p)\right]}_{v_p t_0 < x} \Bigg\} .
\end{aligned}
$$

When we expand the δ functions in (D-4) in the intervals $x < -v_p t_0$, $-v_p t_0 < x < v_p t_0$ and $v_p t_0 < x$, we obtain

(D-9)
$$\begin{aligned}
\delta\left(t - t_0 - \left|t_0 - x/v_p\right|\right) &= \delta\left(t - 2t_0 + x/v_p\right), & v_p t_0 > x \\
&= \delta\left(t - x/v_p\right), & v_p t_0 < x \\
\delta\left(t - t_0 - \left|t_0 + x/v_p\right|\right) &= \delta\left(t - 2t_0 - x/v_p\right), & -v_p t_0 < x \\
&= \delta\left(t + x/v_p\right), & -v_p t_0 > x .
\end{aligned}$$

Finally, upon adding (D-4) (after substituting (D-9)) and (D-8), we obtain our final result:

(D-10)
$$\begin{aligned}
v(x,t) &= \frac{1}{2}\left[\delta(v_p t - x) + \delta(v_p t + x)\right] \\
&- \frac{b}{4v_p}\left\{ \frac{J_1\left[b\left((t-t_0)^2 - (t_0 - x/v_p)^2\right)^{1/2}\right]}{\left[(t-t_0)^2 - (t_0 - x/v_p)^2\right]^{1/2}} \left(t - 2t_0 + x/v_p\right) u_{-1}\left(t - t_0 - \left|t_0 - \frac{x}{v_p}\right|\right) \right. \\
&\left. + \frac{J_1\left[b\left((t-t_0)^2 - (t_0 + x/v_p)^2\right)^{1/2}\right]}{\left[(t-t_0)^2 - (t_0 + x/v_p)^2\right]^{1/2}} \left(t - 2t_0 - x/v_p\right) u_{-1}\left(t - t_0 - \left|t_0 + x/v_p\right|\right) \right\}, \\
& \qquad t \geq t_0 .
\end{aligned}$$

Note that this solution agrees with the solution, (2-110)(a), valid for $t < t_0$, at the instant $t = t_0$, thus, ensuring continuity of $v(x,t)$ at $t = t_0$.

B. Asymptotic Expansion of v(x,t).

We start by expressing (D-1) in exponential form

(D-11)
$$\begin{aligned}
v(x,t) = \frac{1}{8\pi}\int_{-\infty}^{\infty} \Big\{ & \exp j\left[k v_p t_0 + \underset{(A)}{(v_p^2 k^2 + b^2)^{1/2}}(t - t_0) - kx\right] \cdot \left(1 + \frac{v_p k}{(v_p^2 k^2 + b^2)^{1/2}}\right) \\
& + \exp j\left[k v_p t_0 - \underset{(B)}{(v_p^2 k^2 + b^2)^{1/2}}(t - t_0) - kx\right] \cdot \left(1 - \frac{v_p k}{(v_p^2 k^2 + b^2)^{1/2}}\right) \\
& + \exp j\left[-k v_p t_0 + \underset{(C)}{(v_p^2 k^2 + b^2)^{1/2}}(t - t_0) - kx\right] \cdot \left(1 - \frac{v_p k}{(v_p^2 k^2 + b^2)^{1/2}}\right) \\
& + \exp j\left[-k v_p t_0 - \underset{(D)}{(v_p^2 k^2 + b^2)^{1/2}}(t - t_0) - kx\right] \cdot \left(1 + \frac{v_p k}{(v_p^2 k^2 + b^2)^{1/2}}\right) \Big\} dk .
\end{aligned}$$

[illegible] phase with respect to t. Thus,

$$\omega_A = \frac{\partial \varphi_A}{\partial t} = \frac{\partial}{\partial t}\left[k v_p t_0 + (v_p^2 k^2 + b^2)^{1/2}(t - t_0) - kx \right] = (v_p^2 k^2 + b^2)^{1/2}$$

(D-12) $$\omega_B = \frac{\partial \varphi_B}{\partial t} = -(v_p^2 k^2 + b^2)^{1/2}$$

$$\omega_C = \frac{\partial \varphi_C}{\partial t} = (v_p^2 k^2 + b^2)^{1/2}$$

$$\omega_D = \frac{\partial \varphi_D}{\partial t} = -(v_p^2 k^2 + b^2)^{1/2}.$$

Let us consider the (A) integral first. The asymptotic expansions of the others follow similarly. We write the (A) integral in the form

(D-13) $$\int_{(A)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[i b(t-t_0)\left\{ \frac{k v_p t_0}{b(t-t_0)} + \left[\left(\frac{v_p k}{b}\right)^2 + 1\right]^{1/2} - \frac{kx}{b(t-t_0)} \right\}\right] \cdot \left(1 + \frac{v_p k}{(v_p^2 k^2 + b^2)^{1/2}}\right) dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i b(t-t_0) g(k,x,t)} \cdot \left(1 + \frac{v_p k}{(v_p^2 k^2 + b^2)^{1/2}}\right) dk,$$

where

(D-14) $$g(k,x,t) = \frac{k v_p t_0}{b(t-t_0)} + \left[\left(\frac{v_p k}{b}\right)^2 + 1\right]^{1/2} - \frac{kx}{b(t-t_0)}.$$

The asymptotic expansion will be valid for $b(t-t_0) \gg 1$, i.e., for times after switching long compared to the reciprocal of the cutoff frequency, b. Equation (D-13) is of the form studied earlier

(D-15) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\, e^{i b(t-t_0) g(k,x,t)}\, dk$$

with

(D-16) $$A(k) = \frac{1}{4}\left(1 + \frac{v_p k}{(v_p^2 k^2 + b^2)^{1/2}}\right).$$

Thus, its asymptotic expansion is

(D-17) $$\int_{(A)} \sim \frac{\left[1 + \frac{v_p k_s}{(v_p^2 k_s^2 + b^2)^{1/2}}\right] e^{i\left(b(t-t_0)g(k_s,x,t) \mp \pi/4\right)}}{4\sqrt{2\pi}\left(b(t-t_0)\left|\frac{d^2 g(k_s,x,t)}{dk^2}\right|\right)^{1/2}},$$

where the (−) sign is used if $\frac{d^2 g(k_s)}{dk^2} > 0$, and the (+) sign if $\frac{d^2 g(k_s)}{dk^2} < 0$.

The stationary point, k_s, is the solution of

(D-18) $$\frac{dg}{dk} = \frac{v_p t_0}{b(t-t_0)} + \frac{v_p^2 k}{b^2} \cdot \frac{1}{\left[\left(\frac{v_p k}{b}\right)^2 + 1\right]^{1/2}} - \frac{x}{b(t-t_0)} = 0$$

or

(D-19) $$(v_p t_0 - x) + \frac{v_p^2 k_s (t-t_0)}{\left[(v_p k_s)^2 + b^2\right]^{1/2}} = 0.$$

Note that if $(v_p t_0 - x) > 0$, then $k_s < 0$, in order that (D-19) be satisfied (recall that $t > t_0$ in all that we are doing here), and, from (D-12), $\omega = \omega_A > 0$. Thus, we conclude that points in space-time, $(v_p t, x)$, such that $x < v_p t_0$, $t > t_0$, are reached by rays of integral (A) on which $k_s < 0$ and $\omega_s > 0$. Similarly, if $x > v_p t_0$, the corresponding rays have $k_s > 0$, $\omega_s > 0$.

The solution of (D-19) for k_s, and the corresponding value of ω_s, are

(D-20)

(a) $$k_s = \frac{-b\,(v_p t_0 - x)}{v_p\left[v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right]^{1/2}}$$

(b) $$\omega_s = (v_p^2 k_s^2 + b^2)^{1/2} = \frac{b\, v_p (t-t_0)}{\left[v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right]^{1/2}}$$

In addition [illegible] $\frac{d^2 g(k_s)}{dk^2}$ is equal to

(D-21) $$\frac{d^2 g(k_s)}{dk^2} = \frac{\left[v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right]^{3/2}}{v_p b^2 (t-t_0)^3},$$

and $A(k_s)$, from (D-16), becomes

(D-22) $$A(k_s) = \frac{1}{4}\left[\frac{x + v_p t - 2 v_p t_0}{v_p (t-t_0)}\right].$$

Hence, the expression for the asymptotic expansion, (D-17), is

(D-23) (a) $$\int_{(A)} \sim \left(\frac{v_p b}{2\pi}\right)^{1/2} \left(\frac{x + v_p t - 2 v_p t_0}{v_p}\right) \cdot \frac{e^{i\left[\frac{k}{v_p}\left(v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right)^{1/2} - \pi/4\right]}}{4\left[v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right]^{3/4}}$$

Similarly

(D-23) (b) $$\int_{(B)} \sim \left(\frac{v_p b}{2\pi}\right)^{1/2} \left(\frac{v_p t + x - 2 v_p t_0}{v_p}\right) \cdot \frac{e^{-i\left[\frac{k}{v_p}\left(v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right)^{1/2} - \pi/4\right]}}{4\left[v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right]^{3/4}}$$

(c) $$\int_{(C)} \sim \left(\frac{v_p b}{2\pi}\right)^{1/2} \left(\frac{v_p t - x - 2 v_p t_0}{v_p}\right) \cdot \frac{e^{i\left[\frac{k}{v_p}\left(v_p^2 (t-t_0)^2 - (v_p t_0 + x)^2\right)^{1/2} - \pi/4\right]}}{4\left[v_p^2 (t-t_0)^2 - (v_p t_0 + x)^2\right]^{3/4}}$$

(d) $$\int_{(D)} \sim \left(\frac{v_p b}{2\pi}\right)^{1/2} \left(\frac{v_p t - x - 2 v_p t_0}{v_p}\right) \frac{e^{-i\left[\frac{k}{v_p}\left(v_p^2 (t-t_0)^2 - (v_p t_0 + x)^2\right)^{1/2} - \pi/4\right]}}{4\left[v_p^2 (t-t_0)^2 - (v_p t_0 + x)^2\right]^{3/4}}$$

[illegible]

$(v_p t_0 - x) \gtrless 0$, $\omega_s < 0$, those associated with Ⓒ satisfy $k_s \gtrless 0$ if $(v_p t_0 + x) \gtrless 0$, $\omega_s > 0$, and those associated with Ⓓ satisfy $k_s \gtrless 0$ if $(v_p t_0 + x) \lessgtr 0$, $\omega_s < 0$.

Upon adding (D-23) we obtain the total asymptotic representation of $v(x,t)$:

$$v(x,t) \sim \left(\frac{v_p b}{2\pi}\right)^{1/2} \cdot \frac{1}{2} \left\{ \frac{(t - 2t_0 + x/v_p) \cos\left[\frac{b}{v_p}\left(v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right)^{1/2} - \pi/4\right]}{\left[v_p^2 (t-t_0)^2 - (v_p t_0 - x)^2\right]^{3/4}} \right. \tag{D-24}$$

$$\left. + \frac{(t - 2t_0 - x/v_p) \cos\left[\frac{b}{v_p}\left(v_p^2 (t-t_0)^2 - (v_p t_0 + x)^2\right)^{1/2} - \pi/4\right]}{\left[v_p^2 (t-t_0)^2 - (v_p t_0 + x)^2\right]^{3/4}} \right\}.$$

This result could also have been obtained simply by inserting their asymptotic expressions for the Bessel functions in (D-10).

V. Experimental Results.

It is always comforting to know that analytical and numerical results can be confirmed experimentally. Conversely, it is a pleasure to be able to explain experimental results by analyzing a mathematical model of the experimental system.

There have been several experimental studies of pulse propagation in waveguides; the one that we shall discuss here is that of pulse compression in an acoustic waveguide.* Figure 2-26 illustrates the arrangement used in the study.

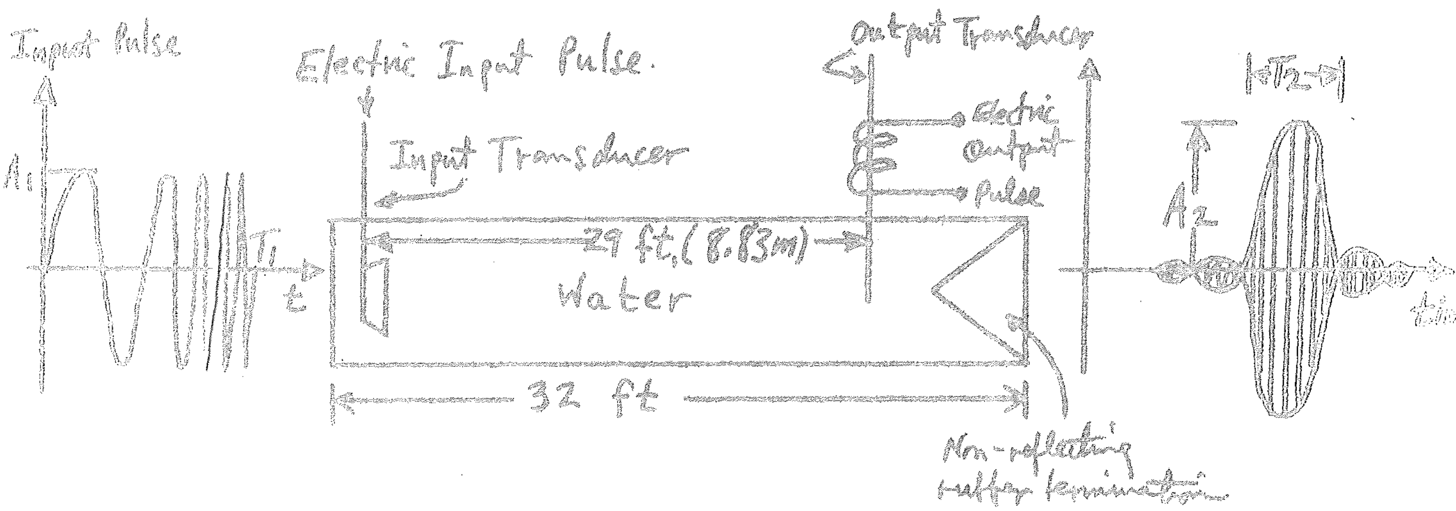

Figure 2-26. Illustrating the acoustic waveguide, FM input electrical pulse and compressed electrical output pulse.

The water filled trough (or acoustic waveguide) carries acoustic, or sound, waves; its propagation characteristics are analogous to those of an electric waveguide. One advantage of using an acoustic waveguide rather than electrical for experiments on transient pulse propagation is that sound waves travel so much slower (5 orders of magnitude) than do electrical that it is easier to measure the critical times and resulting waveshapes associated with acoustical propagation.

* K. Walther, "Pulse Compression in an Acoustic Waveguide", J. Acoust. Soc. of America, Vol. 33, No. 5, May, 1961, pp. 681-686.

The dispersion characteristic of the acoustical waveguide is given in terms of the group delay time, $\tau_g = L/v_g$ where L is the distance from transmitter (input transducer) to receiver (output transducer) and v_g is the group velocity; $v_g = v_p\left(1-(\omega_c/\omega)^2\right)^{1/2}$. For the arrangement of Figure 2-26, L = 8.83 meters, f_c = 13.5 KHz, and v_p = 1473 m/sec. Figure 2-27 (a) is a plot of the group time-delay versus frequency while Figure 2-27 (b) is a plot of group velocity.

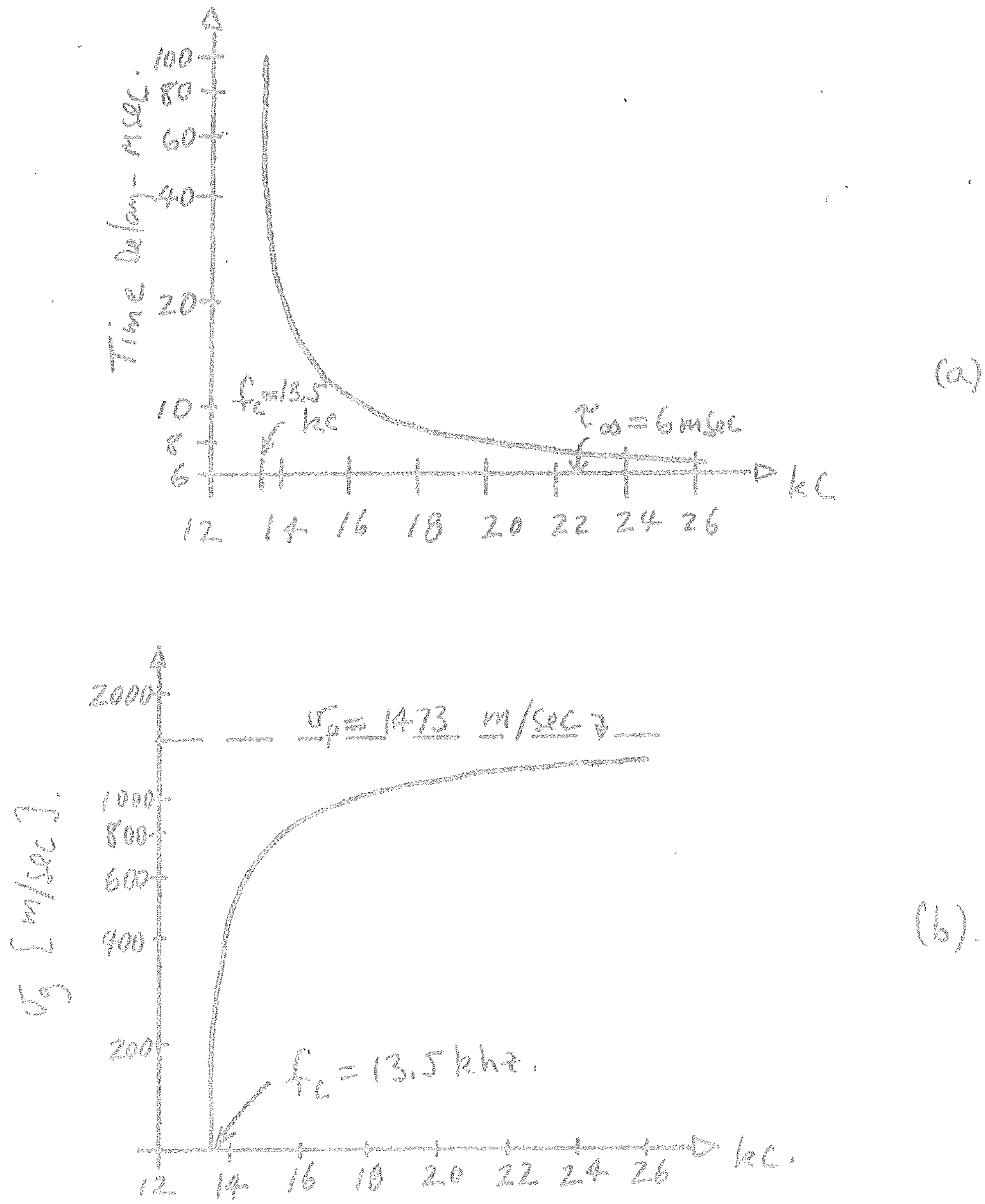

Figure 2-27. Dispersive characteristics of the acoustical waveguide of Figure 2-26. (a) Group time delay for a distance of 8.83 M (29') (b) Group Velocity.

Figure 2-28 illustrates the response of the waveguide, whose dispersion is illustrated in Figure 2-27, when the input consists of a series of rectangular pulses at different carrier frequencies in the range between 13.5 and 20 KH_z. The highly attenuated response at 13.5 KH_z, which happens to be the cut-off frequency of the waveguide, is apparent, as is the sharp rise-time and rather undistorted nature of the pulse at 20 KH_z. The analysis of the last several sections of this chapter indicate why these results must be so; the waveguide responds as does a high-pass filter. In addition the "Fresnel ripples" in the envelope are clearly evident, and so are the precursors, the small responses heralding the arrival of the main signal. The precursors start at 6 msec after the initiation of the input pulse, 6 msec being L/v_p. This is in accord with our earlier theoretical analysis which indicated that the first response of any kind could not arrive with a velocity greater than v_p.

The frequency-modulated input pulse of Figure 2-26 has the instantaneous frequency versus time characteristic of Figure 2-29. This characteristic is suitable for compressing the input pulse at a distance of 8.83m from the source. The frequency range is from 14 to 22 KH_z and the pulse length is 12.6 msec. The pulse characteristic is based on Figure 2-27 (a) and is in agreement with (2-88) when one replaces the normalized variables by their true values.

The compressed output pulses in the waveguide for different distances of the output to input transducers is shown in Figure 2-30. These results are in qualitative agreement with the computer study which resulted in Figure 2-25.

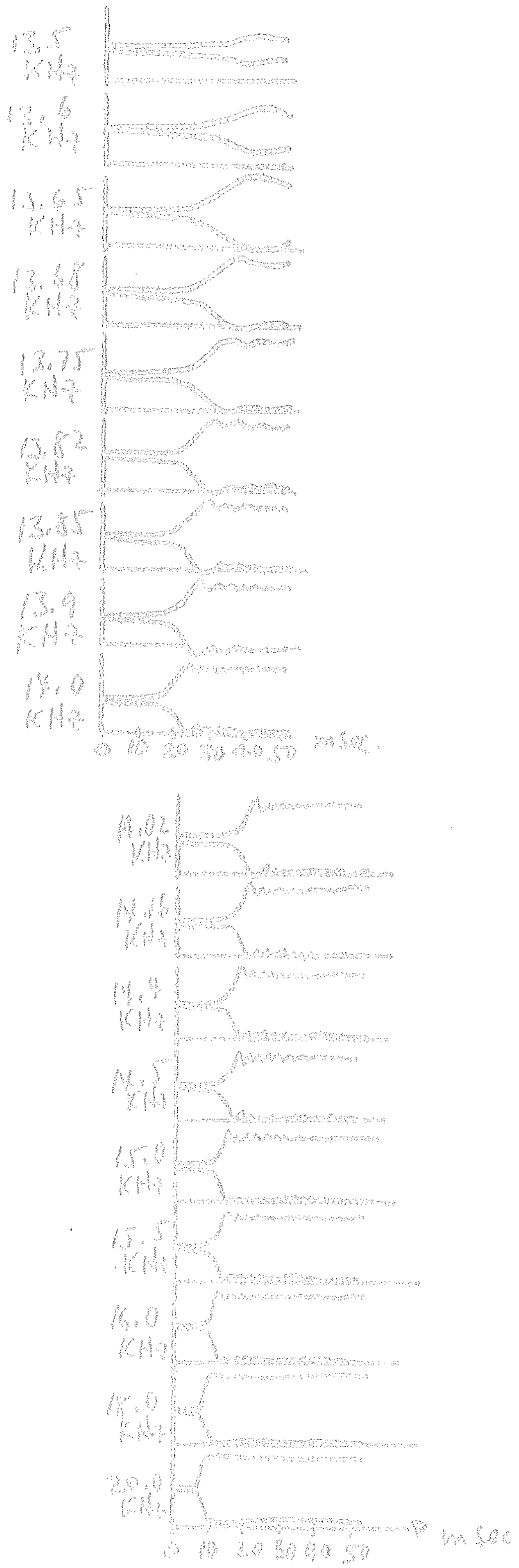

Figure 3-20. Envelope Response to rectangular input pulses at a distance of 8.03m. Time base: 5 msec per division.

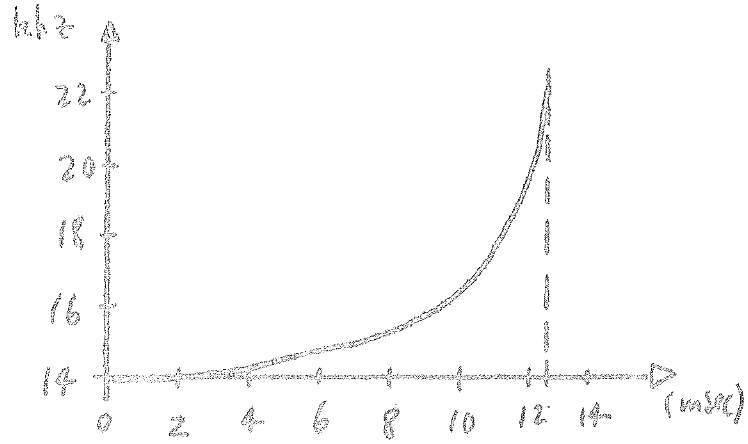

Figure 2-29. Frequency vs. time characteristic of input FM pulse of figure 2-26. The frequency range is 14 to 22 KH_z and the pulse length is 12.6 msec.

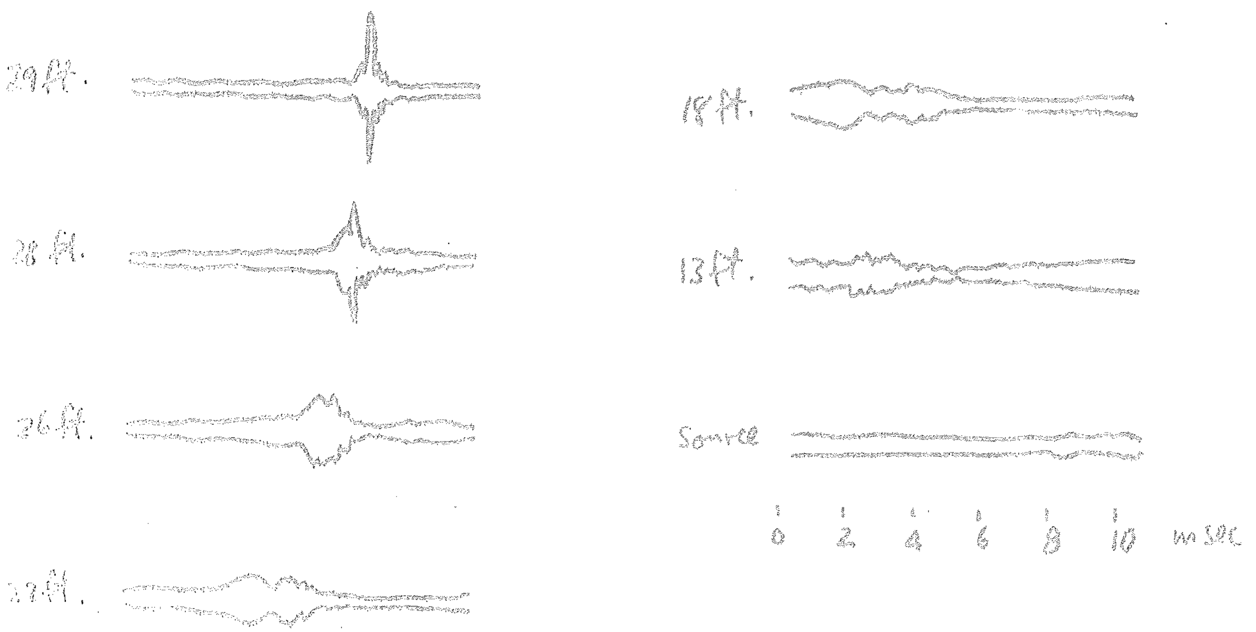

Figure 2-30. Compressed output pulses for different distances of source and receiver transducers. Time base: 1 msec per division.

[illegible] PULSE COMPRESSION*

It is always beneficial to have several alternative ways of interpreting the same phenomena. In Sections 8(B) and 9 we discussed pulse propagation in dispersive media by starting with the convolution integral (2-84), and ultimately ending with numerical results for the propagating waveform (Figure 2-25). A useful alternative description of pulse compression is that of the method of space-time rays first presented in Section 6.

Recall that a space-time ray defines a trajectory in space-time (x, t space) along which a wave packet with fixed center frequency moves. The equation defining a ray is

$$\frac{dx}{dt} = v_g(\omega) = \frac{\partial \omega}{\partial \beta} = \left(\frac{\partial \beta}{\partial \omega}\right)^{-1} . \qquad \text{([illegible])}$$

An important feature of the ray diagram is that the amplitude of the wave propagating in the dispersive system (such as a waveguide) varies inversely with the width of a ray tube (or at least the projection of the width onto the x-axis, see (2-35) and Figure 2-10). Thus, if space-time rays diverge the signal is weakened, whereas if they converge the signal is strengthened. Thus, we may think of pulse compression as the result of space-time rays being focused to a point (x, t) in space-time. We shall return to this property shortly, but let us first dwell on the nature of the frequency modulation of the input signal to have that compression obtain. Again, the dispersive system is taken to be a waveguide whose $\beta(\omega)$ vs. ω diagram is that of Figure 2-11. In Section 2-6 (Figure 2-11) we showed that the direction of the space-time ray corresponding to a wave-packet of (center) frequency ω is that of the normal to the dispersion curve plotted using ω and $v_p\beta$ as the axes, with the positive ω-axis downward (Figure 2-11).

In Figure 2-31 is illustrated the usual dispersion curve for the waveguide, together with an FM-pulse having a decreasing source frequency, and the appropriate ray diagram. Note that the source pulse is represented by an interval along the $v_p t$ -axis because it has a non-zero time duration (i.e., not an impulse) and is located at x=0. In general, we shall take for our source pulse

$$v_s(t) = V_0 \, e^{j\theta(t)} \left[u_{-1}(t) - u_{-1}(t - t_0) \right] , \qquad \text{([illegible])}$$

which is just the complex form of (2-82). The instantaneous frequency $\omega_i(t)$ is given by the time-derivative of $\theta(t)$.

Clearly, Figure 2-31 illustrates the divergence of the rays, that emanate from the interval of the $v_p t$ -axis corresponding to the source pulse, and hence the opposite effect of pulse compression. Thus, for a dispersive system having the dispersion characteristic of a waveguide, an FM-pulse with decreasing frequency cannot be compressed.

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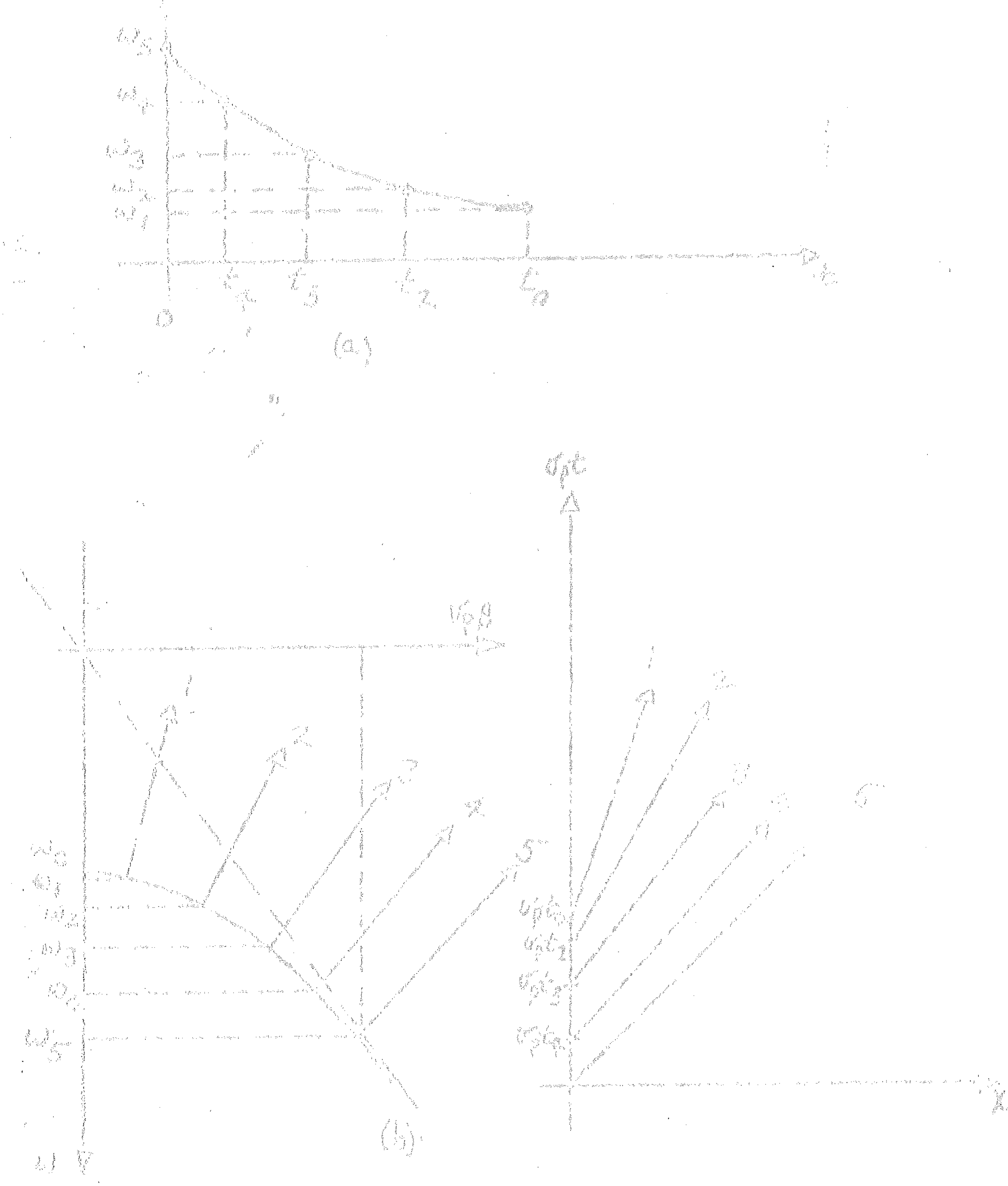
*This section follows L. B. Felsen, "Asymptotic Theory of Pulse Compression in Dispersive Media", IEEE Trans. Ant. Prop., Vol. AP-19, No. 3, May, 1971, pp. 424-432.



7-31. Illustrating the space-time rays produced by a pulse of decreasing frequency, propagating through a [illegible] dispersive system. (a) Source frequency characteristics, (b) space-time rays.

A signal with a decreasing frequency can be compressed, however, in a system having a dispersion curve of opposite curvature to that of Figure 2-31(b). An example is given in Figure 2-32.

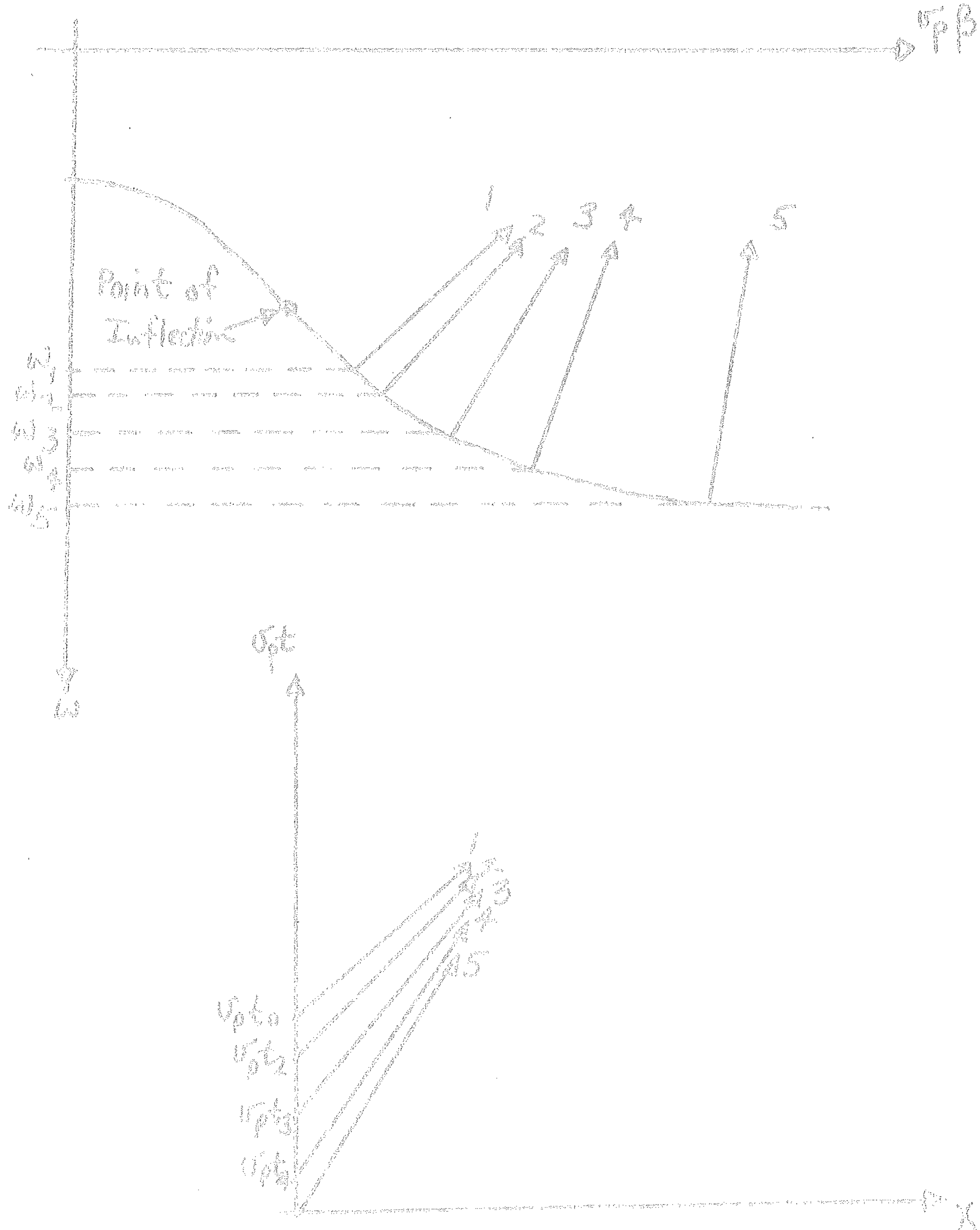


Figure 2-32 Pulse compression does occur for FM signals with decreasing frequency if the dispersion curve has opposite curvature to that of Figure 2-31.

These results tell us whether or not pulse compression is possible, given a dispersion curve and a proposed input source-frequency vs. time characteristic. What we are more interested in is posing the following question and answering it: given a point in space-time (X, T) at which pulse compression is to be achieved, i.e., the space-time rays are to converge at (X, v_pT), and given the dispersion curve, determine the frequency vs. t relationship of the input signal which achieves the requisite compression. This is a problem of <u>signal synthesis</u> and is a vital part in modern communication systems.

In order to answer the question, we apply the preceding graphical constructions in reverse. From the point (X, v_pT) rays are drawn back to the v_pt-axis (x=0 line), intersecting the axis at the various instants of time corresponding to the time instants of the input pulse. Thus, by labeling the rays according to their starting times on the t-axis, we can associate with each of these times its ray, and hence a direction in space-time. We know, however, that directions of rays in space-time are given by normals to the dispersion curve, which means that the rays can also be labeled by frequency (which is the usual way of doing it, and is the say we have done it). Therefore, by searching the dispersion curve for normals lying parallel to the rays we can determine the appropriate frequencies of the rays and, hence, associate the frequencies with their time of occurrence in the source pulse. The method is illustrated in Figure 2-33 for a homogeneous system. If the cut-off frequency, ω_c, varies with x, as in a non-uniform waveguide, we use the same scheme after replacing the inhomogeneous system by a <u>piece-wise homogeneous</u> system (recall Figure 2-13, where we first described the idea of piece-wise homogeneity). Figure 2-34 illustrates the method for an inhomogeneous system.

We would get ideal pulse compression (in which all space-time rays converge to the same point (X, v_pT)), as in Figures 2-33, 2-34, were it not for one thing: the transient rays excited by the sudden beginning and ending parts of the envelope, $[u_{-1}(t) - u_{-1}(t-t_0)]$ of (2-91). Such a pulse generates a broad frequency spectrum, so that the space-time rays emanating from the start and finish of the envelope pulse diverge, as in Figure 2-35. These transient rays account for the pre- and post-cursors surrounding the central pulse of Figure 2-25, and also account for finiteness of the compressed pulse amplitude.

The formal analytical developments used in synthesizing the source signal characteristic start from the ray equation in an inhomogeneous medium,

$$\frac{dx}{dt} = v_g(\omega, x) \tag{2-92}$$

which is simply (2-90) taking into account the variation in group velocity with x. That v_g should depend on x is one manifestation of spatial inhomogeneity.

If we start counting time for a given ray emanating from x=0 at t', then the solution of (2-92) for that ray starting at $(0, v_p t')$ is

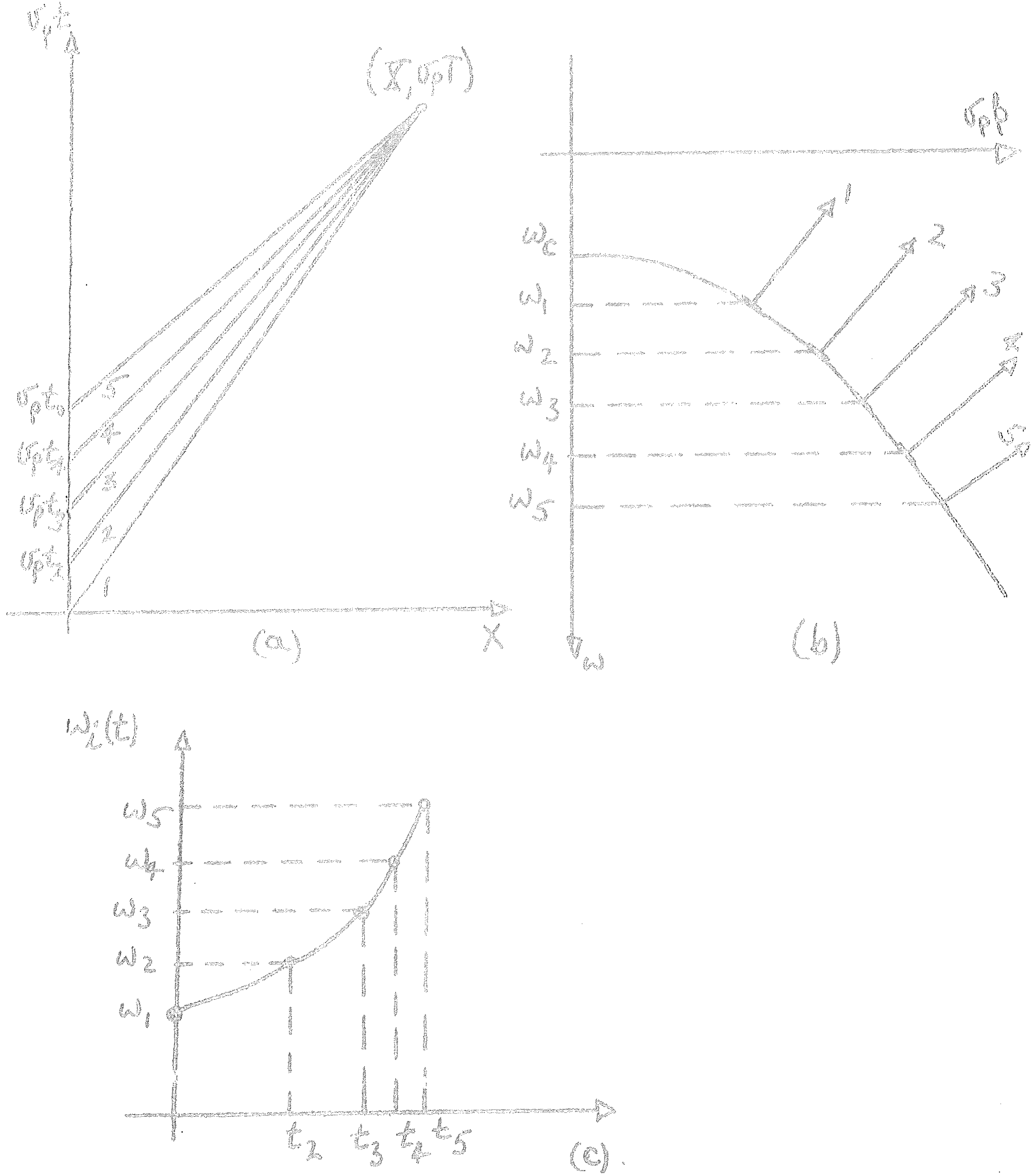


Figure 2-33. Illustrating the determination of the source frequency, $\omega_i(t)$, relation for compression at $(X, v_p T)$ for a homogeneous system. (a) Space-time rays. (b) Normals to dispersion curve. (c) Resultant synthesized signal.

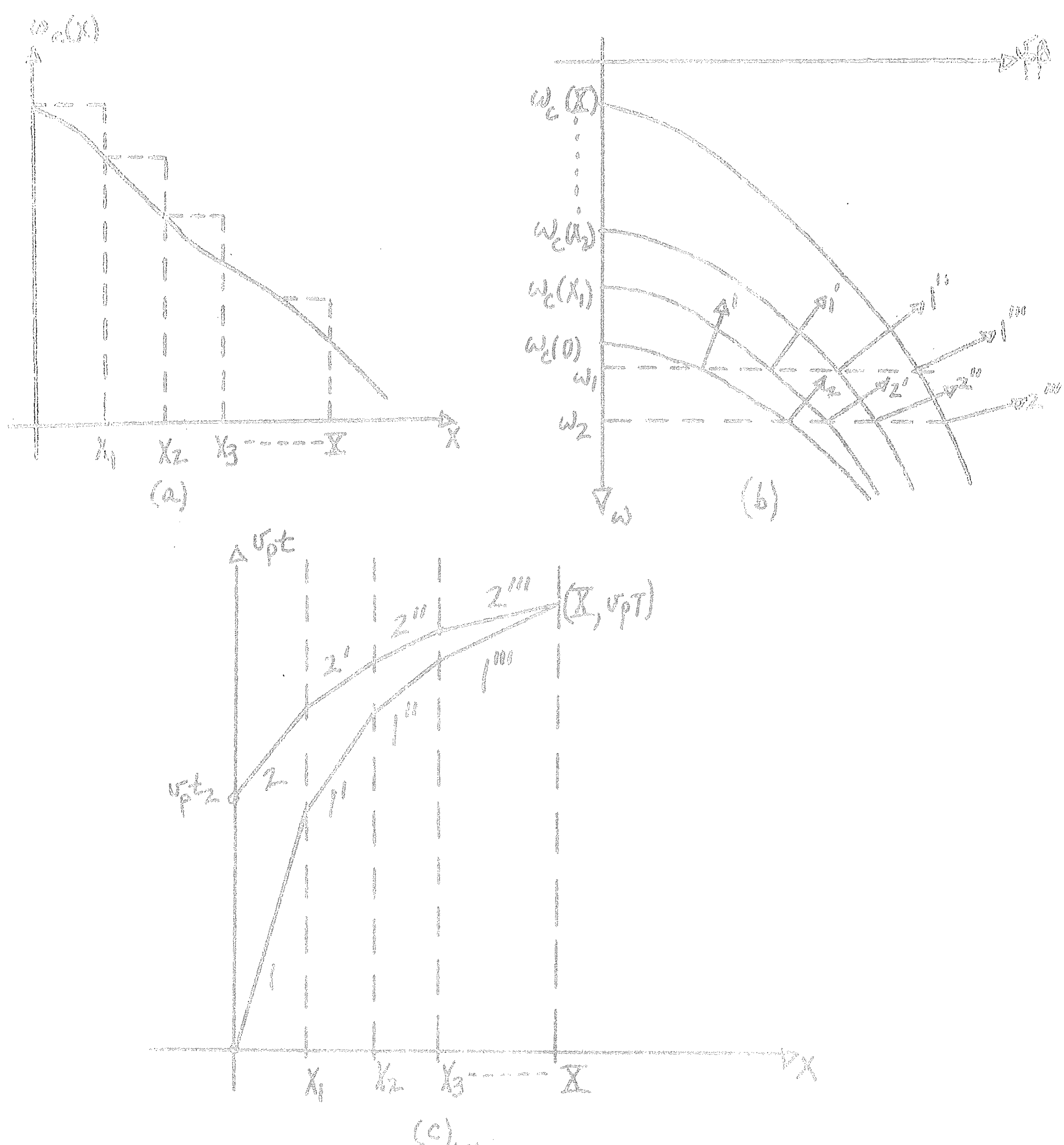


Figure 2-34. Synthesizing $\omega_i(t)$ for an inhomogeneous system. (a) $\omega_c(X)$ and piecewise constant approximation. (b) Dispersion characteristics. (c) Space-time rays.

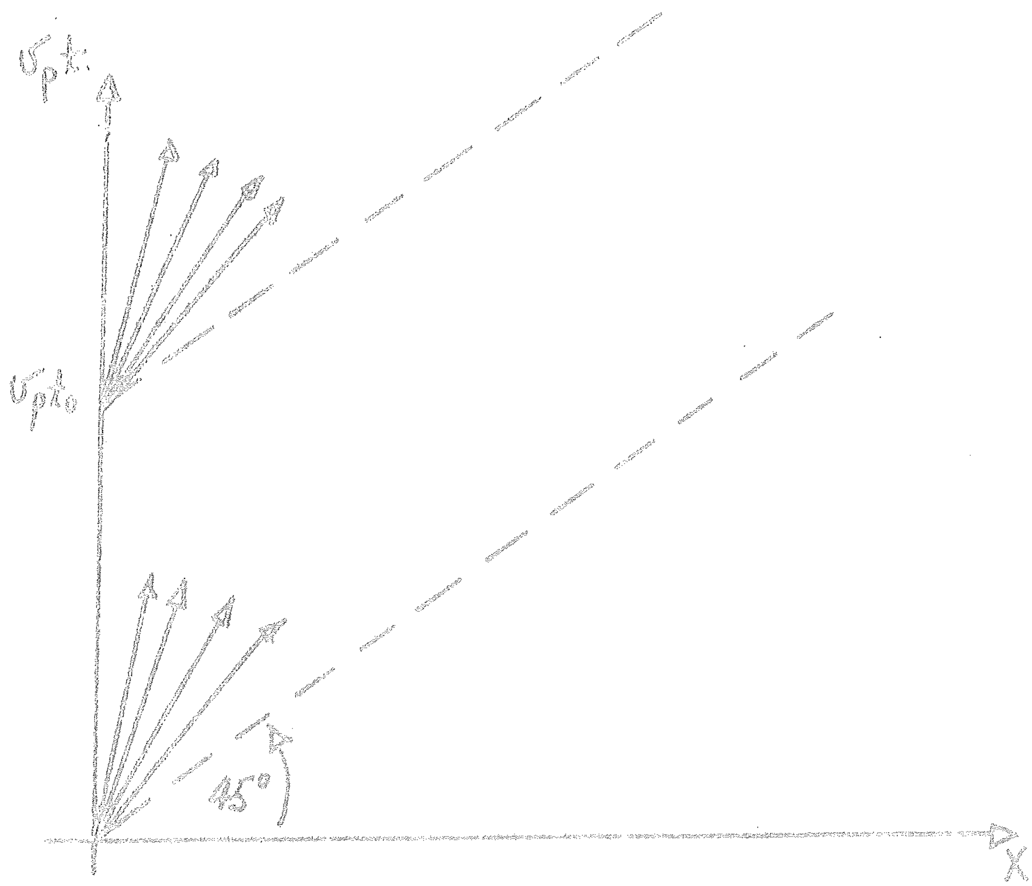


Figure 2-35. Diverging transient rays excited by leading and trailing edges of signal envelope

(2-93) $$t - t' = \int_0^X \frac{dz}{v_g(\omega, z)}$$

Because the ray is to pass through the point $(X, v_p T)$, the integral relation to be satisfied follows from (2-93)

(2-94) $$T - \int_0^X \frac{dz}{v_g(\omega, z)} = t'$$

Knowing the explicit functional relation between v_g, ω and z, (2-94) can be evaluated to determine an implicit relation between ω and t', which, in principle can be solved to obtain the desired relation $\omega_i(t')$ for the instantaneous frequency. For a homogeneous system, in which v_g is independent of x (or z, under the integral signs), the relation has already been derived in Section 8-8(B), equation (2-88), whereas for an inhomogeneous system (2-94) would generally be programmed for a computer. In fact, once the $\omega_i(x)$ characteristic has been approximated by a piecewise constant function, the graphical scheme of Figure 2-34 could be easily implemented on a computer. It is interesting to point out that Figure 2-34 is actually an algorithm for solving (2-94).

11. Time-varying systems

The simplest electrical analog of a dispersive time-variable distributed system is that of the waveguide transmission-line in which the shunt inductances, $\frac{L}{k_c^2 dx}$, of Figure 2-36 varies in time because of a variation in the parameter k_c^2. We will use this model in our study (albeit, a short one) of wave propagation in time-variable systems. Such a study is of interest for applications involving microwave modulation and parametric effects, magnetoelastic delay lines and signal processors, ionized plasmas, nuclear explosions, etc. We will find that many concepts previously derived in this chapter are readily applicable and give much insight into the nature and effects of time variable wave propagation.

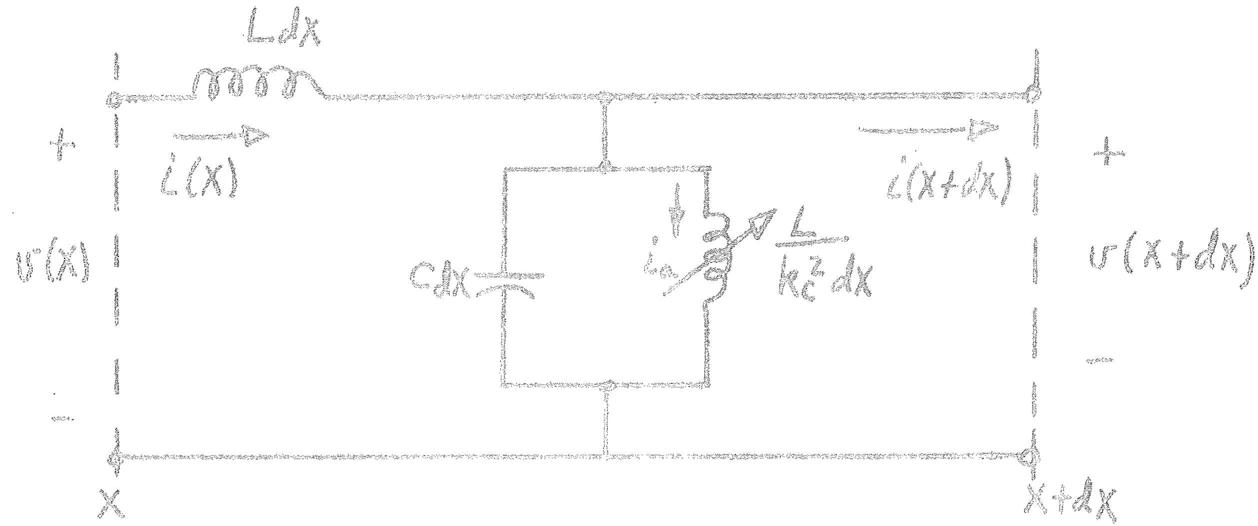


Figure 2-36. A time-variable transmission-line.

Let us proceed to derive the appropriate differential equations. Kirchoff's voltage law yields the usual: $\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}$. We must spend a little more time on the second equation. Kirchoff's current law yields the following equation

$$i(x) - i(x+dx) = C\,dx\,\frac{\partial v(x+dx,t)}{\partial t} + i_a$$

where i_a, the current through the variable inductor and v(x+dx) are related by

$$v(x+dx) = \frac{\partial}{\partial t}\left[\frac{L}{k_c^2\,dx}\,i_a\right].$$

The term in the square brackets, being the product of inductance and current is flux linkage. Hence, this equation is Faraday's law, upon which the volt-amp. differential equation of an inductor is founded.

The right-hand side of the last equation is expanded to give

$$v(x+dx) = (L/k_c^2 dx)\, \partial i_a/\partial t + (i_a/dx)\frac{\partial}{\partial t}\left[L/k_c^2\right]$$

if $\frac{1}{i_a}\cdot\frac{\partial i_a}{\partial t} >> \frac{1}{(L/k_c^2)}\cdot\frac{\partial}{\partial t}\left[\frac{L}{k_c^2}\right]$, then the first term on the right-hand side dominates the second. This condition is equivalent to saying that the normalized time-rate-of-change of current (or frequency of the current) is much greater than that of the varying inductance. We shall make that assumption, although in one of our examples we will consider an abrupt change of the inductance to occur, and this technically violates our basic assumption.

Therefore, upon invoking the slowly-varying (or adiabatic) condition, we have

$$dx \cdot v(x+dx) = \frac{L}{k_c^2(t)} \cdot \frac{\partial i_a}{\partial t},$$

where we are now explicitly stating that k_c^2 is time dependent. When we go back to the first equation for Kirchoff's current law and differentiate it with respect to time, there results

$$\frac{\partial}{\partial t}\left[i(x) - i(x+dx)\right] = C\,dx \cdot \frac{\partial^2 v(x+dx,t)}{\partial t^2} + \frac{\partial i_a}{\partial t}.$$

We now substitute the result obtained above, $\partial i_a/\partial t = \frac{k_c^2(t)}{L} dx\, v(x+dx)$,

and get

$$\frac{\partial}{\partial t}\left[i(x) - i(x+dx)\right] = C\,dx \cdot \frac{\partial^2 v(x+dx,t)}{\partial t^2} + \frac{k_c^2(t)}{L} dx \cdot v(x+dx).$$

Division by dx, with subsequent passage to the limit, $dx \to 0$, yields our second transmission-line equation

$$-\frac{\partial}{\partial t}\left(\frac{\partial i}{\partial x}\right) = C\frac{\partial^2 v}{\partial t^2} + \frac{k_c^2(t)}{L} v(x).$$

When we substitute in the result $\frac{1}{L}\partial^2 v/\partial x^2 = -\partial^2 i/\partial t\,\partial x$

obtained from the very first (Kirchoff's voltage law) equation we reach our goal:

(2-95) $$\frac{\partial^2 v}{\partial x^2} = \frac{1}{v_p^2}\cdot\frac{\partial^2 v}{\partial t^2} + k_c^2(t)\, v,$$

where $v_p^2 = 1/LC$. Clearly, in our model, the cut-off wave number, k_c, varies with time and introduces a variable coefficient into the wave equation, (2-95).

In order to see the connection of (2-95) with the preceding parts of this chapter, we seek a solution for v(x,t) that can be written as a spatial Fourier transform

(2-96) $$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(k,t)e^{-jkx}\,dk \quad ,$$

This differs from the ordinary Fourier transform in that our exponent involves x rather than t. For this reason, k, which has the usual dimensions of (length)$^{-1}$ for the wave number, is often called the spatial frequency. Equation (2-96) simply states that the most general function of x can be synthesized by summing the propagation factor, e^{-jkx}, over all wave numbers (or spatial frequencies). Because $k = 2\pi/\lambda$ where λ is wavelength, it follows that we can synthesize any spatial function by summing elementary space waves over all possible wavelengths. Sometimes (2-96) is called the plane-wave spectral representation of v(x,t). What equation does the space spectrum, V(k,t), satisfy? This is very easily determined by calculating each part of (2-95) by using (2-96). Thus:

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} -k^2 V(k,t)e^{-jkx}\,dk$$

$$\frac{1}{v_p^2}\frac{\partial^2 v}{\partial t^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{v_p^2}\cdot\frac{\partial^2 V(k,t)}{\partial t^2}e^{-jkx}\,dk$$

(2-97) $$k_c^2(t)v = \frac{1}{2\pi}\int_{-\infty}^{\infty} k_c^2(t)\,V(k,t)\,e^{-jkx}\,dk$$

and upon substituting these results into (2-95) there obtains

(2-98) $$\frac{\partial^2 v}{\partial x^2} - \frac{1}{v_p^2}\frac{\partial^2 v}{\partial t^2} - k_c^2(t)v =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-(k^2+k_c^2(t))V(k,t) - \frac{1}{v_p^2}\frac{\partial^2 V(k,t)}{\partial t^2}\right]e^{-jkx}\,dk$$

$$= 0.$$

Since this must hold for every x, we can simply equate the factor multiplying e^{-jkx} in the integrand to zero, and thereby deduce the desired result:

$$\frac{d^2 V(k,t)}{dt^2} + v_p^2\left[k^2 + k_c^2(t)\right]V(k,t) = 0. \tag{2-99}$$

Upon comparing (2-99) with (2-95), we see that we have effectively replaced $\partial^2/\partial x^2$ by $-k^2$, in analogy with the usual replacement in Fourier analysis of $\partial^2/\partial t^2$ by $-\omega^2$.

Equation (2-99) is of the form

$$\frac{d^2 V}{dt^2} + \omega^2(k,t)V = 0 \quad ; \quad \omega^2(k,t) = v_p^2 k^2 + v_p^2 k_c^2(t) = v_p^2 k^2 + \omega_c^2(t) \tag{2-100}$$

This equation is of precisely the form discussed in Appendix C, if we assume that $\omega(k,t)$ is slowly varying, which assumption was made previously in deriving the final form of the transmission-line wave equation, (2-95). Thus, the solution of (2-100) is the WKBJ approximation

$$V(k,t) = \frac{A}{[\omega(k,t)]^{1/2}} e^{j\int_{t'}^{t}\omega(k,\eta)d\eta} + \frac{B}{[\omega(k,t)]^{1/2}} e^{-j\int_{t'}^{t}\omega(k,\eta)d\eta}, \tag{2-101}$$

where A and B are arbitrary constants which must be determined from the initial conditions. Of course, this approximation fails where $\omega(k,t)=0$.

<u>Example 1.</u> Solve (2-100) for the case of an <u>abrupt</u> change in ω_c from $\omega_c = a$, $t < t_0$ to $\omega_c = b$, $t > t_0$, where a and b are constants. The initial condition at t=0 is that $V(k,0)=1$ and $\frac{dV(k,0)}{dt}=0$. The significance of these initial conditions is that there is no initial rate of change of capacitor voltage (see Fig. 2-36), which means that the initial capacitor current is zero, <u>but the initial distribution of voltage along the line is a delta function concentrated at x=0</u>, $v(x,0) = \delta(x)$, because the Fourier transform of $\delta(x)$ is the constant unity, and this agrees with the initial value given above for the Fourier transform $V(k,t)$ of $v(x,t)$ at t=0.

<u>Solution:</u> For $t < t_0$, we must solve

$$\frac{d^2 V(k,t)}{dt^2} + (v_p^2 k^2 + a^2)V(k,t) = 0, \quad V(k,0) = 1, \quad \frac{dV}{dt}(k,0) = 0. \tag{2-102}$$

The solution of (2-102) which satisfies the initial conditions is easily found to be

$$V(k,t) = \cos\left(v_p^2k^2+a^2\right)^{1/2}t \quad , \quad t < t_0. \tag{2-103}$$

For $t > t_0$, we must solve the same equation with a replaced by b:

$$\frac{d^2V(k,t)}{dt^2} + \left(v_p^2k^2+b^2\right)V(k,t) \tag{2-104}$$

with the condition at $t = t_0$ that $V(k,t)$ and $\frac{d}{dt}V(k,t)$ must be continuous at $t = t_0$. V must be continuous because it, being a capacitor voltage, cannot change instantly unless excited by a current impulse. From (2-95) we see that $\partial V/\partial t$ cannot change instantly because that would imply delta function behavior for either $\partial^2V/\partial t^2$ or $k_c^2(t)\,V$, neither of which is permissible, unless $k_c^2(t)$ behaved as a delta function, but that is not the case here: $k_c^2(t)$ suffers only a step change.

It is no problem to satisfy (2-104):

$$V(k,t) = A\sin\left(v_p^2k^2+b^2\right)^{1/2}(t-t_0) + B\cos\left(v_p^2k^2+b^2\right)^{1/2}(t-t_0), \quad t \geq t_0.$$

At $t = t_0$ this result must equal (2-103):

$$\cos\left(v_p^2k^2+a^2\right)^{1/2}t_0 = B,$$

and the derivatives of the two solutions must also be matched at $t = t_0$

$$-\left(v_p^2k^2+a^2\right)^{1/2}\sin\left(v_p^2k^2+a^2\right)^{1/2}t_0 = \left(v_p^2k^2+b^2\right)^{1/2}A$$

or

$$A = -\frac{\left(v_p^2k^2+a^2\right)^{1/2}}{\left(v_p^2k^2+b^2\right)^{1/2}}\sin\left(v_p^2k^2+a^2\right)^{1/2}t_0.$$

Thus, our solution is

$$V(k,t) = \cos\left(v_p^2k^2+a^2\right)^{1/2}t, \qquad 0 \leq t \leq t_0$$

(2-105)

$$V(k,t) = \cos\left(v_p^2k^2+a^2\right)^{1/2}t_0 \cdot \cos\left(v_p^2k^2+b^2\right)^{1/2}(t-t_0) - \frac{\left(v_p^2k^2+a^2\right)^{1/2}}{\left(v_p^2k^2+b^2\right)^{1/2}}\sin\left(v_p^2k^2+a^2\right)^{1/2}t_0 \cdot \sin\left(v_p^2k^2+b^2\right)^{1/2}(t-t_0), \quad t_0 \leq t.$$

In this example we did not need to use the WKBJ approximation because for either of the regions $t \gtrless t_0$, none of the coefficient functions varied with time. That is why we could also have solved this problem by using Laplace transforms on (2-100). The Laplace transform, however, could not have been used on (2-100) if $\omega^2(t)$ varied with time for either $t \gtrless t_0$, as in the next example.

Example 2. Solve (2-100) when $\omega_p(t)$ varies as

(2-106)
$$\omega_p(t) = b\exp(v_p t/h), \quad t > 0$$
$$= 0, \quad t < 0$$

b and h are constants. Dual initial conditions prevail compared to Example 1. (Explain why these initial conditions are the duals of those in Example 1).

Solution:

We must solve

$$\frac{d^2V}{dt^2} + (v_p^2 k^2 + b^2 e^{2v_p t/h})V = 0 \; ; \quad \left.\begin{array}{l} V(k,0) = 0 \\ \dfrac{dV(k,0)}{dt} = 1 \end{array}\right\} \text{Initial Conditions}$$

If, as assumed, the time constant, $h/2v_p$, is long enough, then the coefficient function $\omega^2(t) = (v_p^2 k^2 + b^2 \exp(2v_p t/h))$ is slowly enough varying to permit the WKBJ approximation (2-101):

(2-107)
$$V(k,t) = A\,\frac{\exp\left[j\int_0^t [v_p^2 k^2 + b^2 e^{2v_p\eta/h}]^{1/2} d\eta\right]}{(v_p^2 k^2 + b^2 e^{2v_p t/h})^{1/4}} + B\,\frac{\exp\left[-j\int_0^t (v_p^2 k^2 + b^2 e^{2v_p\eta/h})^{1/2} d\eta\right]}{(v_p^2 k^2 + b^2 e^{2v_p t/h})^{1/4}} .$$

The boundary condition V(k,0) = 0 implies that $0 = A + B \Rightarrow A = -B$ while the condition $\left.\frac{dV(k,t)}{dt}\right|_{t=0} = 1$ may be easily applied when we recall (Appendix C, (C-12)) that

$$\frac{d}{dt}\left(\frac{e^{\pm j\int_0^t \omega(k,\eta)d\eta}}{(\omega(t))^{1/2}}\right) \simeq \pm j(\omega(t))^{1/2} e^{\pm j\int_0^t \omega(k,\eta)d\eta} .$$

Thus, upon applying this result to (2-107), there obtains

$$jA(v_p^2 k^2 + b^2)^{1/4} - jB(v_p^2 k^2 + b^2)^{1/4} = 1,$$

or, by virtue of the fact that A = -B,

$$A = \frac{1}{j2(v_p^2 k^2 + b^2)^{1/4}} = -B.$$

Upon making use of these results, the final form of (2-107) is

$$\overline{j\,2(v_p^2k^2+\omega_b^2)^{1/4}(v_p^2k^2+\omega_b^2e^{2v_pt/h})^{1/4}}$$

(2-108)

$$= \frac{\sin\left(\int_0^t (v_p^2k^2+\omega_b^2e^{2v_p\eta/h})^{1/2}d\eta\right)}{(v_p^2k^2+\omega_b^2)^{1/4}(v_p^2k^2+\omega_b^2e^{2v_pt/h})^{1/4}},$$

Once $V(k,t)$ has been determined one can compute the voltage, $v(x,t)$, by substituting into (2-96). We shall demonstrate such a calculation by using the result of Example 1 with $a = 0$. If a vanishes then $\omega^2(k)$ in (2-100) is given by $\omega^2(k)=v_p^2k^2$, which is the dispersion relation of a non-dispersive, ideal transmission-line. Hence, we will compute the voltage $v(x,t)$ along a transmission-line (or other wave propagating system) which suddenly becomes dispersive.

We start with the result, (2-105), with $a = 0$:

$$V(k,t) = \cos(v_pk)t, \qquad 0 \le t \le t_0$$

(2-109)
$$V(k,t) = \cos(v_pk)t_0\cdot\cos(v_p^2k^2+\omega_b^2)^{1/2}(t-t_0)$$

$$-\frac{v_pk\sin(v_pk)t_0}{(v_p^2k^2+\omega_b^2)^{1/2}}\sin(v_p^2k^2+\omega_b^2)^{1/2}(t-t_0), \qquad t_0 \le t,$$

and substitute this result into (2-96). Let us work with each region, $0 \le t \le t_0$, and $t_0 \le t$, separately. Thus, consider

(2-110) (a)
$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(v_pk)t\,e^{-jkx}\,dk$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(e^{jk(v_pt-x)}+e^{-jk(v_pt+x)}\right)dk$$

$$= \frac{1}{2}\left(\delta(v_pt-x)+\delta(v_pt+x)\right), \qquad 0 \le t \le t_0,$$

where we have used the definition: $\delta(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{\pm jkx}\,dk.$ *

* See G. R. Cooper and C. D. McGillem, Methods of Signal and System Analysis, Holt, Rinehart and Winston, New York, 1967, Chapt. 5.

[illegible] at x = 0, will split in half, partly propagating to the right and partly to the left, each with velocity v_p. This is consistent with our results in Chapter 1.

The evaluation of $v(x, t)$ [illegible] for $t > t_0$ is a little bit more involved. We will leave the details to Appendix D and present the results here:

$$
\begin{aligned}
v(x,t) = {} & \frac{1}{2}\left[\delta(v_p t - x) + \delta(v_p t + x)\right] \\
& - \frac{b}{4v_p}\left\{ \frac{J_1\left[b\left((t-t_0)^2 - (t_0 - x/v_p)^2\right)^{1/2}\right]}{\left[(t-t_0)^2 - (t_0 - x/v_p)^2\right]^{1/2}}\,(t - 2t_0 + x/v_p)\, u_{-1}(t - t_0 - |t_0 - x/v_p|) \right. \\
& \left. + \frac{J_1\left[b\left((t-t_0)^2 - (t_0 + x/v_p)^2\right)^{1/2}\right]}{\left[(t-t_0)^2 - (t_0 + x/v_p)^2\right]^{1/2}}\,(t - 2t_0 - x/v_p)\, u_{-1}(t - t_0 - |t_0 + x/v_p|) \right\}
\end{aligned}
\qquad (2\text{-}110)\ (b)
$$

where $J_1(x)$ is the ubiquitous Bessel function of the first kind, order unity.

Upon comparing (2-110)(a)-(b), we note the presence of the delta functions, whose appearance is not surprising in-as-much as we are finding the impulse response of a non-dispersive-dispersive system and our previous work on impulse responses of such systems, taken separately, had already indicated the presence of the propagating delta functions. Note that they travel with velocity, v_p, both before and after switching. The time-variable system, therefore, appears perfectly transparent to them and they go merrily along as though nothing happened.

Obviously, something has happened, however, and the sudden switching of the state of the system from non-dispersive to dispersive has produced the aforementioned Bessel function "tail".

We are usually interested in the asymptotic response as a function of time for fixed x, i.e., the response as $t \to \infty$. Equation (2-110) (b) indicates that the asymptotic response becomes

$$v(x,t) \sim J_1(bt) \qquad (2\text{-}111)$$

as t becomes large. But the asymptotic form of the Bessel function was given in Chapter I and is

$$J_1(bt) \sim \frac{\cos(bt - 3\pi/4)}{(bt)^{1/2}}, \qquad (2\text{-}112)$$

which indicates that the voltage response "rings" with angular frequency b, the cut-off frequency after switching at $t = t_0$, and decays as $t^{-1/2}$. These results agree with the previously derived asymptotic response of a dispersive (waveguide) transmission-line to an impulse.

Application of stationary phase methods to the determination of the asymptotic expansion will be included in Appendix D, also.

What is more interesting is the nature of the "transmitted" and "reflected" waves at a temporal discontinuity. Thus, consider the two terms of (2-110) (b) containing the unit step-functions, $u_{-1}(t-t_0-|t_0-x/u_p|)$ and $u_{-1}(t-t_0-|t_0+x/u_p|)$. The unit step-function, of course, vanishes when its argument goes negative and is unity for positive values of its argument. Let us, therefore, inquire as to the sign of the arguments of the two unit steps above. Consider first $u_{-1}(t-t_0-|t_0-x/u_p|)$, with argument $t-t_0-|t_0-x/u_p|$. This argument can be broken into two expressions depending on the sign of t_0-x/u_p:

$$
\begin{aligned}
\text{(2-113) (a)} \quad t-t_0-|t_0-x/u_p| &= t-t_0-(t_0-x/u_p) = t-2t_0+x/u_p\,, & t_0-x/u_p > 0 \\
&= t-t_0-(x/u_p-t_0) = t-x/u_p\,, & t_0-x/u_p < 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\text{(2-113) (b)} \quad t-t_0-|t_0+x/u_p| &= t-2t_0-x/u_p\,, & t_0+x/u_p > 0 \\
&= t+x/u_p\,, & t_0+x/u_p < 0.
\end{aligned}
$$

As a function of t, therefore, there are four characteristic transition-lines corresponding to the discontinuities of the unit step-functions of (2-110) (b):

$$
\text{Transition-lines:} \quad
\begin{aligned}
t &= 2t_0 - x/u_p\,, & t_0 &> x/u_p \\
t &= x/u_p\,, & t_0 &< x/u_p \\
t &= 2t_0 + x/u_p\,, & t_0 &> -x/u_p \\
t &= -x/u_p\,, & t_0 &< -x/u_p\,,
\end{aligned}
\qquad \text{(2-114)}
$$

with the first pair corresponding to the step-function $u_{-1}(t-t_0-|t_0-x/u_p|)$ and the second pair to $u_{-1}(t-t_0-|t_0+x/u_p|)$. The important features of (2-110), including (2-114), are shown in Figure 2-37.

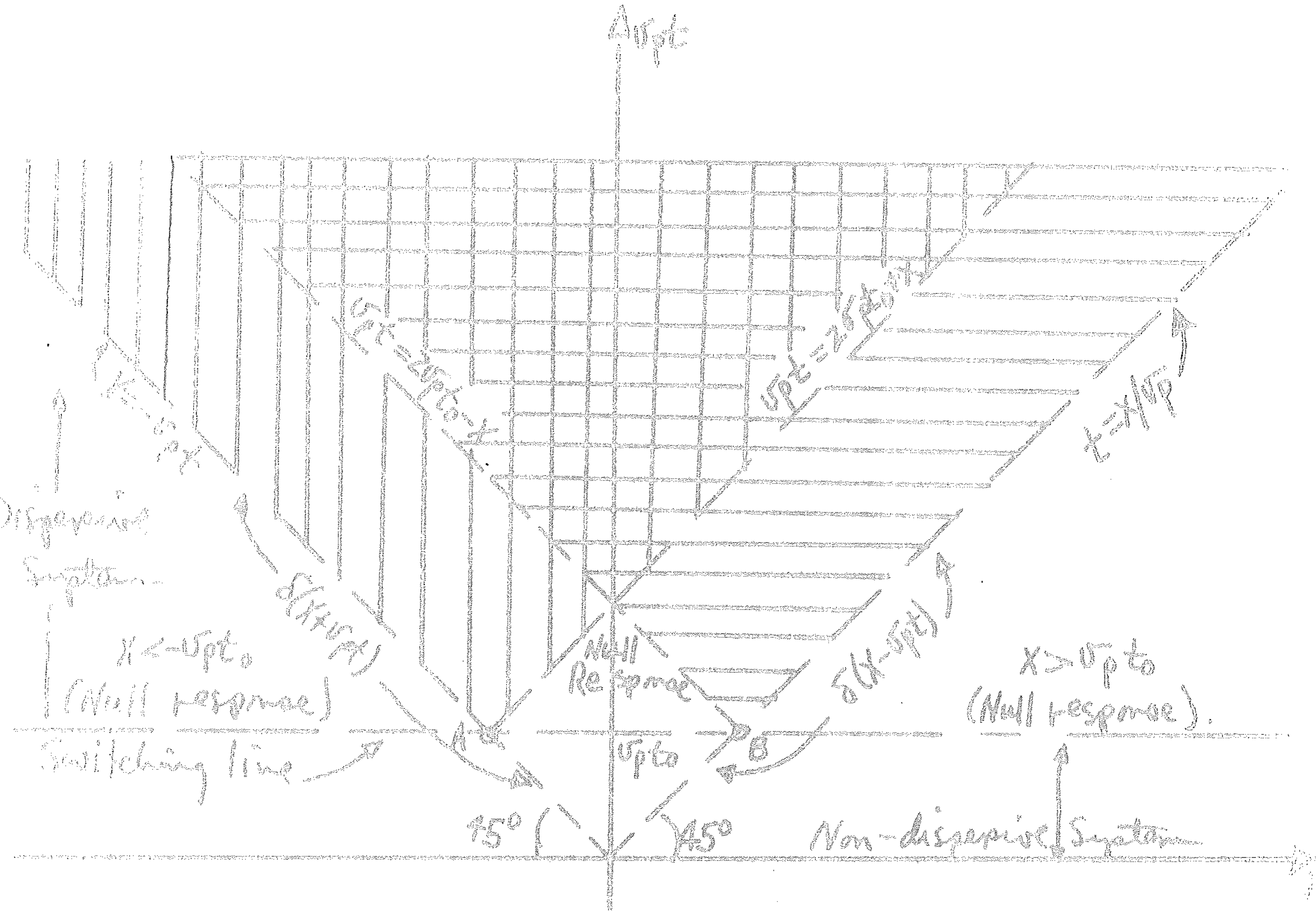


Figure 2-37. Illustrating important regions in (v_pt, x)-space (space-time) for the solution (2-110).

Consider point B, for example. There are two transition-lines emanating from this point, one extending in the (+) x-direction ($x = v_pt$) and one in the (-) x-direction ($v_pt = 2v_pt_0 - x$). The former is the boundary of space-time rays propagating in the (+) x-direction, after switching, and corresponds to <u>transmitted waves</u>. The latter is the boundary of space-time rays propagating in the (-) x-direction, after switching, and corresponds to <u>reflected waves</u>. Thus, we have transmission and reflection, not from a spatial discontinuity, as in the usual case, but from the temporal discontinuity at t_0.

In order to better understand the behavior of the system it is necessary to resort to stationary-phase arguments and the concomitant space-time rays. Thus, let us return to the general expression (2-96) and substitute for V(k,t) the WKBJ expression (2-101) for a general time-variation of the medium

$$v(x,t) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[a(k,t)e^{-j\left(kx-\int_{t'}^{t}\omega(k,\eta)d\eta\right)} + b(k,t)e^{-j\left(kx+\int_{t'}^{t}\omega(k,\eta)d\eta\right)}\Big]d\eta \qquad (2\text{-}115)$$

We recall from our introduction to the stationary-phase principle that when $a(k,t)$, $b(k,t)$ are relatively slowly varying functions of k (compared to the variation of the exponential phase factors) then the major contribution to $v(x,t)$ comes from the vicinity of k-space about the stationary-phase point, k_s, at which the exponents (or phases) are stationary:

$$x \pm \int_{t'}^{t} \frac{\partial \omega(k,\eta)}{\partial k} d\eta = 0 \quad , \quad \text{at} \quad k = k_s . \tag{2-116}$$

We easily determine the differential equations for the space-time rays by differentiating (2-116) with respect to t:

$$\frac{dx}{dt} \pm \frac{\partial \omega}{\partial k}(k_s, t) = 0 . \tag{2-117}$$

Conversely, if we start with (2-117), the solution for the space-time rays is (2-116).

The expression $\partial\omega/\partial k$ is the ubiquitous group velocity and is determined once the dispersion relation $\omega = \omega(k,t)$ is known. Thus, the ray equation (2-117), states that the slope of the space-time ray along which $k = k_s$ is constant is equal to the group-velocity evaluated at $k = k_s$. Note, however, that due to the temporal variations of the system (and, hence, of ω), the group velocity will vary with time, in general.

In the case of a simple discontinuity in the dispersion relation, due to an instantaneous change in the form of ω versus k, as in Example 1, we may decompose the interval of integration in the phase factors of (2-115) into those intervals over which $\omega(k,t)$ is slowly varying and then insist on continuity of phase at the transition point. Thus, if t_0 is a point of discontinuity of $\omega(k,t)$ then we write

$$\begin{aligned} \int_{t'}^{t} \omega(k,\eta) d\eta &= \int_{t'}^{t_0} \omega_1(k,\eta) d\eta + \int_{t_0}^{t} \omega_2(k,\eta) d\eta , \qquad t \geq t_0 \\ &= \int_{t'}^{t} \omega_1(k,\eta) d\eta , \qquad t_1 \leq t \leq t_0 . \end{aligned} \tag{2-118}$$

where $\omega_1(k,t)$ is the dispersion relation prior to switching at t_0, and $\omega_2(k,t)$ is that after. ω_1 and ω_2 are smooth, slowly varying functions of time.

The corresponding equations to (2-116) and (2-117) are

$$\begin{aligned} & x \pm \int_{t'}^{t} \frac{\partial \omega_1}{\partial k}(k,\eta) d\eta = 0, \qquad t' \leq t \leq t_0 \\ & x \pm \int_{t'}^{t_0} \frac{\partial \omega_1}{\partial k}(k,\eta) d\eta \pm \int_{t_0}^{t} \frac{\partial \omega_2}{\partial k}(k,\eta) d\eta = 0, \qquad t_0 \leq t. \end{aligned} \tag{2-119}$$

$$\frac{dx}{dt} \pm \frac{\partial \omega_2}{\partial k}(k,t) = 0, \qquad t_0 \leq t.$$

Thus, it is the instantaneous dispersion curve that determines the ray trajectories in the regions $t \leq t_0$ and $t \geq t_0$.

We must always keep in mind that the ray theory as we have developed it is valid only in an asymptotic sense, i.e., only for distances well away from the source or for large times. Again, "large" or "well away" is usually taken relative to a characteristic number, such as wavelength or frequency.

Example 3: Determine the space-time ray diagram for the system of Figure 2-37.

Solution:

We use (2-119) and (2-120) with $t'=0$, $\omega_1(k,t) = v_p k$, $\omega_2(k,t) = (v_p^2 k^2 + b^2)^{1/2}$, where b is a constant. Thus, the stationary point condition is

(2-121) (a) $x \pm v_p t = 0, \qquad 0 \leq t \leq t_0$

(b) $$x \pm v_p t_0 \pm \frac{v_p^2 k_s (t - t_0)}{(v_p^2 k_s^2 + b^2)^{1/2}} = 0, \qquad t_0 \leq t.$$

The solution for (2-121)(b) is

(2-122) $$k_s(x,t) = \pm \frac{b\,|x \pm v_p t_0|}{v_p\left[v_p^2 (t - t_0)^2 - (x \pm v_p t_0)^2\right]^{1/2}}, \qquad t \geq t_0,$$

The corresponding frequency is

(2-123) $$\omega_{2s}(x,t) = \left(v_p^2 k_s^2(x,t) + b^2\right)^{1/2} = \frac{b v_p (t - t_0)}{\left[v_p^2 (t - t_0)^2 - (x \pm v_p t_0)^2\right]^{1/2}}, \qquad t \geq t_0.$$

The ray equations are

(2-124)

(a) $\frac{dx}{dt} = \pm v_p, \qquad 0 \leq t \leq t_0$

(b) $$\frac{dx}{dt} = \pm \frac{v_p^2 k_s}{(v_p^2 k_s^2 + b^2)^{1/2}} = \pm v_p^2 \left(\frac{k_s}{\omega_s}\right), \qquad t \geq t_0.$$

Because k_s is constant on a given ray, (2-124) (b) indicates that the slope of the ray trajectory changes with a change in frequency. Figure 2.38 illustrates the construction of the ray diagram from this analysis.

[illegible] straight lines, due to the temporal homogeneity, k is fixed with value k_s on the entire ray. Thus, the reflected and transmitted rays at either A or B correspond to the same value of $k(=k_s)$ but have frequencies which are just the negatives of each other.

This latter fact is rather interesting. It indicates that the wave returning to the source location (x=0), i.e., the "reflected" wave, because it oscillates with exactly the negative frequency of the transmitted wave, is the time reversed form of the transmitted wave. In other words, if the transmitted wave has a time dependence, at some point, of f(t), then the reflected wave has time dependence f(-t). The spatial distribution of the wave is unchanged; the reflected wave is simply received in reversed order at x = 0.

The analytical proof is straightforward. Let f(t) have (temporal) Fourier transform, $F(\omega)$, and let g(t) have Fourier transform $F(-\omega)$, i.e., the frequency components of g(t) are just the negative of f(t). We will show that g(t) = f(-t). Thus, by the hypotheses

$$(2\text{-}125) \quad (a)\ f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega, \quad (b)\ g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(-\omega) e^{j\omega t} d\omega .$$

In (2-125) (b), make the change of variable $\xi = -\omega$, so that (2-125) (b) becomes

$$(2\text{-}126) \quad g(t) = \frac{1}{2\pi}\int_{\infty}^{-\infty} F(\xi) e^{-j\xi t}(-d\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi) e^{-j\xi t} d\xi = f(-t)$$

which concludes the proof.

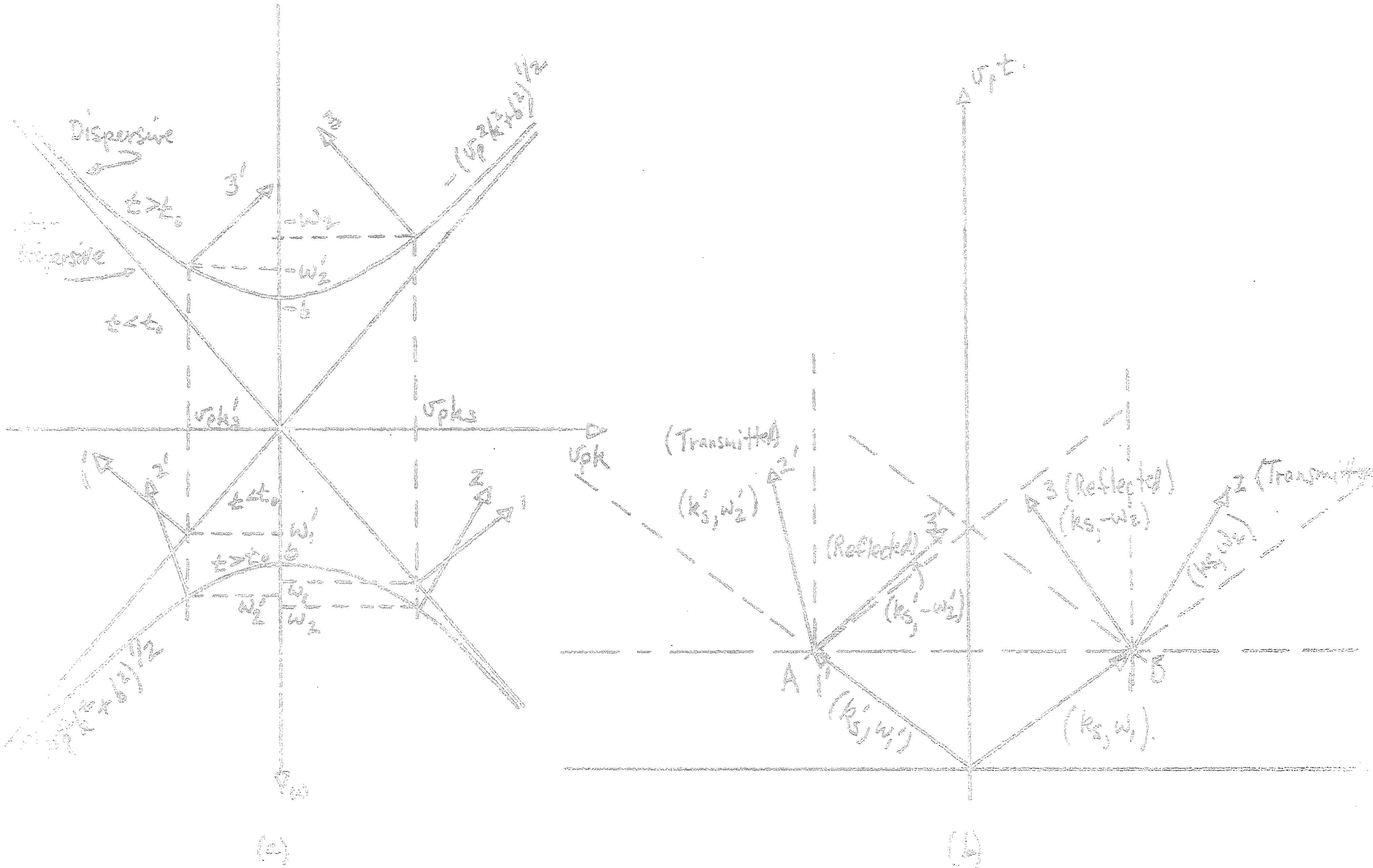
Example 4. Discuss the space-time ray diagram of the slowly-varying system of example 2.

Solution: From the statement of Example 2, we have $\omega_c(t) = b\exp(v_p t/h)$ and $\omega(k,t) = (v_p^2 k^2 + b^2 \exp(2v_p t/h))^{1/2}$.

The ray equation is, by (2-117),

$$(2\text{-}127) \quad \frac{dx}{dt} = \frac{\partial\omega(k_s,t)}{\partial k} = \frac{v_p^2 k_s}{(v_p^2 k_s^2 + b^2 \exp(2v_p t/h))^{1/2}} .$$

Along the ray defined by this differential equation, k_s is treated as a constant valued parameter. We note, however, the explicit dependence of the right-hand side on time, t. This means that the rays are bent, i.e., dx/dt is not constant.



Dispersive
Non-Dispersive
3'
1'
2'
1
2
3
t > t₀
t < t₀
vpk
vpk'₃
vpk₂
A
B
(Transmitted)
(Reflected)
3 (Reflected)
2 (Transmitted)
(a)
(b)

[illegible] dealing with this particular situation. [illegible] is to try to integrate the equation directly and look for a closed form solution, while the second is to approximate the right-hand side by a piece-wise constant function and thereby determine the ray trajectory by a straightline approximation. This example is amenable to both methods.

In order to demonstrate the first method, write (2-127) as

$$(2\text{-}128)\qquad x = \int_0^t \frac{v_p^2 k_s\, d\eta}{\left(v_p^2 k_s^2 + b^2 \exp\left(\frac{2 v_p \eta}{h}\right)\right)^{1/2}},$$

corresponding to the ray starting at the origin at $t = 0$, and make the change of variable $\xi = \left(1 + \frac{b^2}{v_p^2 k_s^2} \exp(2 v_p \eta / h)\right)^{1/2}$.

It is straightforward to transform (2-128) into the form

$$x = h \int_{\eta=0}^{t} \frac{d\xi}{(\xi^2 - 1)} = \frac{h}{2} \int_{\eta=0}^{t} \left[\frac{1}{\xi - 1} - \frac{1}{\xi + 1}\right] d\xi = \frac{h}{2} \ln\left[\frac{\xi - 1}{\xi + 1}\right]\Bigg|_{\eta=0}^{t}$$

$$= \frac{h}{2} \ln\left\{\left[\frac{\xi(k_s,t) - 1}{\xi(k_s,t) + 1}\right] \cdot \left[\frac{\xi(k_s,0) + 1}{\xi(k_s,0) - 1}\right]\right\}.$$

[illegible] $\xi = \frac{\omega}{v_p |k_s|}$, so that the ray trajectory becomes

$$(2\text{-}129)\qquad x = \frac{h}{2} \ln\left\{\frac{\omega(k_s,t) - v_p |k_s|}{\omega(k_s,t) + v_p |k_s|} \cdot \frac{\omega(k_s,0) + v_p |k_s|}{\omega(k_s,0) - v_p |k_s|}\right\}.$$

Incidentally, this can be solved to yield the stationary point k_s as a function of x and t

$$(2\text{-}130)\qquad k_s = \pm \frac{b}{v_p} \frac{\exp(v_p t / 2h) \sinh(x/h)}{\left(2\left[\cosh(v_p t/h) - \cosh(x/h)\right]\right)^{1/2}}.$$

Equation (2-129) can be plotted in $(x, v_p t)$-space, keeping k_s fixed, thereby generating a field of space-time rays (one ray per k_s-value).

Closed form solutions, such as (2-129), are not easily come by in practice and one usually resorts to the second, or approximate, method for determining rays. If we plot $\omega_c(t)$ versus t, and then approximate the smoothly continuous curve by a piecewise constant function, as in Figure 2-39(a), one can then generate a collection of dispersion curves by means of the formula $\omega_n(k) = (v_p^2 k^2 + \omega_{cn}^2)^{1/2}$, where $\{\omega_{cn}\}$

[illegible] of [illegible] in the various sub-intervals shown in Figure 2-39 (a). $\omega_n(k)$ is the dispersion curve corresponding to t in the nth subinterval (Figure 2-39(b)). Now we simply apply our usual construction techniques and generate the straightline approximation to the smoothly bending ray, as in Figure 2-39 (c). This is simply the "finite difference" method of solving the differential equation (2-127), or approximating the integral (2-128).

There are several interesting applications of time-variable systems as described in this section. We shall list a few of them.*

(1) Transient frequency shifting, which is due to shifting the dispersion curves, conserving the wave number, k, during the operation, is a means for frequency sweeping or coding RF pulses.

(2) Variable recall storage, is a means of storing a signal for recall at a later time by converting the input signal into a mode with very low velocity, due to switching the medium, and holding it in that state until recall is desired.

(3) Gating results when a signal is eliminated due to switching the medium from one state to another with greater dissipation.

(4) Time scaling occurs owing to the conservation of the wave vector k. Because the spatial distribution of a signal is conserved when the system is abruptly changed in time, if the group velocity is changed then the modulation envelope will propagate past a spatially fixed detector at a different rate and the observed waveform will have an altered time scale (much like recording a signal and replaying it at a different rate).

(5) Time reversal is virtually the same phenomenon as time scaling except that (4) uses a transmitted wave whereas (5) utilizes the reflected wave. Since the reflected wave will have the same spatial distribution as the incident wave (because k is conserved) but travels in the opposite direction, the waveform observed at a detector fixed near the launching point will be reversed in time.

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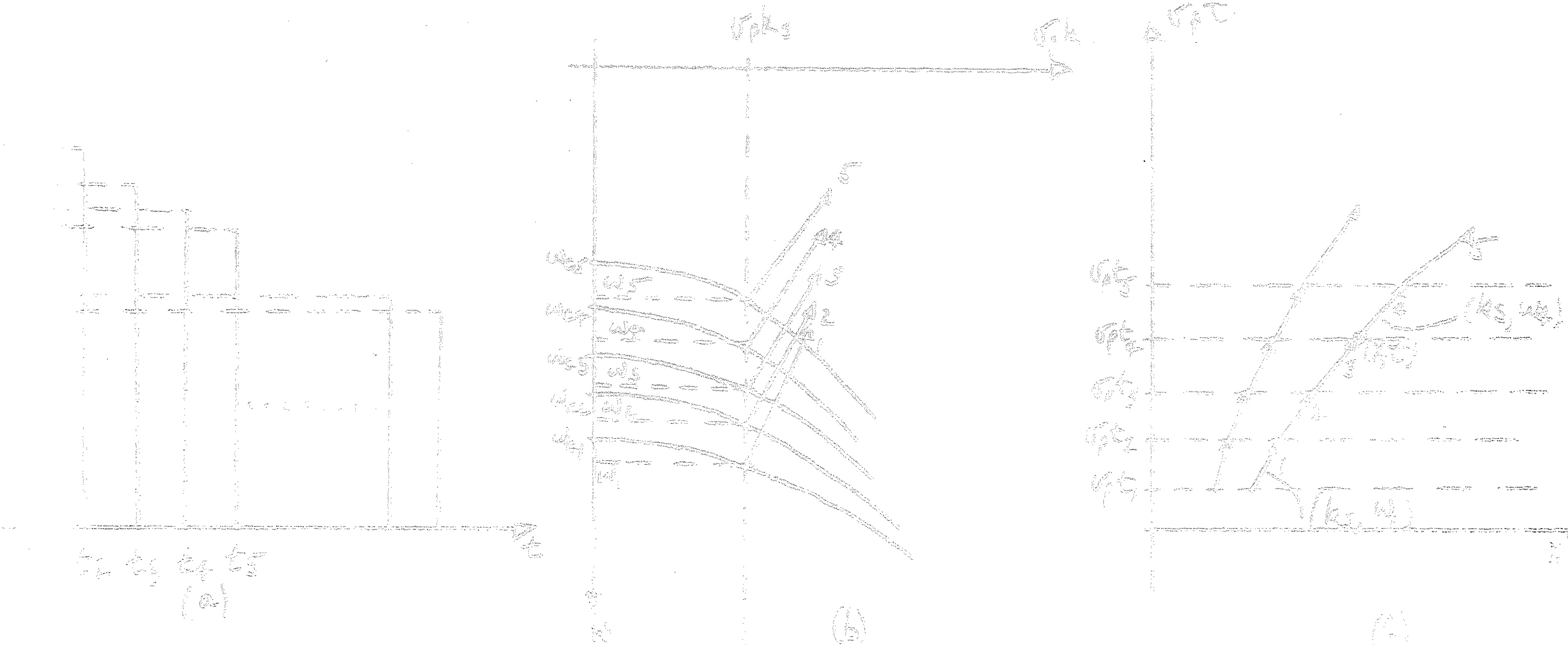
* Suggested further reading should be done in B. A. Auld, J. H. Collins and H. R. Zapp, "Signal Processing in a Nonperiodically Time-Varying Magnetoelastic Medium", Proc. IEEE, Vol. 56, No. 3, March, 1968, pp. 258-272; S. M. Rezende and F. R. Morgenthaler, "Magnetoelastic Waves in Time-Varying Magnetic Fields, (I, II)", J. Ap. Phys., Vol. 40, No. 2, Feb. 1969, pp. 524-545.



-19. Approximation (2-127) or (2-128) by a piecewise constant procedure. Only the portion [illegible] shown. (a) Piecewise constant [illegible] (b) [illegible] (c) Space-time [illegible]

PULSE PROPAGATION ON A NONLINEAR TRANSMISSION-LINE

In this chapter we are going to investigate a transmission-line which is radically different from those considered in the preceding chapters. The transmission-line to be considered here models an animal nerve axon (or neuron). Its mathematical description involves a non-linear wave equation, whereas those of the preceding chapters involved linear equations only.

The class of non-linear equations is far wider than that of linear ones and the model we consider here is only a small part of that class. As such, however, it demonstrates some truly interesting and remarkable facts about non-linear transmission-lines--at least about that one which models the neuron.

1. Some Physiological Preliminaries.

The neuron, or nerve cell, is the basic unit of the nervous system, and the number of neurons in the nervous system has been estimated as 10^{10}. As complex as it is, the nerve cell can be thought of as consisting of three fundamental parts: (1) the cell body, (2) dendrite, and (3) axon. Figure 3-1 illustrates the manner in which these three parts fit together. A brief description follows.

Cell Body: The cell body, being the widest part of the entire neuron (its diameter may be as much as 0.1 mm), houses the nucleus of the cell and much of the metabolic machinery which does the chemical processing for the cell. The axon originates at the cell body.

Dendrites: If we think of the cell body as the trunk of a tree, then the dendrites form the tree branches. They are submicroscopic at their tips but because of their great number they contain a very large surface area which is useful in performing their role of sensors.

Axon: The axon begins with a thin segment connection to the cell body. This segment eventually broadens into a myelin covered "cable" (myelin is a fatty substance, whose loss from the axon, it is conjectured, causes multiple sclerosis). There are breaks, called the nodes of Ranvier, in the myelin sheath, and the myelin-covered portions between the nodes are internodes. Internodes are 1-2 mm long.

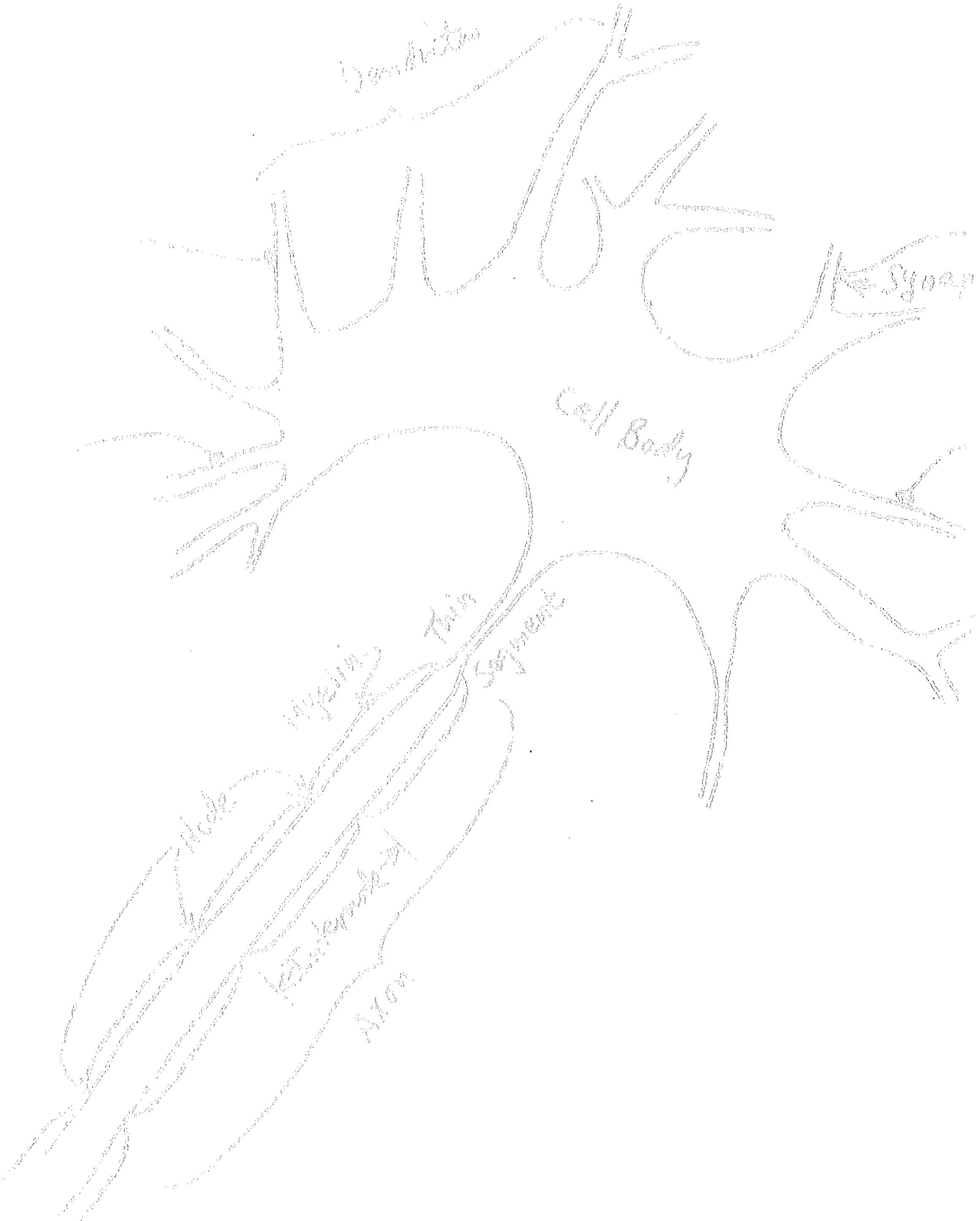


Fig. [illegible] Illustrating the nerve cell. (Internodal length [illegible] mm, nodal length [illegible] μ.)

The axon is the longest part of the neuron and may extend for several feet before it branches and distributes end bulbs at the tips of its branches over the surface of the dendrites and cell bodies of other neurons. The junction of an end bulb with the surface of the neuron beneath it is called the synapse. Apparently, the end bulbs secrete substances that affect the membrane, or cell wall, of the dendrites and cell body and generates the nerve impulse which propagates along the axon. Transmission of such impulses across the synaptic junction enables one neuron to communicate with another.

In the nerves of animals the axon is essentially a cylinder covered with a membrane, which is very thin (50–100 Å). The membrane is in contact with the tissues and fluids that surround the neuron, except at the internodes where it is covered with myelin. It is this membrane which transforms the axon into an active transmission-line.

We should point out that the myelinated axon just described is characteristic of the higher order animals, having appeared later in evolution. There are animals, such as the squid, whose axons are unmyelinated, thereby exposing the entire cell membrane to the extra-cellular fluid environment.

4.2. Modeling the Squid Axon.

The squid axon has been the one subjected to most investigations because it is easily accessible to the experimenter. We start by modeling it as the membrane covered cylinder of diameter D (Figure 3-2(a)). D is of the order of one millimeter for giant squid and the length of the axon may be many centimeters.

Figure 3-2 (b) shows an enlarged cross-section of the membrane and the plasma consisting of sodium (Na), potassium (K), and chlorine (Cl) ions both inside and outside of the membrane.

The ratios of the ionic concentrations are:

$$\frac{(Na)_i}{(Na)_o} = 1{:}10\,, \quad \frac{(Cl)_i}{(Cl)_o} = 1{:}10\,, \quad \frac{(K)_i}{(K)_o} = 40{:}1.$$

When the membrane is not generating a nerve impulse there is a resting potential of the order of 70 mv. A nerve impulse is generated in response to a voltage change such that the inside voltage is raised (positve change) 10m-volts relative to the outside change (i.e., so that the outside is now about 60 mv above the inside).

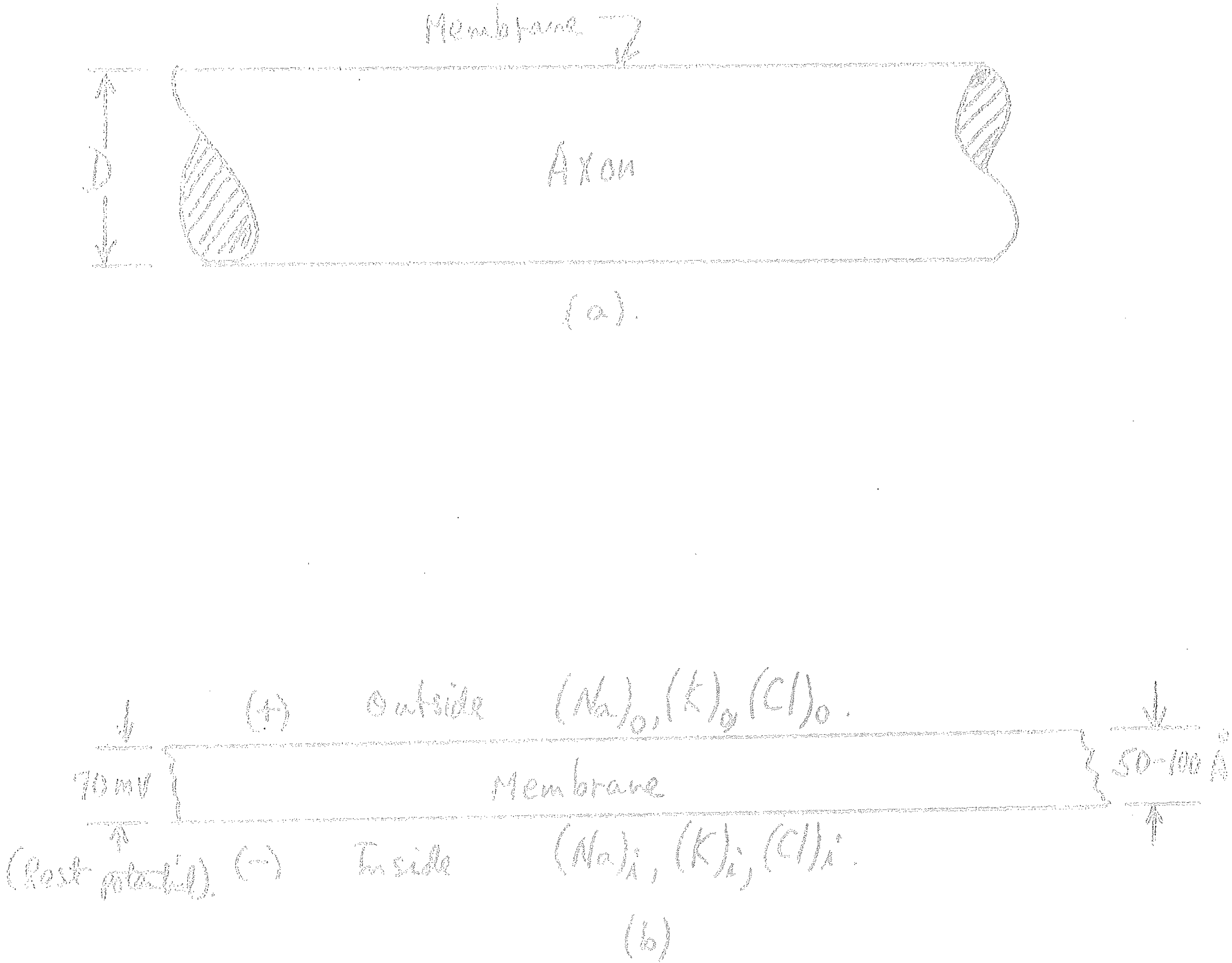


Figure 3-2. The axon and membrane to be modelled. (a) cylindrical axon with covering membrane. (b) Cross-section of membrane showing presence of potassium, sodium and chlorine ions. D is of the order of 1 mm for the giant squid and 10 μ for higher order mammals.

The mathematical model starts by assuming that conditions are uniform along the axon, i.e., the axon is said to be in a "space" or "voltage" clamp which keeps the outside and inside voltages constant along the length of the axon.

If we then measure the current flowing through the membrane we would find that it consists of four parts: (1) a displacement current flowing through the equivalent membrane capacitance, (2) a potassium ion current, (3) a sodium ion current, and (4) a leakage ion current due to other ions, as well, possibly, as chloride. The equations for these currents are given by

(a) $i_C = C \frac{dv}{dt}$ (displacement current)

(1-1) (b) $i_K = g_K n^4 (v - v_K)$ (potassium ion current)

(c) $i_{Na} = g_{Na} m^3 h (v + v_{Na})$ (sodium ion current)

(d) $i_L = g_L (v - v_L)$. (leakage ion current).

where v is the voltage across the membrane (positive when the outside is more positive than the inside) measured with respect to the resting potential. v_K, v_{Na} and v_L are fixed potentials having the approximate values: $v_K = 12$ mV, $v_{Na} = -115$ mV and $v_L = -11$ mV. These voltages are determined by the concentration ratios of their respective ions and are measured relative to the rest potential, also. The conductance parameters, g_K, g_{Na}, g_L, are fixed and have been experimentally measured to be: $g_K = 36 \cdot 10^{-3}\ \mho/cm^2$, $g_{Na} = 120 \cdot 10^{-3}\ \mho/cm^2$, $g_L = 0.3 \cdot 10^{-3}\ \mho/cm^2$. Finally, h, m and n are dimensionless "activation levels" which vary between 0 and 1. They correspond, respectively, to sodium inactivation, sodium activation and potassium activation. C, of course, is the membrane capacitance.

The net current is given by $i = i_C + i_K + i_{Na} + i_L$; an equivalent circuit is shown in Figure 3-3.

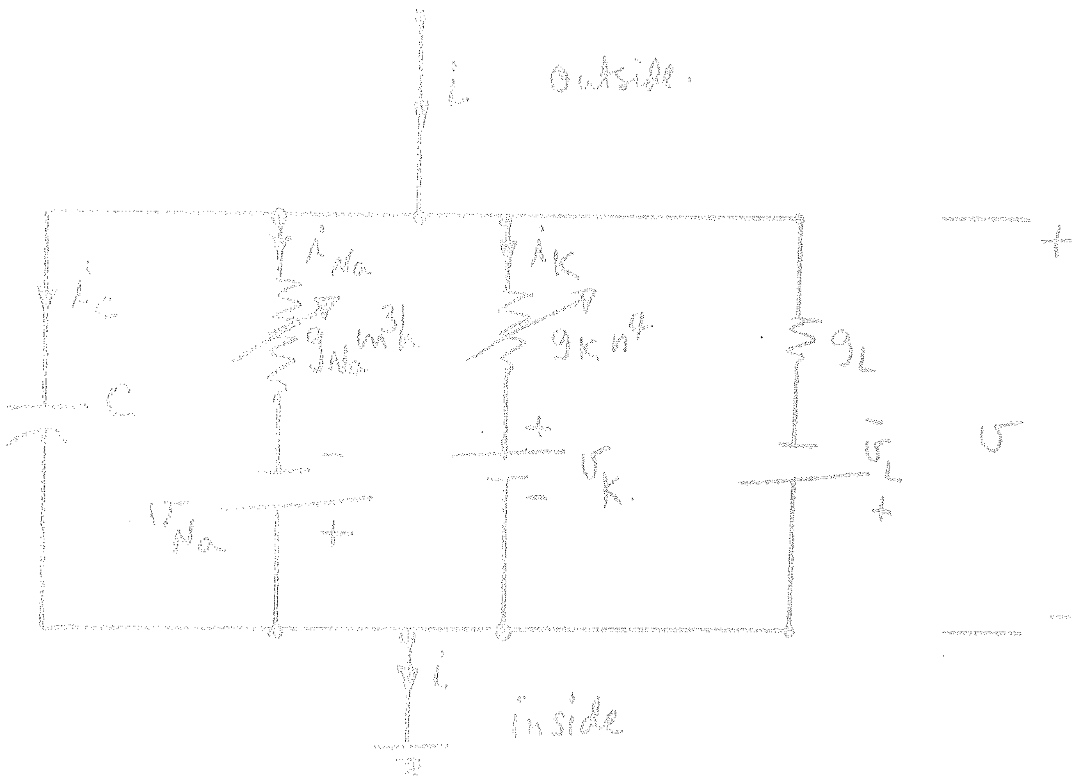


Figure 3-3. The axon membrane under space-clamped conditions; the entire axon acts as a lumped (non-linear) system. All voltages are measured with respect to the rest potential.

As we shall soon see, because the activations depend on the terminal voltage, v, the branches containing the sodium and potassium "batteries" are connected to non-linear conductances $g_{Na}\ hm^3$, $g_K n^4$.

The parameters h, m, n satisfy the kinetic equations

(a) $$\frac{dh}{dt} + [\alpha_h(v) + \beta_h(v)]h = \alpha_h(v)$$

(1-2) (b) $$\frac{dm}{dt} + [\alpha_m(v) + \beta_m(v)]m = \alpha_m(v)$$

(c) $$\frac{dn}{dt} + [\alpha_n(v) + \beta_n(v)]n = \alpha_n(v),$$

with the α's and β's given empirically by the following non-linear functions of v (times in milliseconds and voltages in millivolts):

(1-3) $$\alpha_h(v) = 0.07\exp\left(\frac{v}{20}\right), \quad \beta_h(v) = \frac{1}{\exp\left[\left(\frac{v+30}{10}\right)\right]-1}$$

$$\alpha_m(v) = \frac{0.1\,(v+25)}{\exp\left[\frac{v+25}{10}\right]-1}, \quad \beta_m(v) = 4\,\exp\left(\frac{v}{18}\right)$$

$$\alpha_n(v) = \frac{0.01(v+10)}{\exp\left[\frac{v+10}{10}\right]-1}, \quad \beta_n(v) = 0.125\exp\left(\frac{v}{80}\right).$$

Thus, the system (1-2), together with (1-3) and Kirchoff's current law

(1-4) $$i = C\frac{dv}{dt} + g_K n^4(v-v_K) + g_{Na}\, h m^3(v-v_{Na}) + g_L(v-v_L)$$

determines a non-linear system of four equations in the four unknowns v, h, m, n (assuming i to be given). Later we will replace this system by one that is more manageable.

Figure 3-3, together with the above system, describe the lumped-parameter model of the axon membrane under spatial clamping, i.e., such that conditions are uniform along the axon length. Obviously in this situation no signals "propagate" down the axon because of the assumed uniformity of voltage and current along the axon. What is needed in order to describe nerve impulse propagation is a "transmission-line" model for the axon and its associated wave equation.

In deriving the wave equation we assume that the axon consists of a continuous distribution of infinitesimal circuits, each of the form of Figure 3-3, with the following changes. Because there will be a current flowing along the axon (the "longitudinal" current as opposed to the four "transverse" current components shown in Figure 3-3), we must include series resistances (per unit length), R_o, R_i, to account for a longitudinal voltage drop outside and inside the axon, respectively. In addition, C, g_L, g_K and g_{Na} will now be understood to be, respectively, membrane capacitance per-unit-area and conductances per-unit-area, so that if the axon is taken to be a cylinder of diameter D (as in Figure 3-2 (a)) then the net capacitance over a length dx along the axon is $C(\pi D)dx$ and the conductances are $g_L(\pi D)dx$, $g_K(\pi D)dx$ and and $g_{Na}(\pi D)dx$.

Thus, the incremental section of the squid nerve axon under non-space clamped, or propagating, conditions is shown in Figure 3-4.

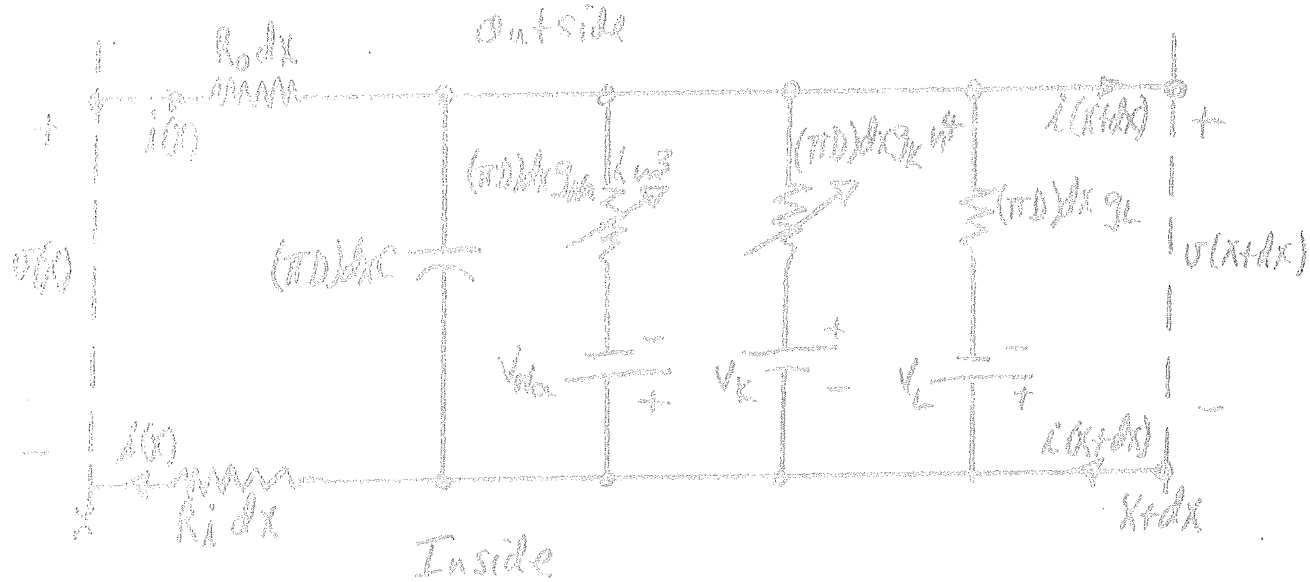


Figure 3-4. Incremental section of non-space clamped (or propagating) axon.

Let us now apply Kirchoff's laws to derive the transmission-line equations for the <u>active</u>, <u>distributed</u> system of Figure 3-4. Kirchoff's voltage law yields

$$v(x) = i(x)(R_o+R_i)dx + v(x+dx),$$

which upon rearrangement, division by dx and passage to the limit, $dx \to 0$, gives us the first equation

$$\text{(1-5)}\quad \text{(a)}\qquad \frac{\partial v}{\partial x} = -i(R_o+R_i).$$

Next, Kirchoff's current law reads

$$i(x) = (\pi D)dx\,C\,\frac{\partial v(x+dx)}{\partial t} + (\pi D)dx\,g_{Na}hm^3\left(v(x+dx)+V_{Na}\right) + (\pi D)dx\,g_K n^4\left[v(x+dx)-V_K\right] + (\pi D)dx\,g_L\left[v(x+dx)+V_L\right] + i(x+dx).$$

$$\frac{i(x+dx)-i(x)}{dx}$$

and pass to the limit $dx \to 0$, this equation yields the second transmission-line equation

$$(1\text{-}5)(b) \quad \frac{\partial i}{\partial x} = -\pi D\left[C\frac{\partial v}{\partial t} + g_{Na} h m^3 (v+V_{Na}) + g_K n^4 (v-V_K) + g_L (v+V_L)\right].$$

The two first-order transmission-line equations, (1-5)(a),(b), can be combined to yield a single second-order wave equation simply by differentiating (1-5)(a) with respect to x and substituting the result into (1-5)(b), in the usual way, with the result

$$(1\text{-}6) \quad \frac{\partial^2 v}{\partial x^2} = -(R_0+R_i)\frac{\partial i}{\partial x} = (R_0+R_i)\pi D\left[C\frac{\partial v}{\partial t} + g_{Na} h m^3 (v+V_{Na}) + g_K n^4 (v-V_K) + g_L (v+V_L)\right].$$

The partial differential equation, (1-6), together with the ordinary system (1-2), yield a system of four non-linear equations in the four unknowns v, h, m, n:

$$(1\text{-}7) \quad (a) \quad \frac{\partial^2 v}{\partial x^2} = (R_0+R_i)\pi D\left[C\frac{\partial v}{\partial t} + g_{Na} h m^3 (v+V_{Na}) + g_K n^4 (v-V_K) + g_L (v+V_L)\right]$$

$$(b) \quad \frac{\partial h}{\partial t} + [\alpha_h(v) + \beta_h(v)]h = \alpha_h(v)$$

$$(c) \quad \frac{\partial m}{\partial t} + [\alpha_m(v) + \beta_m(v)]m = \alpha_m(v)$$

$$(d) \quad \frac{\partial n}{\partial t} + [\alpha_n(v) + \beta_n(v)]n = \alpha_n(v),$$

where the partial derivatives are used in (b)-(d) because h, m, n may depend explicitly on x because v does.

The coefficients $\{a_j^{(k)}\}, \{b_j^{(k)}\}$, $k, j = 0, \dots, n$, are given in (1-3).

Thus, we have obtained the general mathematical model for the squid axon. This model is called the <u>Hodgkin-Huxley (HH)</u> model in honor of its expositors. Because of its complexity, we replace it by a simpler model called the <u>Bonhoeffer-Van der Pol (BVP)</u> model which we shall now describe.

3-3. <u>A Simplified Model</u>.

Let us start our discussion with, oddly enough, the circuit of Figure 3-5. The elements, a capacitor, biased tunnel-diode, inductor and resistor, are all familiar to electrical engineers. The circuit equations are derived using Kirchoff's laws:

(1-8)

$$\text{(a)} \quad i_T = C\frac{dv}{dt} - i - f(e)$$

$$\text{(b)} \quad L\frac{di}{dt} + iR = -v = e - E_0 ,$$

Because $v = -e + E_0$, we can eliminate v in favor of e in (1-8) (a) and rewrite the system as

(1-9)

$$\text{(a)} \quad i_T = -C\frac{de}{dt} - i - f(e)$$

$$\text{(b)} \quad L\frac{di}{dt} + iR + E_0 = e .$$

This is a coupled system of non-linear equations (the non-linearity stems from the characteristic, f(e) vs. e, in Figure 3-6) in the two variables (e, i). Let us expand f(e) in a power series about the operating point e_0 shown in Figure 3-6:

(1-10)

$$f(e) = f(e_0) + f'(e_0)(e-e_0) + f''(e_0)\frac{(e-e_0)^2}{2!} + f'''(e_0)\frac{(e-e_0)^3}{3!} + \cdots$$

$$\simeq i_0 - \frac{1}{\rho}(e-e_0) + \frac{(e-e_0)^3}{3\rho K^2} , \qquad (\rho > 0,\ K > 0),$$

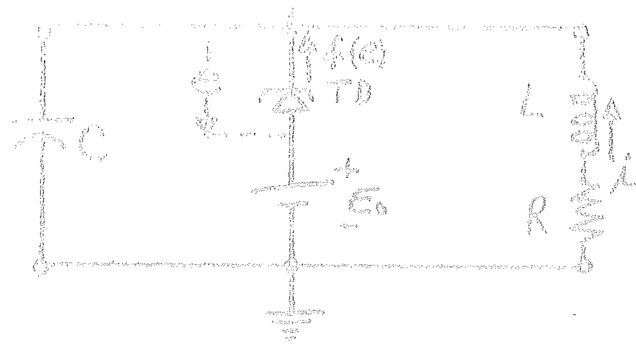


Figure 3-5. A circuit used to s mulate BHP model.

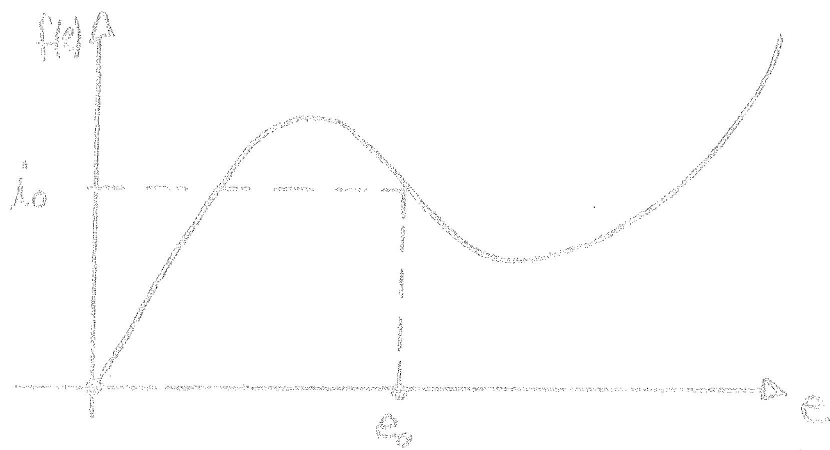


Figure 3-6. The voltage-current terminal characteristics of the tunnel-diode.

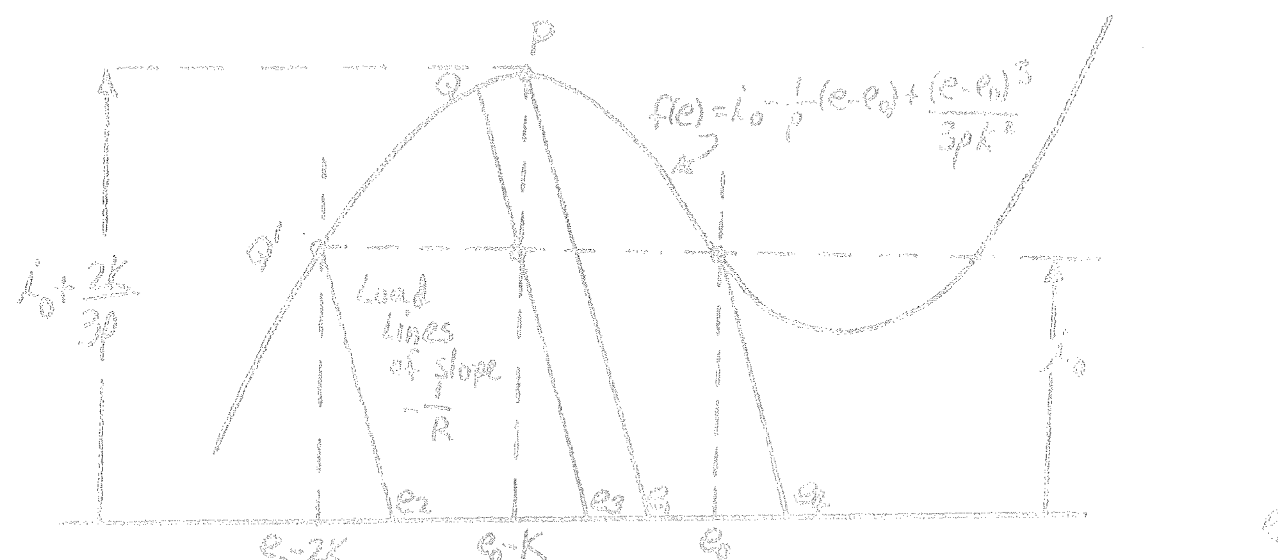


Figure 3-7. Illustrating the biasing conditions. When the bias voltage, , is set between e_1 and e_2, the circuit of Figure 3-5 is monostable, whereas, if = e_4, the circuit is unstable and oscillates spontaneously.

[illegible] third-order term. In (1-10), $i_0 = f(e_0)$, $f'(e_0) = -\frac{1}{\rho}$, $f''(e_0) = 0$, $f'''(e_0) = \frac{2}{\rho K^2}$. The assumption that the second derivative of f(e) vanishes at e_0 implies that f(e) has a point of inflection at e_0.

Upon substituting (1-10) into (1-9) (a), there results

$$(1\text{-}10)\qquad i_T = -C\frac{de}{dt} - \left[i + i_0 - \frac{1}{\rho}(e-e_0) + \frac{(e-e_0)^3}{3\rho K^2}\right].$$

We can put (1-9) and (1-10) into a dimensionless form by introducing the variables

$$(1\text{-}11)\qquad \tau = \frac{t}{\sqrt{LC}},\quad u = -\frac{(e-e_0)}{K},\quad w = \frac{\rho}{K}(i+i_0),\quad I_T = \frac{\rho}{K}i_T,$$
$$a = \frac{i_0 R + (e_0 - E_0)}{K},\quad b = \frac{R}{\rho},\quad c = \frac{1}{\rho}\sqrt{\frac{L}{C}}.$$

Then after a little bit of differentiation and algebra (1-9) (b) and (1-10) emerge as

(1-12)

(a) $I_T = \frac{1}{c}\frac{du}{d\tau} - w - (u - u^3/3)$

(b) $c\frac{dw}{d\tau} + bw = a - u$.

Let us take for the values of the constants, a, b, c :

(1-13) (i) $0 < b < 1$, (ii) $b < c^2$, (iii) $1 - \frac{2}{3}b < a < 2$,

or in terms of the actual circuit parameters

(1-14)

(i) $R < \rho = -\frac{1}{f'(e_0)}$

(ii) $\frac{L}{R} > \rho C = -C/f'(e_0)$

(iii) $e_2 < E_0 < e_1$ (See Figure 3-7).

[illegible] the circuit of Figure 3-[illegible] acts as a monostable multivibrator.

The system (1-12), together with the conditions (1-13) or (1-14), is called the Bonhoeffer-Van der Pol (BVP) system. It will be used in place of the Hodgkin-Huxley (HH) system of equations (1-12)-(1-4). The philosophy behind replacing the (HH) system by the (BVP) system is not to necessarily gain quantitative accuracy by using a simpler model (BVP) but rather that the (BVP) model gives good qualitative agreement with many known neurophysiological phenomena. Because the (BHP) model utilizes only two state-variables (u,w), whereas the (HH) equations involved four state-variables (v,h,m,n) we have achieved a tremendous simplification. In addition, the form of (1-12) is classical, being the Lienard-Levinson-Smith equation, of which Van der Pol's equation is a famous special case (a=b=0). Thus, the properties of (1-12) are rather well known.

Now, just as (1-2) - (1-4) were the ordinary differential equations for the space-clamped nerve in the (HH) model, so are (1-12) the corresponding equations in the (BVP) model. When we remove the space clamp so that the neuron becomes a transmission-line, an incremental section of this line, in the (BVP) model, becomes that shown in Figure 3-8.

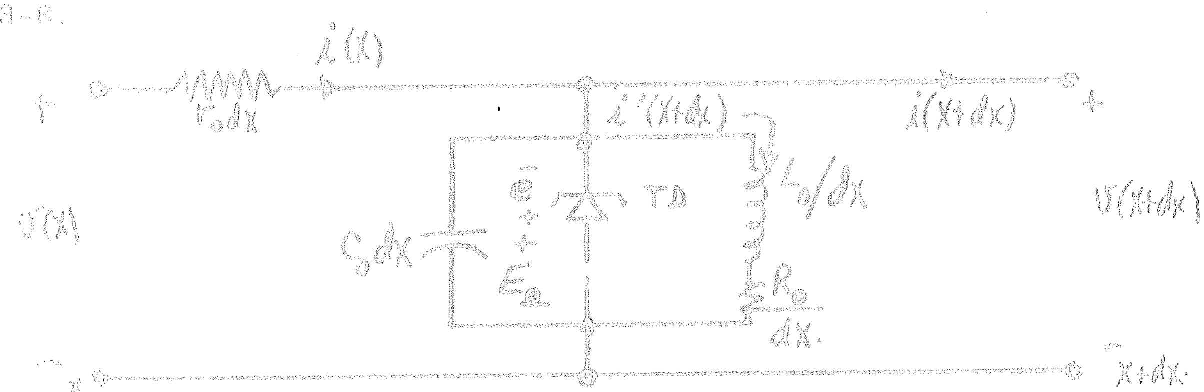


Figure 3-8. An incremental section of an active transmission-line simulating the squid neuron in the (BVP) model. r is the net (inside + outside) longitudinal resistance per-unit-length.

[illegible]

[illegible]ine of Figure 3-8 are easily derived using (1-12). Comparing (1-12) and Figure 3-5 with Figure 3-8 we see that we must make the following replacements in the constants of (1-12):

	Replace	C	by $C_0 dx$
		L	" L_0/dx
		R	" R_0/dx
		ρ	" ρ_0/dx
(1-15)		K	" K_0
		α	" $\sqrt{\frac{L_0}{C_0}} \cdot \frac{1}{\rho_0} = \alpha_0$
		a	" $a_0 = \frac{f(e_0)R_0 + (e_0 - E_0)}{K}$
		b	" $\frac{R_0}{\rho_0} = b_0$

Note that in (1-12), a involves the product $i_0 R$, where $i_0 = f(e_0)$. In Figure 3-8, because there is assumed to be a "continuous distribution" of tunnel diodes, we must think of there being an effective tunnel-diode current per-unit-length of f(e), so that now i_0 becomes $f(e_0)dx$, and the product $i_0 R$ appearing in the definition of a becomes $f(e_0)dx \cdot \frac{R}{dx} = R f(e_0)$; thus, a is unchanged in form.

Equations (1-12) for the distributed system now become, upon substituting (1-15) into (1-12)

$$\text{(a)} \quad I_\tau = \frac{\rho_0}{dx} \cdot \frac{1}{K_0} i_\tau = \frac{1}{\alpha_0} \frac{\partial u}{\partial \tau} - w - \left(u - u^3/3\right) \qquad \text{(1-16)}$$

$$\text{(b)} \quad \alpha_0 \frac{\partial w}{\partial \tau} + b_0 w = a_0 - u.$$

three-branch circuit of Figure 3-5. From Figure 3-8, it is obvious that $i_p = i(x) - i(x+dx)$, so that (1-16) (a) becomes

$$\frac{\rho_0}{K_0}\left[\frac{i(x) - i(x+dx)}{dx}\right] = \frac{1}{\mu_0}\cdot\frac{\partial u}{\partial \tau} - w - (u - u^3/3),$$

the left side of which, as $dx \to 0$, approaches $-\frac{\rho_0}{K_0}\cdot\frac{\partial i}{\partial x}$.

Thus, we have

$$\text{(1-17)} \quad -\frac{\rho_0}{K_0}\cdot\frac{\partial i}{\partial x} = \frac{1}{\mu_0}\cdot\frac{\partial u}{\partial \tau} - w - (u - u^3/3).$$

We can easily derive a relation between v and i by applying Kirchoff's voltage law to Figure 3-8. The result, as we have derived many times before, is

$$\frac{\partial v}{\partial x} = -i r_0 .$$

Upon differentiating this equation with respect to x, we get $\frac{\partial^2 v}{\partial x^2} = -r_0\frac{\partial i}{\partial x}$, and when this result is substituted into (1-17), it follows that

$$\frac{\rho_0}{r_0 K_0}\cdot\frac{\partial^2 v}{\partial x^2} = \frac{1}{\mu_0}\cdot\frac{\partial u}{\partial \tau} - w - (u - u^3/3)$$

Because $u = \frac{(e_0 - E_0) + v}{K_0}$ (see (1-11) and Figure 3-8), we have $\frac{1}{K_0}\cdot\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[\frac{v}{K_0}\right] = \frac{\partial^2 u}{\partial x^2}$, so that the result above can be expressed in terms of u and w:

$$\text{(1-18)} \quad h_0\frac{\partial^2 u}{\partial x^2} = \frac{1}{\mu_0}\cdot\frac{\partial u}{\partial \tau} - w - (u - u^3/3),$$

where $h_0 = \rho_0/r_0$. Note that h_0 has dimensions of $(\text{length})^2$, so that (1-18) remains dimensionless. Finally, when we couple (1-18) with (1-16) (b), we obtain the (BVP) model equations

$$\text{(a)} \quad h_0\frac{\partial^2 u}{\partial x^2} = \frac{1}{\mu_0}\frac{\partial u}{\partial \tau} - w - (u - u^3/3)$$

$$\text{(b)} \quad \mu_0\frac{\partial w}{\partial \tau} + b_0 w = a_0 - u.$$

$$(1\text{-}19)\quad \text{(a)}\ \ h_0 \frac{\partial^2 u}{\partial x^2} \ [illegible]\ \kappa_0 \frac{\partial u}{\partial \tau}\ [illegible]\ (u - u^3/3)$$

$$\text{(b)}\ \ \kappa_0 \frac{\partial w}{\partial \tau} + b_0 w = a_0 - u.$$

This system is easier to simulate and analyze than the (HH) model equations (1-7), and may well be the simplest mathematical model of the nerve axon.

It will facilitate matters to eliminate w in favor of u, thereby expressing the model, (1-19), by a single third-order partial-differential equation. Let us first set $R_0 = 0$ ($b_0 = 0$) and differentiate (1-19) (a) with respect to τ. The right-hand side will contain $\partial w/\partial \tau$, which, by virtue of (1-19) (b) and the assumption that $R_0 = 0$, becomes $\frac{\partial w}{\partial \tau} = \frac{a_0 - u}{\kappa_0}$. When this is substituted into (1-19) (a) we obtain

$$(1\text{-}20)\quad \kappa_0 h_0 \frac{\partial^3 u}{\partial x^2 \partial \tau} = \frac{\partial^2 u}{\partial \tau^2} + \kappa_0 (u^2 - 1)\frac{\partial u}{\partial \tau} + u - a_0$$

where $\kappa_0 > 0$, $1 < a_0 < 2$, $h_0 > 0$.

We can get rid of a few of the constants in (1-20) by again (!) making some changes of variables. Thus: let

$$(1\text{-}21)\quad s = \frac{x}{\sqrt{\kappa_0 h_0}},\quad z = \frac{2a_0}{a_0^2 - 1}(a_0 - u),\quad \mu = \kappa_0 (a_0^2 - 1)$$

$$\varepsilon = \frac{a_0^2 - 1}{4a_0^2}.$$

Straightforward differentiation and algebra yield

$$(1\text{-}22)\quad \frac{\partial^3 z}{\partial s^2 \partial \tau} = \frac{\partial^2 z}{\partial \tau^2} + \mu\left(1 - z + \varepsilon z^2\right)\frac{\partial z}{\partial \tau} + z, \quad \mu > 0,\quad 0 < \varepsilon < \frac{1}{4}.$$

This is the final form of the (BVP) model, which can now be used to simulate neuron action.

Let us now analyze the following (boundary excitation) problem letting $\mu = 10$, $\varepsilon = 0.1$:

(1-23) (a) Solve: $\dfrac{\partial^3 z}{\partial \xi^2 \partial \tau} = \dfrac{\partial^2 z}{\partial \tau^2} + 10\left(1 - z + 0.1 z^2\right)\dfrac{\partial z}{\partial \tau} + z, \quad \xi \geq 0, \tau \geq 0.$

(b) Initial conditions: $z(\xi,\tau) = 0, \quad \dfrac{\partial z(\xi,\tau)}{\partial \tau} = 0, \quad$ for $\tau = 0$.

(c) Boundary condition: $z(\xi,\tau) = F(\tau), \quad$ for $\xi = 0$.

In (1-23) (c), we take

$$(1\text{-}24) \qquad F(\tau) = \frac{z_0}{2}\left(1 - \cos\frac{2\pi\tau}{\tau_0}\right), \quad 0 \leq \tau \leq \tau_0$$
$$= 0, \quad \tau_0 \leq \tau.$$

The problem we have posed in (1-23), involving a non-linear partial-differential equation, is not amenable to solution by the usual tools with which we are familiar, such as Laplace and Fourier transforms. Progress in the analysis of such problems is usually made through the use of computers. Results of such numerical studies indicate rather radical (and highly interesting!) behavior. Let us list some of them:

A. Signal Amplification. For the case $z_0 = 5$, $\tau_0 = 3$ in (1-24), the steady-state pulse amplitude (which is highly distorted from the input cosinusoidal pulse of (1-24)) is about 2.5 times the input pulse amplitude (see Figure 3-9), thus demonstrating the amplification capability of the neuron.

B. Signal Attenuation. On the other hand for the case $z_0 = 20$, $\tau_0 = 1$ the steady-state pulse is attenuated, its steady-state amplitude being about the same as in the first case. Figure 3-10 illustrates this case.

3. Signal Elimination. An input pulse for which $z_0 < \xi$, $\tau_0 > ?$ is eliminated during propagation, i.e., its steady-state value is zero (Figure 5-11).

These are amazing results and are due to the non linearity of the system. They are a manifestation of the apparent "signal shaping" property of the neuron. Let us summarize the computer findings:

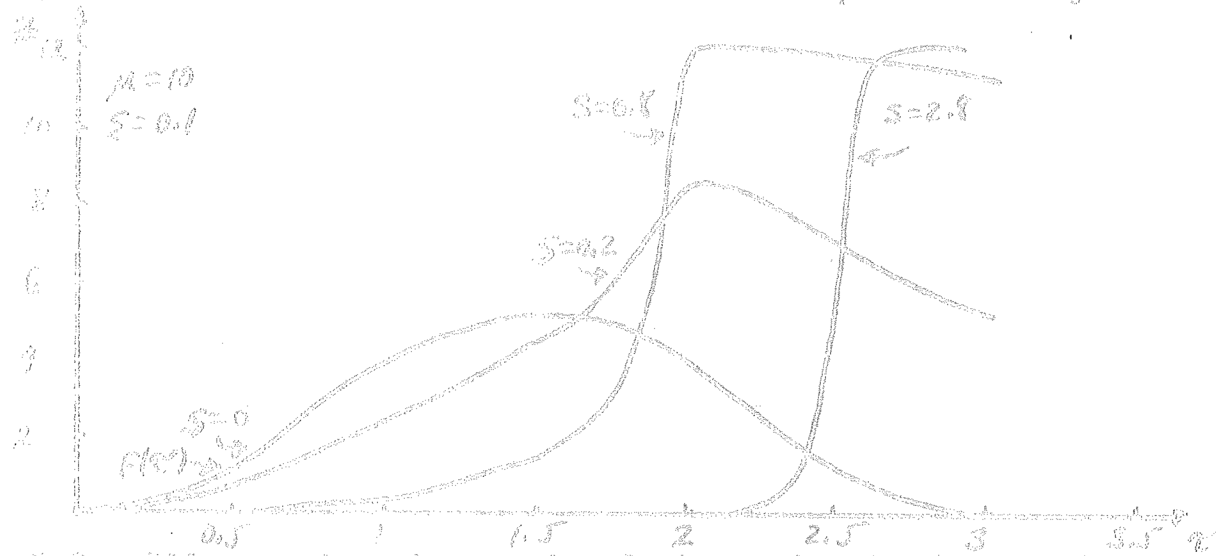


Figure 5-9. Illustrating that a signal above threshold but below the asymptotic value is amplified during transmission.

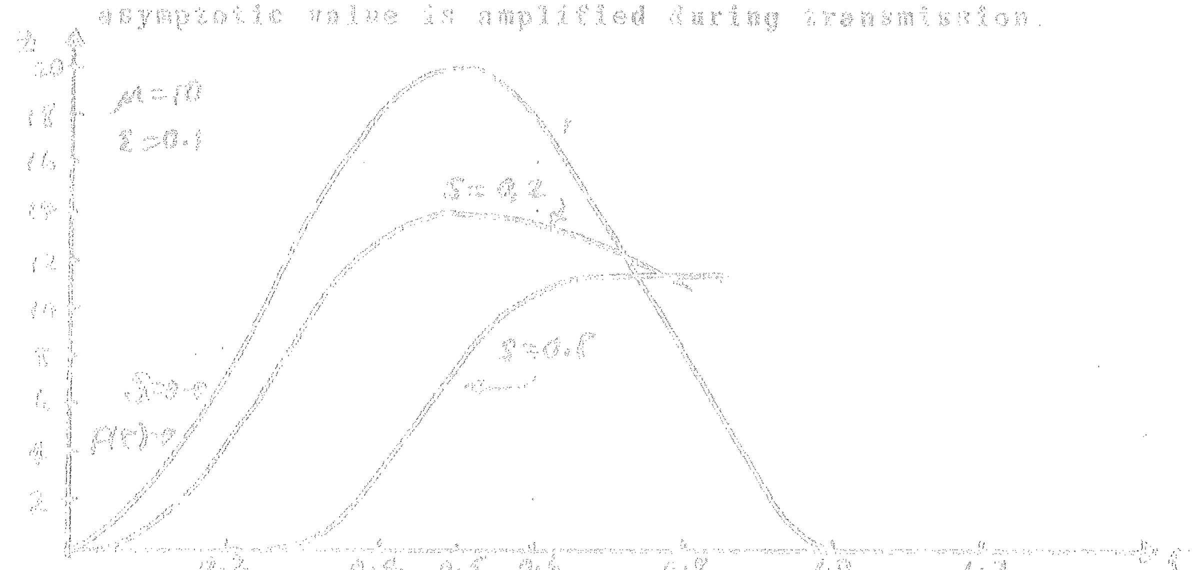


Figure 5-10. Illustrating that a signal above the asymptotic value is attenuated during transmission.

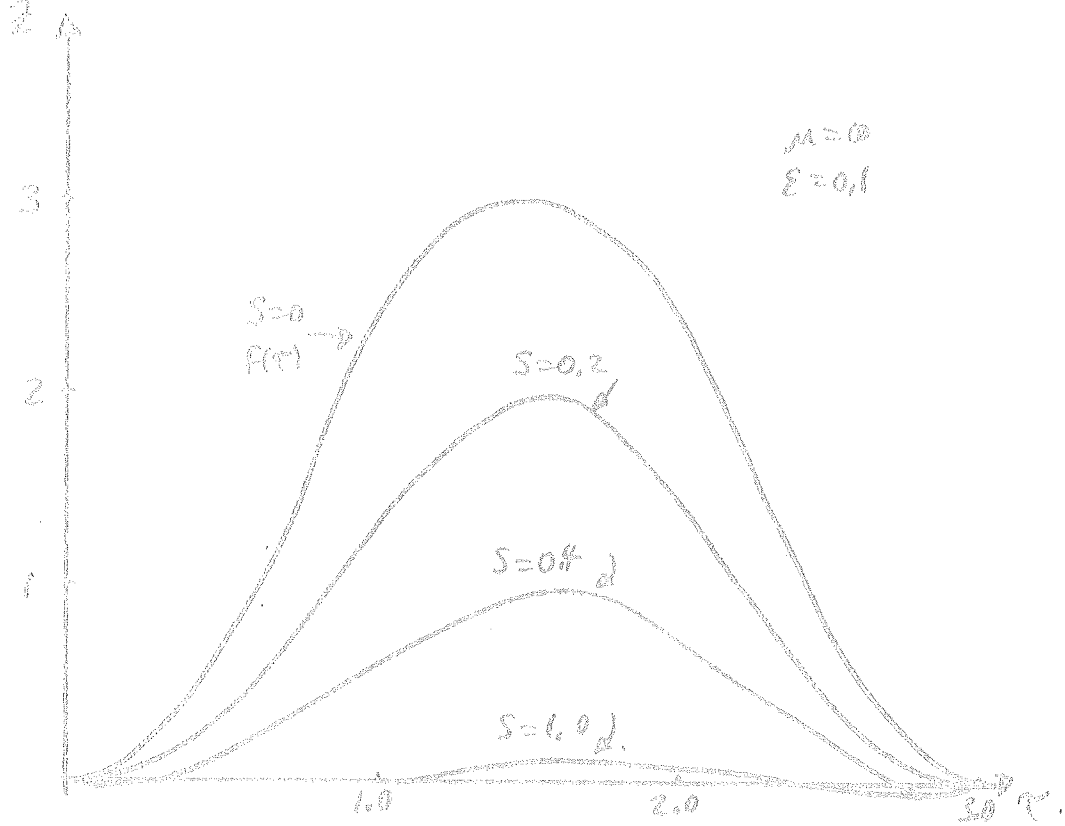


Figure 3-11. Illustrating that a signal below threshold is eliminated during transmission.

(1) A signal with an amplitude between a certain "asymptotic" value and a certain "threshold" value is amplified during transmission.

(2) whereas, a signal whose amplitude is greater than the above asymptotic value will be attenuated during transmission to the same asymptotic (steady-state) value as in (1), and

(3) a signal lower than the threshold value in (1) is simply eliminated during transmission (apparently nature doesn't want to trouble nervous systems with every little signal that comes along).

These results are summarized in Figure 3-12; Figure 3-13 is a schematic display of pulse propagation in space-time.

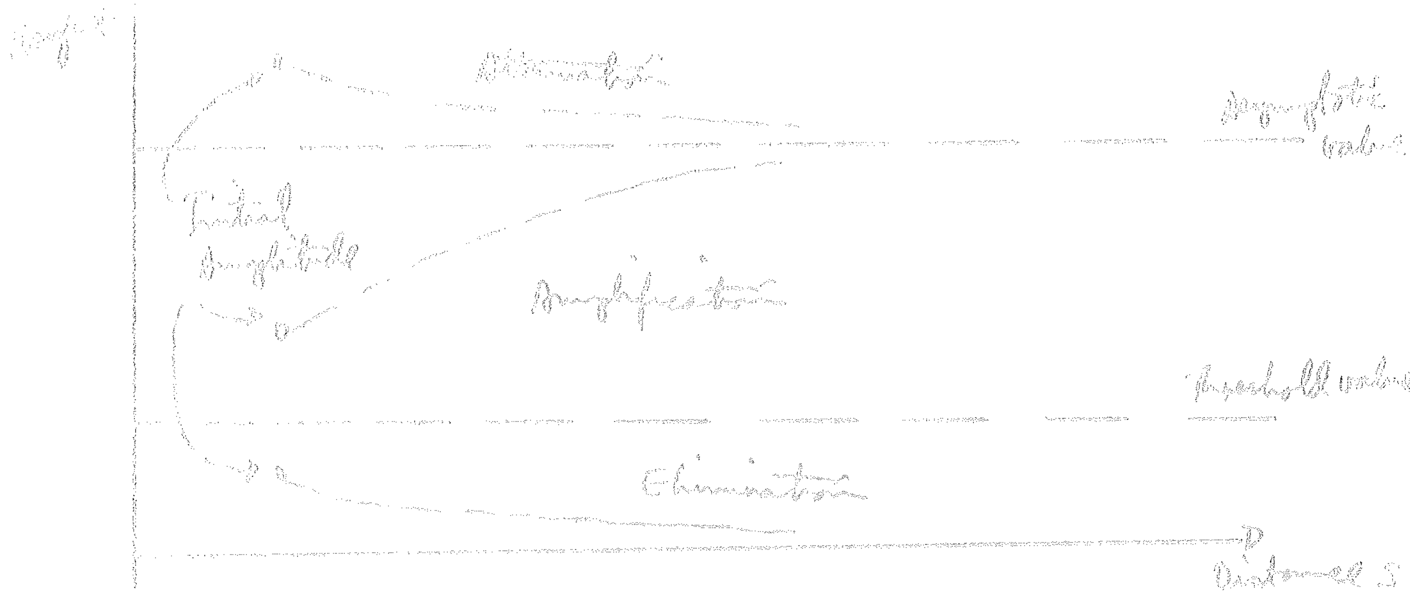


Figure 3-12. Amplification, attenuation, or elimination of neuronal signals.

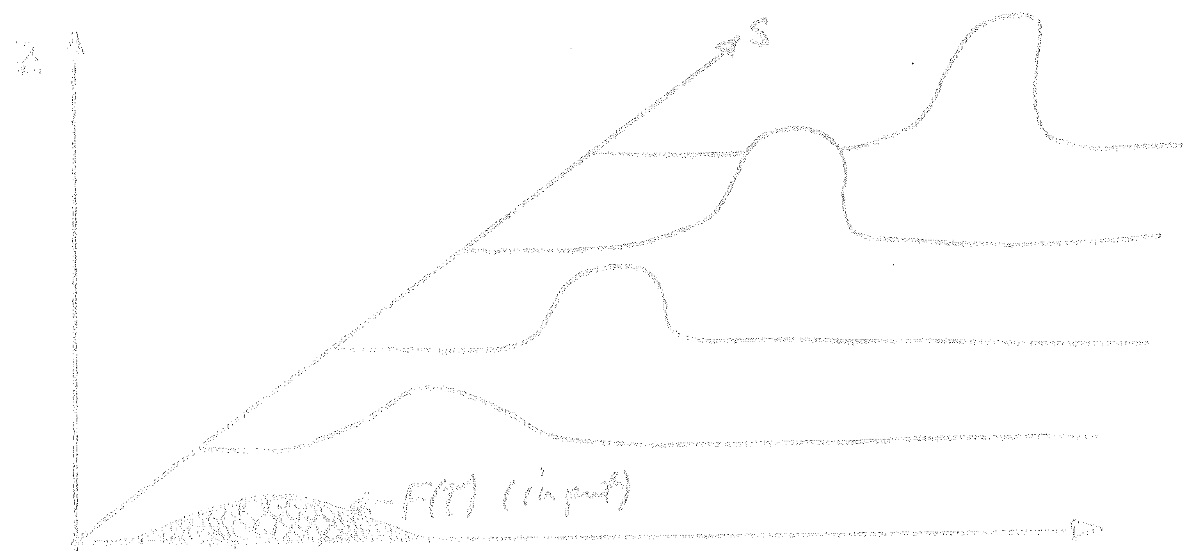


Figure 3-13. Schematic of pulse propagation in space-time.

The properties listed above are all quite interesting, and artificial neurons are the subject of considerable research interest because of their possible connection with digital circuitry and other engineering systems (not to mention the improved understanding of biological systems that is concommitant with their study).

Let us now search for that waveform (if it exists), which, in the steady-state, propagates down the line with no distortion. The preceding results seem to indicate that such a waveform should exist. Such a waveform will be called an <u>asymptotic waveform</u>, or, because it is propagating in a steady-state mode, is also referred to as the <u>dynamic steady-state</u> (dss) waveform.

In our discussion of distortionless transmission in Chapt. I, we concluded that if the input signal at $s = 0$, $z(0,\tau)$, is given as $F(\tau)$, then the signal at any point $s > 0$ is given by $F(\tau - s/\theta)$, where θ is the velocity of propagation of the undistorted signal. Here, because s and τ are dimensionless, θ is also dimensionless.

Let us call $\tau - s/\theta$, ξ, so that we are going to look for a solution of the non-linear wave equation, (1-22), which can be written as a function of the single variable $\xi = \tau - s/\theta$, with θ constant (i.e., independent of s or τ). Thus, we write $z(s,\tau) = F(\tau - s/\theta)$ and substitute this into (1-22), making use of the following relations:

(1-24) $$\frac{\partial z}{\partial s} = \frac{\partial F}{\partial s} = \frac{dF}{d\xi} \cdot \frac{\partial \xi}{\partial s} = -\frac{1}{\theta} \cdot \frac{dF}{d\xi}$$

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial^2 F}{\partial s^2} = \frac{1}{\theta^2} \cdot \frac{d^2 F}{d\xi^2}$$

$$\frac{\partial^3 z}{\partial s^2 \partial \tau} = \frac{1}{\theta^2} \cdot \frac{d^3 F}{d\xi^3} \cdot \frac{\partial \xi}{\partial \tau} = \frac{1}{\theta^2} \frac{d^3 F}{d\xi^3}$$

$$\frac{\partial z}{\partial \tau} = \frac{\partial F}{\partial \tau} = \frac{dF}{d\xi} \cdot \frac{\partial \xi}{\partial \tau} = \frac{dF}{d\xi}$$

$$\frac{\partial^2 z}{\partial \tau^2} = \frac{d^2 F}{d\xi^2} \cdot \frac{\partial \xi}{\partial \tau} = \frac{d^2 F}{d\xi^2}.$$

linear) differential equation in the independent variable ξ :

$$\delta \frac{d^3F}{d\xi^3} - \frac{d^2F}{d\xi^2} - \mu(1-F+\epsilon F^2)\frac{dF}{d\xi} - F = 0, \tag{1-25}$$

where $\delta = 1/\theta^2$. At this time B (or θ) is an unknown constant; part of the objective here is to determine its value.

One solution of (1-25) is $F(\xi) \equiv 0$, the quiescent state. The solution we are looking for must satisfy the following conditions: for fixed observation time, τ, the solution must vanish as $s \to \infty$, while at a fixed observation point, s, the solution must vanish as $\tau \to \infty$. The first case corresponds to $\xi \to -\infty$, while the latter to $\xi \to +\infty$. Hence, our asymptotic (or dss) solution must satisfy $F(\xi) \to 0$ as $\xi \to \pm\infty$.

The determination of such a solution for the third-order system (1-25) usually requires numerical experimenting on a digital computer, but we can at least get an analytic start in the following way. Because we require that our solution approach the quiescent state $F(\xi) \equiv 0$, we inquire about the nature of $F(\xi)$ in the neighborhood of $F(\xi) = 0$, i.e., how does $F(\xi) \to 0$? For small values of $F(\xi)$, (1-25) can be linearized by discarding products of F and its derivatives, thus:

$$\delta \frac{d^3F}{d\xi^3} - \frac{d^2F}{d\xi^2} - \mu\frac{dF}{d\xi} - F = 0, \tag{1-26}$$

Being a linear differential equation, (1-26) possesses solutions of the form $Ae^{\lambda\xi}$, where λ satisfies

$$\delta\lambda^3 - \lambda^2 - \mu\lambda - 1 = 0. \tag{1-27}$$

[illegible] accepted, depending on the following consideration. If the test solution returns to zero exponentially (i.e., without oscillation, because both negative roots as mentioned above are real) as $\xi \to +\infty$, then it is the desired solution and the corresponding value of B determines the (dSS) velocity by means of $\theta = 1/\sqrt{B}$. If the solution does not behave properly as $\xi \to \infty$ for the proposed value of B, it is discarded and the test repeated for a new value of B. The idea is sketched in Figure 3-14.

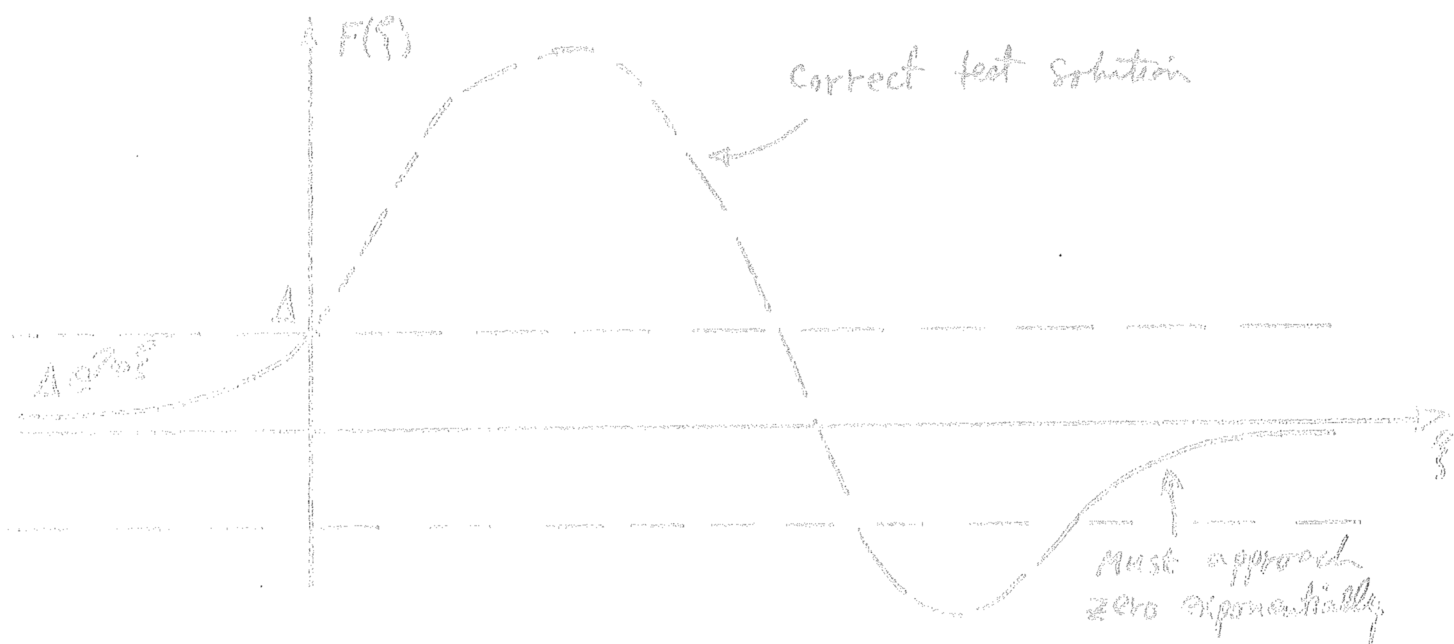


Figure 3-14: The correct (dSS) solution, if it exists, of (1-25) must start and return to the quiescent state, $F(\xi) \equiv 0$, exponentially if the test value of B is to be allowed.

Using this procedure, Nagumo, Arimoto, and Yoshizawa*, generated the pulses in Figure 3-15 and concluded that for $B \cong 30.44488$, a physically realizable, stable (in the sense that if perturbed the system returns

* Much of our development follows their paper "An Active Pulse Transmission-line Simulating Nerve Axon", Proc. IRE, Vol. 50, No. 10, pp. 2061-2070, Oct. 1962.

range above. From (3-21) we see that μ is proportional to e_0, which in turn, by (3-15), is inversely proportional to R_0, the absolute value of the resistance of the tunnel diode at the point (e_0, i_0). Thus, by decreasing the resistance (making the slope at (e_0, i_0) steeper) we increase the velocity of the steady state waveform.

3-6. Lumped Parameter Model

A lumped parameter circuit which approximates the continuous line of Figure 3-8 is shown in Figure 3-17.

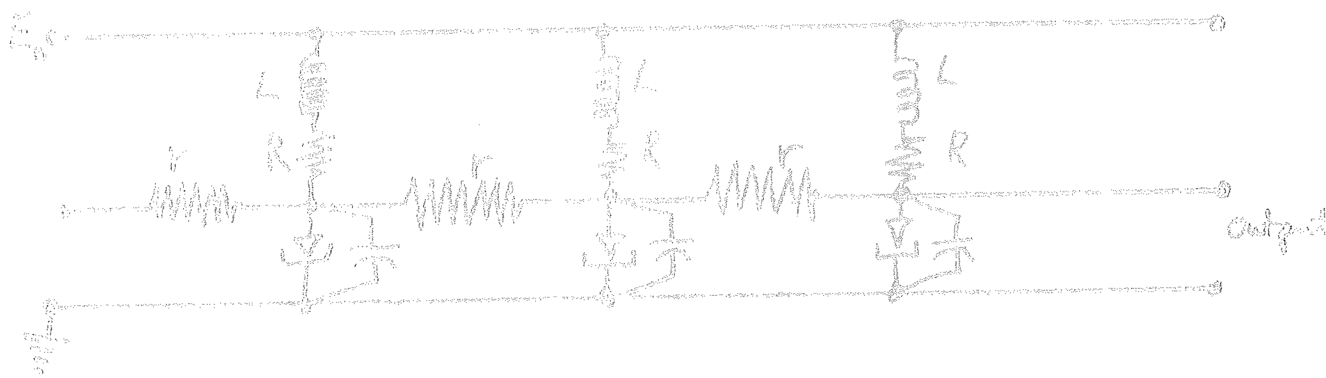
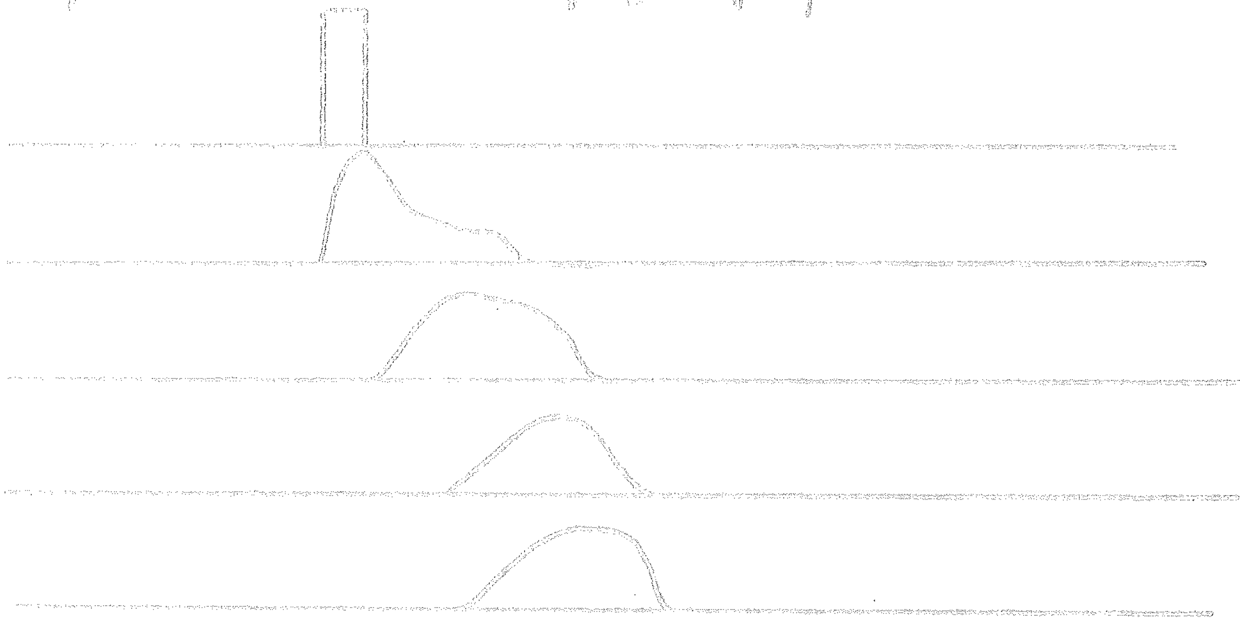


Figure 3-17. Lumped parameter approximation of artificial neuron line.

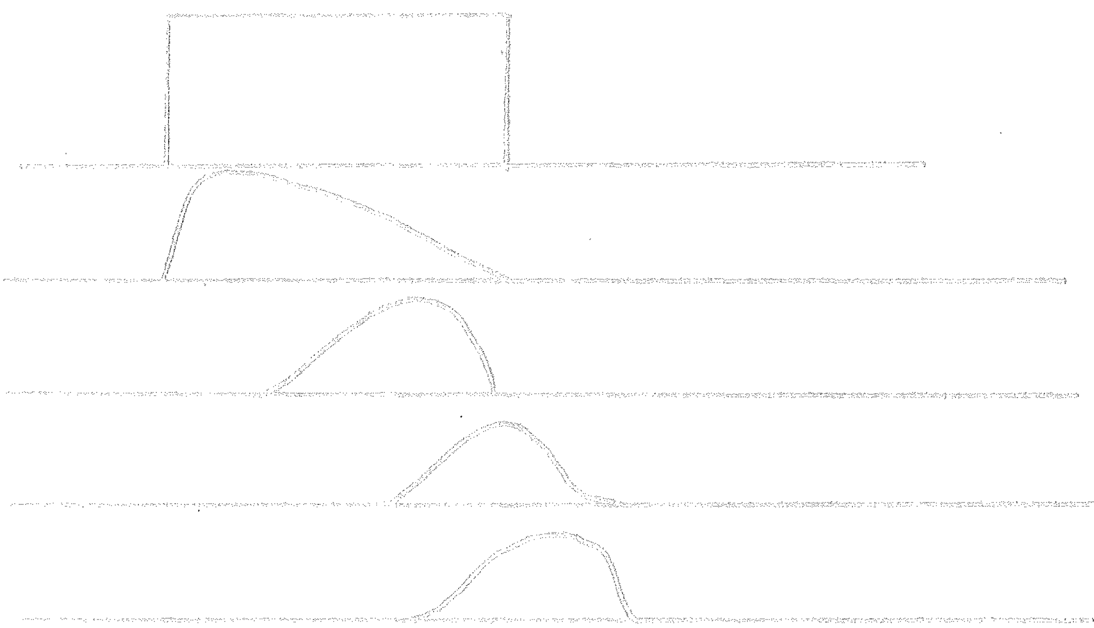
In experiments performed on a nine-stage circuit like the one shown in Figure 3-17, Nagumo, et al,* dramatically displayed the wave shaping properties of an artificial neuron, both with respect to amplitude shaping (amplification attenuation, elimination) and pulse width shaping. Some of their results are shown in Figure 3-18. The original reference should be consulted for further details. In each figure, the top waveform is the input signal, while the second, third fourth and fifth waveforms are those observed at the first, fourth, seventh and ninth stages of the circuit of Figure 3-17, respectively.

* Their values were $L = 4mh$, $R = 70\Omega$, $r = 500\Omega$, $E_0 = 100mv$, $C = 0.01\mu f$.

[illegible], note also that the asymptotic pulse height is reached after one stage of shaping.



(a)



(b)

Figure 3-18. Illustrating experimental results on the wave shaping properties of the lumped parameter artificial neuron of Figure 3-17. (a) A narrow signal (6 μsec) is widened in the course of transmission and (b) a wide signal (32 μsec) is narrowed.

[illegible]

The analytical results obtained in this chapter have been for the unmyelinated axon, i.e., an axon with an active membrane exposed to electrochemical effects continuously along its length. As we have previously mentioned, however, higher order animals and others, possess a myelinated axon which apparently evolved later in biology. Such an axon, illustrated in Figure 3-1, cannot have chemical activity continuously along the line because of the myelin sheath. Rather, the active membrane, and its associated electrochemical "plant", appear at nodes which are concentrated at points separated by the myelin covered internodes.

The incremental equivalent circuit of the myelinated axon is given in Figure 3-19. Note that the internodes are modeled as a linear passive RCG transmission-line. The nodes are lumped elements (due to the point-like concentration of a node. A node length may be of the order of a micron, while that of an internode 2 mm) which are described by the non-linear, Hodgkin-Huxley system (1-2)-(1-4). Thus, the non-linearities of the system are not distributed, as in Figure 3-4 and equation (1-7), but are lumped at regular intervals along the axon. The resulting system, which is easier to program on a computer, is a lumped-distributed transmission-line.

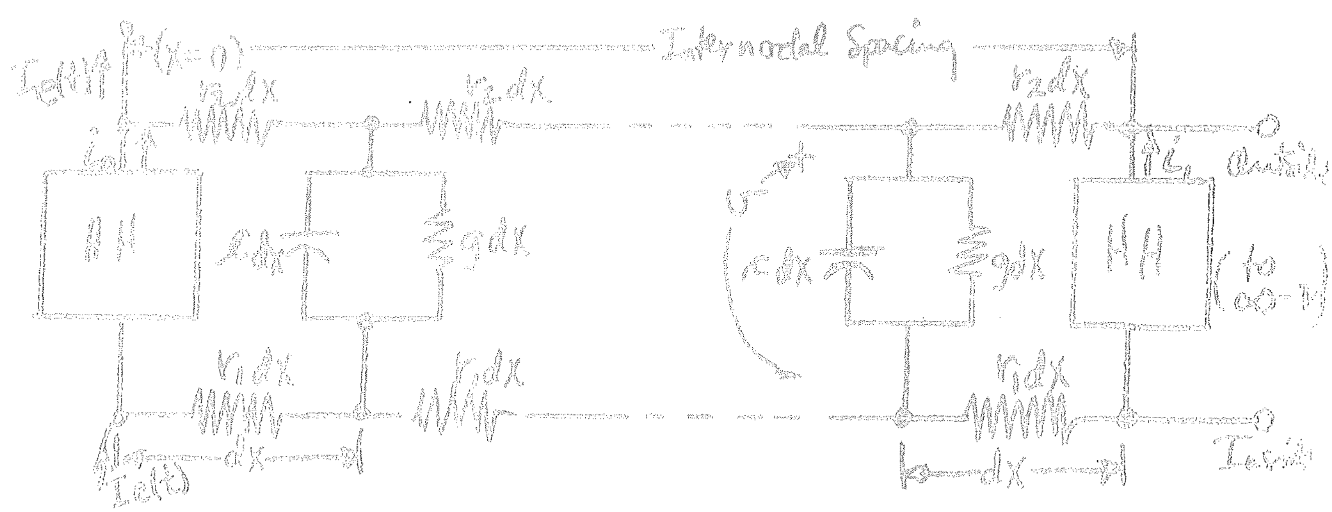


Figure 3-19. Incremental sections of a myelinated neuron. The blocks marked HH stand for the circuit of Figure 3-3.

<u>Table 1.</u>

$v(x,t)$ = potential difference across myelin sheath.

$v_j(t)$ = potential difference across membrane at node j.

$h_j(t)$, $m_j(t)$, $n_j(t)$ = activation variables (HH-model) at node j.

$I_0(t)$ = input excitation current (assumed given) at $x = 0$.

$i_j(t)$ = outward membrane current at node j.

r_1 = internal longitudinal resistance per-unit-length of myelin sheath.

r_2 = external " " " "

c = myelin capacitance per-unit-length.

g = " conductance " "

C_N = total membrane capacitance (same at each node).

g_K, g_{Na}, g_L = total membrane conductances (same at each node).

L = node spacing (fixed).

v_K, v_{Na}, v_L = fixed membrane potentials (same at each node).

The transmission-line equation for the myelin covered internodes is

(1-29) (a) $$c\frac{\partial v}{\partial t} = \left(\frac{1}{r_1 + r_2}\right)\frac{\partial^2 v}{\partial x^2} - g v.$$

The Hodgkin-Huxley model (Figure 3-3) contributes (1-2)-(1-4) at each node. Thus, for node j (j = 0, 1, 2, ...).

(1-29)

(b) $$i_j(t) = C_N\frac{dv_j(t)}{dt} + g_K n_j^4(t)(v_j - v_K) + g_{Na} h_j(t) m_j^3(t)(v_j - v_{Na}) + g_L(v_j - v_L).$$

(c) $$\frac{dh_j(t)}{dt} + [\alpha_h(v_j) + \beta_h(v_j)]h_j(t) = \alpha_h(v_j)$$

(d) $$\frac{dm_j(t)}{dt} + [\alpha_m(v_j) + \beta_m(v_j)]m_j(t) = \alpha_m(j)$$

(e) $$\frac{dn_j(t)}{dt} + [\alpha_n(v_j) + \beta_n(v_j)]n_j(t) = \alpha_n(j).$$

The boundary condition at x = 0 follows from Kirchoff's current law

(1-29) (f) $$i_0(t) = I_e(t) - \frac{\partial v/\partial x \,|_{x=0}}{(r_1 + r_2)}$$

In deriving this result we utilized the continuity of voltage boundary condition which states that the voltage along the line must approach the jth node voltage as x approaches the jth node:

(1-29) (g) $$\lim_{x \to jL} v(x,t) = V_j(t)$$

There must be a discontinuity in $\partial v/\partial x$ at the jth node due to the shunt node current

(1-29) (h) $$i_j(t) = \left(\frac{1}{r_1 + r_2}\right)\left[\lim_{x \to jL_-} \frac{\partial v}{\partial x} - \lim_{x \to jL_+} \frac{\partial v}{\partial x}\right].$$

The boundary conditions (1-29) (f)-(h) permit us to excite the nerve axon at x = 0 by the current stimulus, $I_e(t)$, propagate the resulting wave down the myelinated internodes, and "continue" the wave past one node and on to the next.

In order to complete the mathematical model, one must state the initial conditions. If we assume that the system is initially quiescent (at rest) then all voltages are initially zero as well as all time derivatives. Thus, from (1-29) (c)-(e) we have

(1-29) (i) $$v_j(0) = v(x,0) = 0, \quad \text{all } x$$

(j) $$h_j(0) = \frac{\alpha_h(0)}{(\alpha_h(0) + \beta_h(0))}$$

(k) $$m_j(0) = \frac{\alpha_m(0)}{(\alpha_m(0) + \beta_m(0))}$$

(l) $$n_j(0) = \frac{\alpha_n(0)}{(\alpha_n(0) + \beta_n(0))}.$$

[illegible] occurs due to the initial voltage values of [illegible].

We now have a mathematically determinate system model in (1-29) suitable for digital computation. Once the form of $I_e(t)$ is given, one can approximate, for computational purposes, the differential system of (1-29) by difference equations with small (but non-zero) space and time increments. A typical value for the space increment would be 1/8 to 1/10 th of the internodal spacing. The value of the time increment depends upon the value of the spatial increment due to questions of computational stability in numerical work. One must always be certain that he selects enough time points (or "sample instants") to adequately describe the time variation of all signals. This requires usually sampling at a rate of at least twice the highest frequencies anticipated in any of the input or time responses.

It is interesting to note that such computations have been made* with the resulting observation of the occurrence of an asymptotic (dynamic steady-state, dSS) waveform which propagates (or jumps - which is the meaning of saltatory) from node to node. The waveform does not reach its asymptotic state until about three nodes from the initiating node (i=0 in Figure 3-19).

The internodal waveforms, however, are not reproduced, or asymptotic, but seem to be affected by the potentials at two separate nodes. Results for certain values of the parameters listed in Table [illegible] are shown in Figure 3-20.

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* R. FitzHugh, "Computation of Impulse Initiation and Saltatory Conduction in a Myelinated Nerve Fiber", Biophysical Journal, Vol. 2, 1962, pp. 11-21.

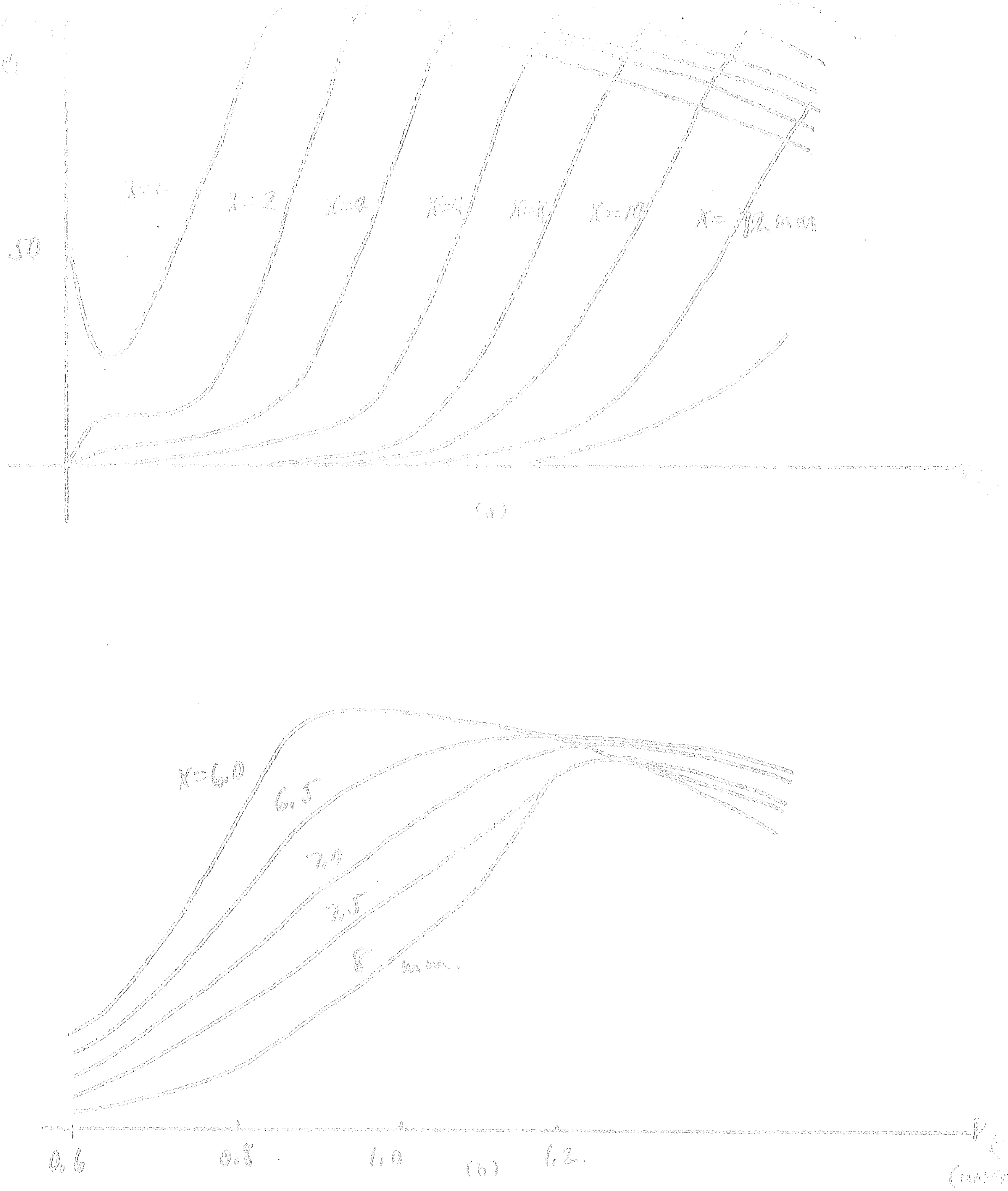


Figure 3-20. Response of the myelinated axon of Figure 3-19 (except that the neuron extends indefinitely to the left as well as to the right) when the excitation, $I_q(t)$, is a square wave pulse of 30 ma. peak, extending for 20 μsec. duration. (a) Response at nodes (2mm spacing between nodes). (b) Internodal response of same excitation.

H. Freygang, Jr., "Some Functions of Nerve Cells in Terms of an Equivalent Network", Proc. IRE, Vol. 47, No. 11, Nov. 1959. pp. 1862-1869.

J. W. Moore, "Electronic Control of Some Active Bioelectric Membranes", Proc. IRE, Vol. 47, No. 11, Nov. 1959, pp. 1869-1880.

J. Nagumo, S. Arimoto and S. Yoshizawa, "An Active Pulse Transmission-line Simulating Nerve Axon" Proc. IRE Vol. 50, No. 10, Oct. 1962, pp. 2061-2070.

Proc. IEEE, Vol. 56, No. 6, June 1968, Special Issue on Studies of Neural Elements and Systems.

R. Suzuki, "Mathematical Analysis and Application of Iron-Wire Neuron Model", IEEE Trans. Bio-Medical Engineering, Vol. BME-14, No. 2, April 1967, pp. 114-124.

R. Fitzhugh, "Impulses and Physiological States in Theoretical Models of Nerve Membrane", Biophysical Journal, Vol. 1, No. 6, July 1961, pp. 445-466.

R. Fitzhugh, "Computation of Impulse Initiation and Saltatory Conduction in a Myelinated Nerve Fiber, Biophysical Journal, Vol. 2, No. 1, Jan. 1962, pp. 11-21.

A. Scott, Active and Non-linear Wave Propagation in Electronics, Wiley-Interscience, New York, 1970.

S. Lefschetz, Differential Equations: Geometric Theory, Interscience, New York, 1957.

R. Plonsey, Bioelectric Phenomena, McGraw-Hill, New York, 1969.

SCATTERING MATRIX AND THE ALGEBRA OF TRANSMISSION-LINES*

In transmission-lines, or microwave circuits involving waveguides (which can also be modeled as transmission-lines), the things that are directly measurable by means of a small probe used to sample the voltage or electric field strength in a waveguide are the standing-wave ratio, location of a voltage (or field) minimum, and power. Knowledge of the first two quantities leads directly to the reflection coefficient. In addition, the transmission coefficient, which relates a transmitted wave in one line to the incident wave from another connecting line, can also be measured directly. The reflection and transmission coefficients are two parameters which relate scattered (transmitted and reflected) waves to an incident wave. They are, therefore, part of a scattering operator which is extremely useful in the theory of guided wave propagation and transmission-lines.

With any transmission-line or obstacle (such as a lumped parameter circuit) we associate a "scattering operator". Such a scattering operator relates the scattored fields directly to the incident field. The treatment of wave propagation and scattering in terms of scattering operators may well be called "an algebraic approach to wave propagation". The

* This chapter is based larged on Chapter 3 of our thesis "Scattering of an Electromagnetic Wave from a Slab with a Randomly Inhomogeneous Dielectric", Purdue University, Aug. 1964. See also R. Redheffer, "Difference Equations and Functional Equations in Transmission-line Theory", Chapt. 12 in E.F. Beckenbach, Ed., Modern Mathematics for the Engineer, Second Series, McGraw-Hill, 1961, and R.E. Collin, Foundations for Microwave Engineering, Chapt. 4, McGraw Hill, 1966.

theory of invariant imbedding is based on this approach. This theory complements the classical approach based directly on the transmission-line equations. To be more specific, we may say that in Kirchoff's laws (upon which the transmission-line equations are based) the fundamental quantities are the voltages and currents which satisfy certain differential equations and boundary conditions. In the algebraic approach the determination of the scattering operator, which relates field quantities at two points in space, is the prime consideration.

As we shall show, the scattering matrix satisfies a Ricatti system of ordinary differential equations with an initial condition. It is typical of an invariant imbedding approach to boundary value problems that one ends up with an initial value problem.

The historical development of the subject of difference equations, i.e., the algebraic approach, in wave scattering has been traced back to Stokes (Redheffer, 1961). The introduction of a principle of invariance in the treatment of radiative transfer problems was made by Ambarzumian (Chandrasekhar, 1960). Ambarzumian's principle was extended and refined by Chandrasekhar, again within the context of radiative transfer problems. Bellman and Kalaba (1959,1960) generalized the idea in order to apply it to a variety of problems in mathematical physics, especially in connection with numerical solutions, and coined the term, "invariant imbedding". They have also suggested the use of such techniques in the numerical investigation of wave propagation in random media. The systematic application of the principle to one-dimensional wave propagation problems is due primarily to Redheffer. Additional references are given in Redheffer's paper.

This chapter is essentially an exposition of Redheffer's paper with additional comments relative to our problem.

The Scattering Matrix

Much of what follows may be likened to two-port network theory if "line" or "obstacle" replaces "network" and "inputs" is replaced by "incident waves from the left and right" and "outputs" is replaced by "reflected and transmitted waves."

Let the obstacle extend from x_1 to x_2 as shown in Fig. 1. v_1 is the complex amplitude of the right-going wave at x_1; v_2 is the complex amplitude of the left-going wave at x_1; v_3 is the complex amplitude of the right-going wave at x_2; and v_4 is the complex amplitude of the left-going wave at x_2. v_1 and v_4 are the incident waves, and v_2 and v_3 are the scattered (transmitted or reflected) waves.

The scattering matrix, $S(x_1,x_2)$, of the obstacle extending from x_1 to x_2 is defined by the vector-matrix relation

$$\begin{pmatrix} v_3 \\ v_2 \end{pmatrix} = \begin{bmatrix} t(x_1,x_2) & \rho(x_1,x_2) \\ r(x_1,x_2) & \tau(x_1,x_2) \end{bmatrix} \begin{pmatrix} v_1 \\ v_4 \end{pmatrix}, \qquad (4\text{-}1)$$

and the 2-by-2 matrix appearing here is the scattering matrix. Its complex elements are: t, the transmission coefficient from x_1 to x_2; ρ, the right-hand reflection coefficient; r, the left-hand reflection coeffickent; and τ, the transmission coefficient from x_2 to x_1. Thus the scattering matrix acts as an operator relating outgoing, or scattered, waves at x_1 and x_2 to incoming, or incident, waves at x_1 and x_2.

We obtain an "initial condition" for $S(x_1,x_2)$ by arguing that if ... als x_2, then the slab is of zero thickness, which means that, ...si ly, there is no scatterer present (recall that the lines on ...e of the obstacle are identical). Therefore, v_1 equals v_3, and ... From this we deduce that $S(x_1,x_1)$ is the identity matrix, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

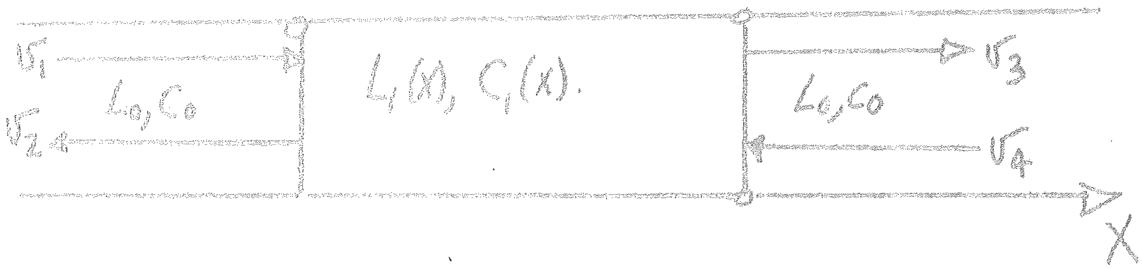


Figure 4-1. An obstacle extending from x_1 to x_2 with Incident and Scattered Waves.

Next we wish to find the rule for combining scattering matrices of two adjacent obstacles, as shown in Fig. 4-2. We have the equations

$$(4\text{-}2)\qquad \begin{pmatrix} v_5 \\ v_4 \end{pmatrix} = S_2 \cdot \begin{pmatrix} v_3 \\ v_6 \end{pmatrix} = \begin{bmatrix} t_2 & \rho_2 \\ r_2 & \tau_2 \end{bmatrix} \begin{bmatrix} v_3 \\ v_6 \end{bmatrix}$$

$$\begin{pmatrix} v_3 \\ v_2 \end{pmatrix} = S_1 \cdot \begin{pmatrix} v_1 \\ v_4 \end{pmatrix} = \begin{bmatrix} t_1 & \rho_1 \\ r_1 & \tau_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_4 \end{bmatrix} .$$

We eliminate v_3 and v_4 from this system of four equations and solve for v_5 and v_2 in terms of v_1 and v_6 to get

$$(4\text{-}3)\qquad \begin{pmatrix} v_5 \\ v_2 \end{pmatrix} = \begin{bmatrix} \dfrac{t_1 t_2}{1-\rho_1 r_2} & \rho_2 + \dfrac{\rho_1 t_2 \tau_2}{1-\rho_1 r_2} \\ r_1 + \dfrac{r_2 t_1 \tau_1}{1-\rho_1 r_2} & \dfrac{\tau_1 \tau_2}{1-\rho_1 r_2} \end{bmatrix} \begin{pmatrix} v_1 \\ v_6 \end{pmatrix} .$$

This matrix is called the composite scattering matrix of the two slabs. It defines the "star-product" of two scattering matrices

$$(4\text{-}4)\qquad \begin{pmatrix} t_1 & \rho_1 \\ r_1 & \tau_1 \end{pmatrix} * \begin{pmatrix} t_2 & \rho_2 \\ r_2 & \tau_2 \end{pmatrix} = \begin{bmatrix} \dfrac{t_1 t_2}{1-\rho_1 r_2} & \rho_2 + \dfrac{\rho_1 t_2 \tau_2}{1-\rho_1 r_2} \\ r_1 + \dfrac{r_2 t_1 \tau_1}{1-\rho_1 r_2} & \dfrac{\tau_1 \tau_2}{1-\rho_1 r_2} \end{bmatrix} .$$

Thus, the star-product is not an ordinary matrix-product.

It is interesting to show how the same result may be obtained by using signal-flow graphs (Sabbagh, 1961, Ch. 21). The appropriate signal-flow graph is shown in Fig. 4-3. This diagram is made by simply writing out the four equations implicit in (4-2) and using arrows to point away from independent variables towards dependent variables. In the diagram all arrows point away from nodes v_1 and v_6, while nodes v_2 and v_5 are receivers only.

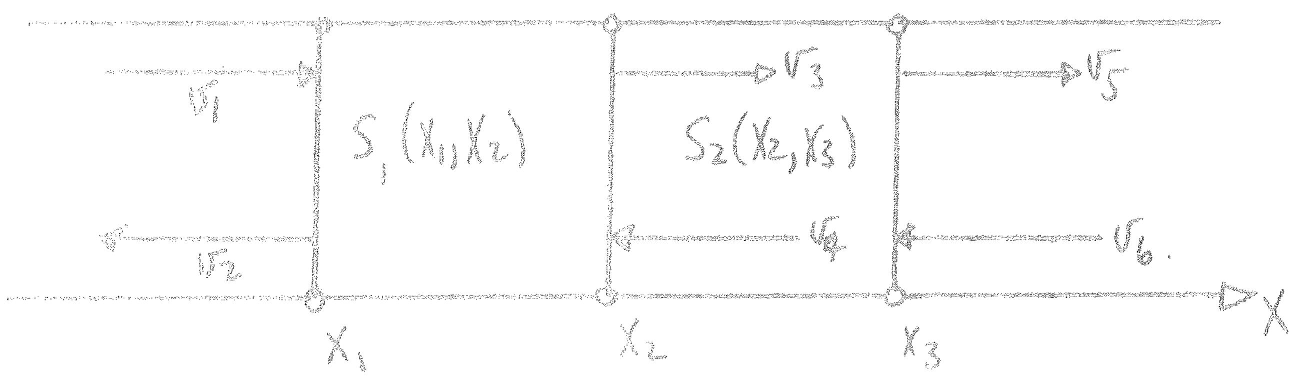


Figure 4-2. Two Adjacent Obstacles with Incident and Scattered Waves

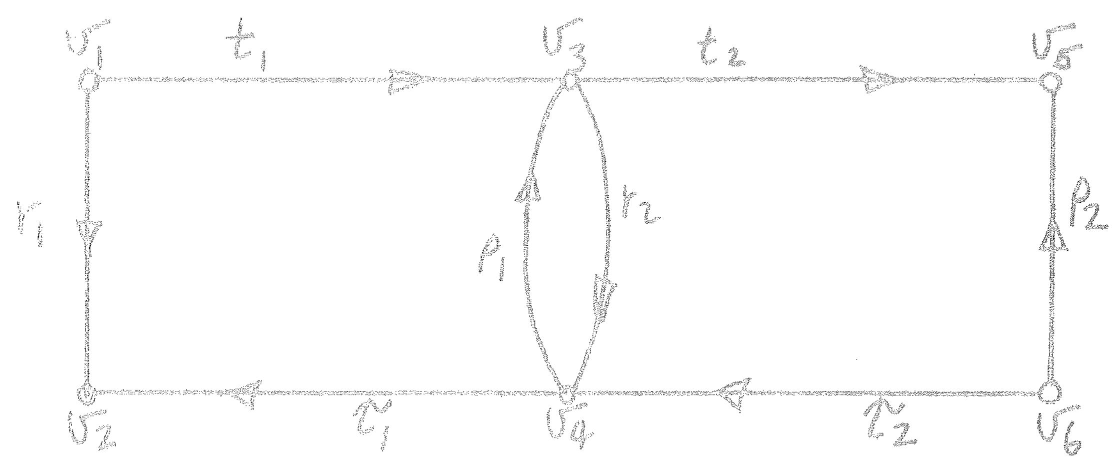


Figure 4-3. The Signal-Flow Graph for Two Adjacent Obstacles

[illegible] v_2 and v_5 in terms of v_1 and v_6. There are two forward paths from v_1 to v_2 -- one with transmittance, r_1, and the other with transmittance, $t_1 \tau_1 r_2$. There is a feedback loop of transmittance, $\rho_1 r_2$, touching this latter path, and it introduces a denominator term $1 - \rho_1 r_2$. Thus, the net transmittance from v_1 to v_2 is $r_1 + \dfrac{t_1 \tau_1 r_2}{1-\rho_1 r_2}$. There is only one path from v_6 to v_2, and it also has the same feedback loop touching it. Thus, the transmittance from v_6 to v_2 is $\dfrac{\tau_1 \tau_2}{1-\rho_1 r_2}$. Similarly, the transmittance from v_1 to v_5 is

$\dfrac{t_1 t_2}{1-\rho_1 r_2}$, and that from v_6 to v_5 is $\rho_2 + \dfrac{\rho_1 t_2 \tau_2}{1-\rho_1 r_2}$.

These, of course, are the results obtained by explicit algebraic elimination of variables.

Example 1: Find the scattering matrix of a length, l, of a transmission-line with propagation constant k.

Solution: The total voltage phasor at any point x is given by

$$V(x) = V_1 e^{-jkx} + V_2 e^{jkx},$$

where V_1 is the amplitude of the positive going wave at x=0 and V_2 the amplitude of the negative going wave at x=0.

If $V_2=0$, then there is only a positive going wave, which at x=l has the value $V_1 e^{-jkl}$. Hence, the transmission coefficient t is given by

$$t = \frac{V_1 e^{-jkl}}{V_1} = e^{-jkl}.$$

Similarly, if $V_1=0$, then there is only a negative going wave, which at x=0 has the value V_2 and at x=l has the value $V_2 e^{jkl}$. Hence, in this case, τ is also given by

$$\tau = \frac{V_2}{V_2 e^{jkl}} = e^{-jkl} = t.$$

The reflection coefficients vanish. Hence

$$S(0,l) = \begin{pmatrix} e^{-jkl} & 0 \\ 0 & e^{-jkl} \end{pmatrix}.$$

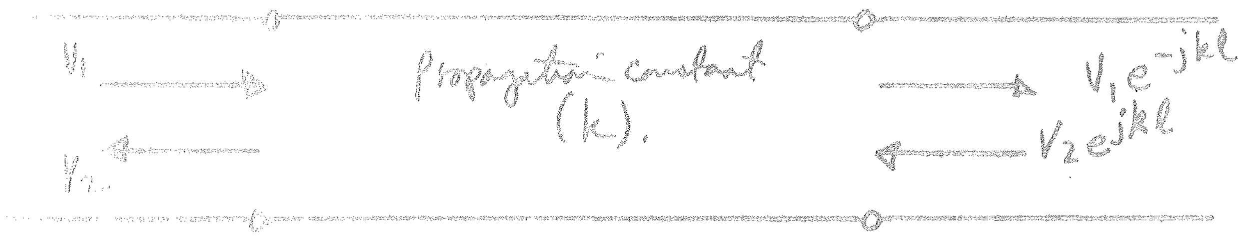


Figure 4-4. Determining the scattering matrix of a length, l, of transmission-line.

The power carried by a voltage wave, V(x), along a transmission-line whose characteristic impedance is z_c (or characteristic admittance is $y_c = 1/z_c$) is

(4-5) $$P = \frac{1}{2}|V|^2 \cdot Y_c = \frac{\frac{1}{2}|V|^2}{Z_c}.$$

Thus, if V^+ stands for a voltage wave incident upon an obstacle and V^- for a voltage wave reflected from an obstacle, then the incident power $P_+ = \frac{1}{2}|V_+|^2 Y_c$, and the reflected power is $P_- = \frac{1}{2}|V^-|^2 Y_c$.

The power carried by a wave clearly depends upon the nature of the transmission-line, i.e., upon its characteristic impedance or admittance. In order to make the power independent of the transmission-line, we define <u>normalized voltages</u>

(4-6) $$\bar{V} = \frac{V}{\sqrt{Z_c}} = V\sqrt{Y_c}$$

so that power now becomes

(4-7) $$P = \frac{1}{2}|\bar{V}|^2,$$

which is independent of the actual transmission-line. Equivalently, what we are doing when we work with normalized voltages and define power as the square of those voltages, as in (4-7), is saying that the characteristic impedances of all lines are unity. The reason for doing this is to ensure a certain symmetry in the scattering matrix. In fact, we

Consider the lossless obstacle of Figure 4-5.

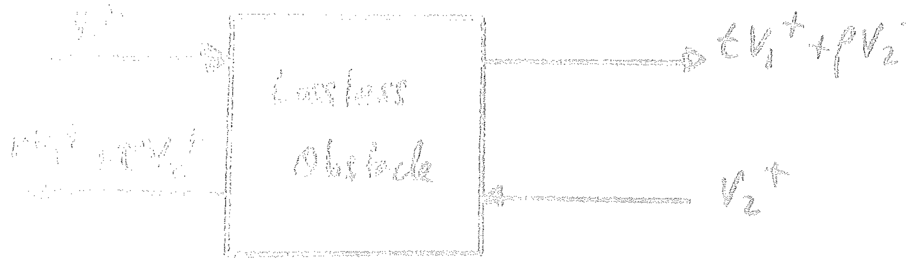


Figure 4-5. Incident and reflected waves upon a lossless obstacle.

If $V_2^+ = 0$ (we are now dropping the bar over the symbols but we are using normalized voltages), then the power balance equation becomes

(4-8) $$|V_1^+|^2 = |t|^2 |V_1^+|^2 + |r|^2 |V_1|^2$$

or

(4-9) (a) $$1 = |t|^2 + |r|^2.$$

Similarly

(4-9) (b) $$1 = |\tau|^2 + |\rho|^2.$$

Equations (4-9) are conditions satisfied by a lossless obstacle. If the obstacle is lossy then (4-9) is replaced by

(4-10)

(a) $$1 \geq |t|^2 + |r|^2$$

(b) $$1 \geq |\tau|^2 + |\rho|^2.$$

Now suppose that the obstacle of Figure 4-5 is cascaded with its mirror image, so that the resulting arrangement is shown in Figure 4-6.

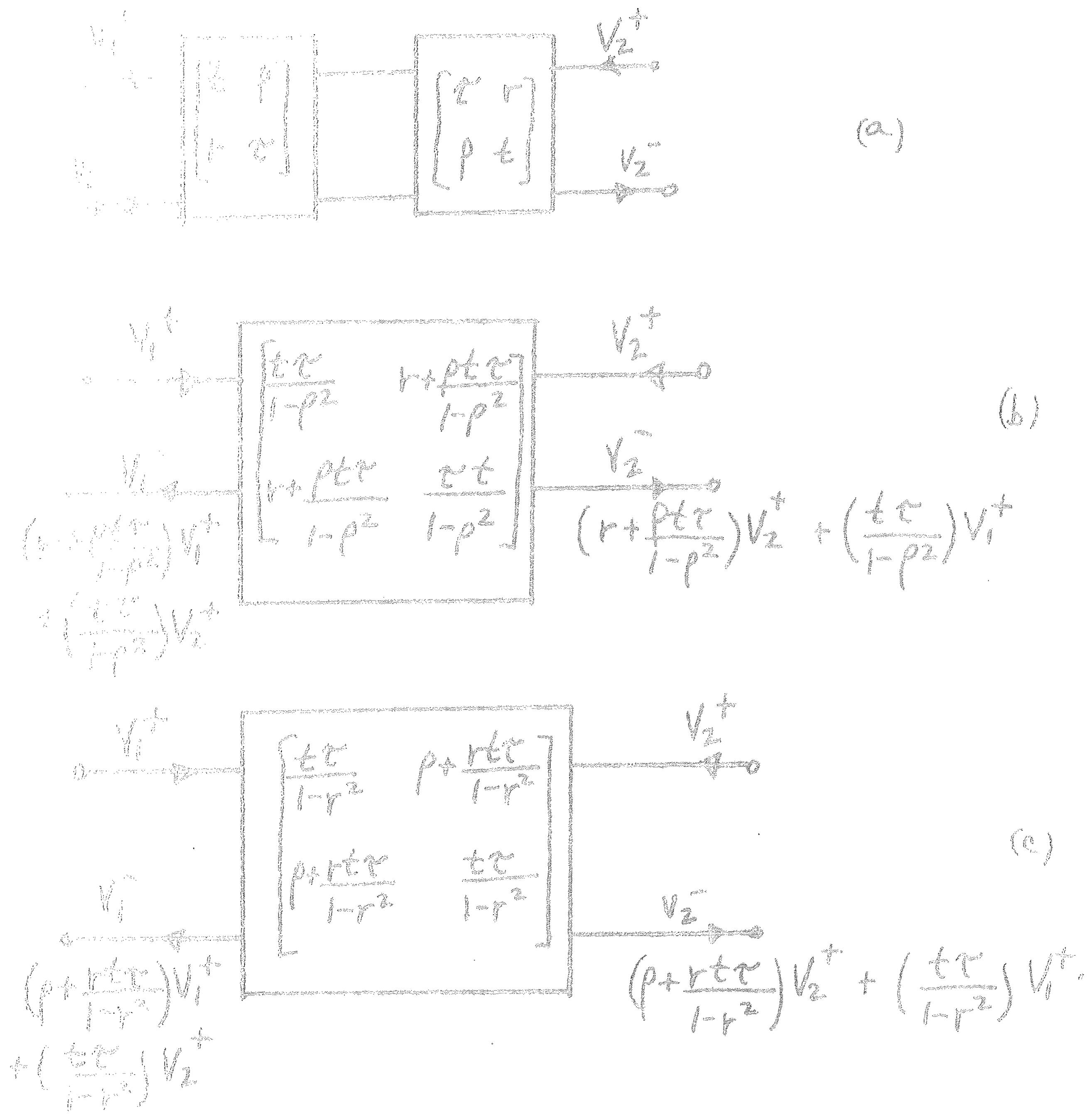


Figure 4-6. Cascaded arrangements of two mirror-imaged obstacles. (a) Original arrangement (b) combined scattering matrix utilizing (4-3) or (4-4). (c) combined scattering matrix when each of the obstacles of (a) are turned end for end.

In Figure 4-6(b), let $V_2^+=0$, so that the magnitude of the transmitted wave, divided by $|V_1^+|$, is

$$\frac{|t||\tau|}{|1-\rho^2|} \tag{4-11}$$

The [illegible] takes on its maximum value when ρ is a pure real number, so that $\rho^2 = |\rho|^2$. (This is easily proved by drawing a phasor diagram and considering the minimum distance between ρ^2 and 1). Thus

$$(4\text{-}12) \qquad \frac{|t||\tau|}{|1-\rho^2|} \leq \frac{|t||\tau|}{1-|\rho|^2} = \frac{|t||\tau|}{|\tau|^2} = \frac{|t|}{|\tau|} \leq 1.$$

The final inequality follows because for a passive obstacle, the transmission coefficient must be bounded by unity.

If we repeat these arguments using Figure 4-6(c), we obtain the inequality

$$(4\text{-}13) \qquad \frac{|t||\tau|}{|1-r^2|} \leq \frac{|t||\tau|}{1-|r|^2} = \frac{|t||\tau|}{|t|^2} = \frac{|\tau|}{|t|} \leq 1.$$

Hence, the only way that the semi-inequalities $|t| \leq |\tau|$, $|\tau| \leq |t|$ can be resolved is if

$$(4\text{-}14) \qquad |t| = |\tau|.$$

Therefore, we have deduced that the magnitudes of the two transmission coefficients of a lossless network are equal. If the phases are equal, $\angle t = \angle \tau$, then $t = \tau$ and the network (or obstacle) is called reciprocal.

Again, these conclusions hold only if we define the incident and scattered voltages to be normalized.

Example 2. Determine the normalized and non-normalized scattering matrices for the junction of two transmission-lines whose characteristic impedances are z_1 and z_2.

Solution: Using non-normalized values, the solution for the voltage and current in line No. 1 (on the left) is

$$I_1 = \frac{V_1^+}{Z_1} e^{-jk_1x} - \frac{V_1^-}{Z_1} e^{jk_1x}$$

and for line No. 2

$$V_2 = V_2^- e^{-jk_2x} + V_2^+ e^{jk_2x}$$

$$I_2 = \frac{V_2^-}{Z_2} e^{-jk_2x} - \frac{V_2^+}{Z_2} e^{jk_2x}$$

At the junction, $x=0$, we have $V_1=V_2$, $I_1=I_2$, which implies that

$$V_1^+ + V_1^- = V_2^+ + V_2^-$$

$$\frac{V_1^+}{Z_1} - \frac{V_1^-}{Z_1} = \frac{V_2^-}{Z_2} - \frac{V_2^+}{Z_2}$$

or

$$V_1^- = \frac{Z_2 - Z_1}{Z_2 + Z_1} V_1^+ + \frac{2Z_1}{Z_1 + Z_2} V_2^+$$

$$V_2^- = \frac{2Z_2}{Z_1 + Z_2} V_1^+ + \frac{Z_1 - Z_2}{Z_1 + Z_2} V_2^+ .$$

Hence, the non-normalized (i.e., using non-normalized voltages and currents) scattering matrix elements become

$$t = \frac{2Z_2}{Z_1 + Z_2}, \quad \tau = \frac{2Z_1}{Z_1 + Z_2}, \quad r = \frac{Z_2 - Z_1}{Z_2 + Z_1} = -\rho .$$

Note that $\tau \neq t$.

The normalized scattering matrix elements are obtained by dividing the equations for V_1^- and V_2^- by, respectively, $\sqrt{Z_1}$, $\sqrt{Z_2}$:

$$\frac{V_1^-}{\sqrt{Z_1}} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \cdot \frac{V_1^+}{\sqrt{Z_1}} + \frac{2Z_1}{Z_1 + Z_2} \cdot \frac{V_2^+}{\sqrt{Z_1}}$$

$$\frac{V_2^-}{\sqrt{Z_2}} = \frac{2Z_2}{Z_1 + Z_2} \cdot \frac{V_1^+}{\sqrt{Z_2}} - \frac{Z_2 - Z_1}{Z_2 + Z_1} \cdot \frac{V_2^+}{\sqrt{Z_2}} ,$$

Defining $V^{\pm}_{1,2}/\sqrt{z_{1,2}}$ to be $\bar{V}^{\pm}_{1,2}$, these equations become

$$\bar{V}_1^- = \frac{z_2 - z_1}{z_2 + z_1}\,\bar{V}_1^+ + \frac{2\sqrt{z_1}\sqrt{z_2}}{z_1 + z_2}\,\bar{V}_2^+$$

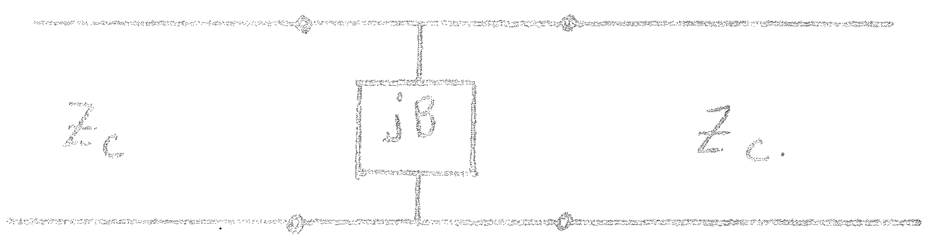
$$\bar{V}_2^- = \frac{2\sqrt{z_1}\sqrt{z_2}}{z_1 + z_2}\cdot\bar{V}_1^+ - \frac{z_2 - z_1}{z_2 + z_1}\cdot\bar{V}_2^+.$$

Hence,

$$\bar{t} = \frac{2\sqrt{z_1}\sqrt{z_2}}{z_1 + z_2} = \bar{\tau}\,, \quad \bar{r} = \frac{z_2 - z_1}{z_1 + z_2} = -\bar{\rho}.$$

Note that in the normalized variables $\bar{t} = \bar{\tau}$. Because we will usually be working in normalized variables, unless otherwise stated, we will suppress the bar over the symbols. When necessary to distinguish normalized and non-normalized variables, we will, of course, use the bar notation.

Example 3. Determine the scattering matrix of the shunt susceptance shown in the figure.



Solution: Because the shunt susceptance is assumed to be imbedded in a transmission-line of uniform characteristic impedance, z_c, the non-normalized and normalized scattering parameters are identical.

If the output line is matched (so that $V_2^+ = 0$ when $V_1^+ \neq 0$), then it is easy to see that

$$r = \frac{z_L - z_c}{z_L + z_c} = \frac{\frac{1}{1/z_c + jB} - z_c}{\frac{1}{1/z_c + jB} + z_c} = \frac{Y_c - Y_c - jB}{Y_c + Y_c + jB}$$

$$= \frac{-jB}{2Y_c + jB} = \rho.$$

[illegible] $\rho = \rho$. In order to determine the [illegible] coefficients, we observe that for a shunt element the voltages at each port must be equal. Hence,

$$V_1^+ + V_1^- = V_1^+(1+r) = V_2^- = V_1^+ \cdot t.$$

Hence,

$$t = 1+r = \frac{2Y_c + jB - jB}{2Y_c + jB} = \frac{2Y_c}{2Y_c + jB} = \tau.$$

The Ricatti System

By using these results we can derive the differential equations satisfied by the elements of the scattering matrix. Upon referring to Fig. 4-2 and (4-4) we have

$$\text{(4-15)} \quad \begin{bmatrix} t(x_1,x_2) & \rho(x_1,x_2) \\ r(x_1,x_2) & \tau(x_1,x_2) \end{bmatrix} * \begin{bmatrix} t(x_2,x_3) & \rho(x_2,x_3) \\ r(x_2,x_3) & \tau(x_2,x_3) \end{bmatrix}$$

$$= \begin{bmatrix} t(x_1,x_3) & \rho(x_1,x_3) \\ r(x_1,x_3) & \tau(x_1,x_3) \end{bmatrix}$$

where $x_1 \leq x_2 \leq x_3$. The elements are defined by their arguments; thus, $t(x_1,x_2)$ is the transmission coefficient for the slab extending from x_1 to x_2, $t(x_2,x_3)$ is the transmission coefficient for the slab extending from x_2 to x_3, and $t(x_1,x_3)$ is the transmission coefficient for the composite slab extending from x_1 to x_3. Similar definitions hold for the other elements. Thus, we have for the elements of the composite scattering matrix

$$\text{(a)} \quad t(x_1,x_3) = \frac{t(x_1,x_2)\cdot t(x_2,x_3)}{1-\rho(x_1,x_2)\,r(x_2,x_3)}, \quad t(x_1,x_1) = 1,$$

$$\text{(b)} \quad \rho(x_1,x_3) = \rho(x_2,x_3) + \frac{\rho(x_1,x_2)\,t(x_2,x_3)\,\tau(x_2,x_3)}{1-\rho(x_1,x_2)\,r(x_2,x_3)}, \quad \rho(x_1,x_1) = 0.$$

(4-16)

$$\text{(c)} \quad r(x_1,x_3) = r(x_1,x_2) + \frac{r(x_2,x_3)\,t(x_1,x_2)\,\tau(x_1,x_2)}{1-\rho(x_1,x_2)\,r(x_2,x_3)}, \quad r(x_1,x_1) = 0.$$

$$\text{(d)} \quad \tau(x_1,x_3) = \frac{\tau(x_1,x_2)\,\tau(x_2,x_3)}{1-\rho(x_1,x_2)\,r(x_2,x_3)}, \quad \tau(x_1,x_1) = 1.$$

[illegible] of the quantities

$$(3\text{-}17)\quad \begin{aligned} &(a)\quad a(x_1) = \frac{\partial \rho}{\partial x}(x_1, x_1) \\ &(b)\quad b_1(x_1) = \frac{\partial t}{\partial x}(x_1, x_1) \\ &(c)\quad b_2(x_1) = \frac{\partial \tau}{\partial x}(x_1, x_1) \\ &(d)\quad c(x_1) = \frac{\partial r}{\partial x}(x_1, x_1) \end{aligned}$$

where differentiation is with respect to the second variable.

By using these results, we shall derive the differential equation satisfied by $\rho(x_1, x)$; the others follow similarly. Eq. (4-16)(b) is expanded as

$$\rho(x_1,x_3) = \rho(x_2,x_3) + \rho(x_1,x_2)\,t(x_2,x_3)\,\tau(x_2,x_3)\cdot\left[1+\rho(x_1,x_2)\,r(x_2,x_3) + \rho^2(x_1,x_2)\,r^2(x_2,x_3) + \cdots\right].$$

Then using the initial conditions for t, ρ, r, τ, together with (4-17) and Taylor's theorem, we have

$$\rho(x_1,x_3) = a(x_2)\cdot(x_3-x_2) + \rho(x_1,x_2)\left[1+\left(b_1(x_2)+b_2(x_2)\right)\cdot(x_3-x_2)\right]\cdot \left[1+\rho(x_1,x_2)\cdot c(x_2)\cdot(x_3-x_2)\right] + O(x_3-x_2),$$

where we have expanded all terms with arguments (x_2, x_3) in a Taylor series about x_2. Rearranging and dividing by x_3-x_2 gives

$$\frac{\rho(x_1,x_3)-\rho(x_1,x_2)}{x_3-x_2} = a(x_2) + \left(b_1(x_2)+b_2(x_2)\right)\cdot\rho(x_1,x_2) + c(x_2)\,\rho^2(x_1,x_2) + O(x_3-y_2).$$

... Ricatti's differential equation:

$$(4\text{-}18)(a) \quad \frac{\partial \rho}{\partial x}(X_1, x) = a(x) + [b_1(x) + b_2(x)] \cdot \rho(X_1, x) + c(x) \cdot \rho^2(X_1, x); \qquad \rho(X_1, X_1) = 0.$$

... equations are

$$\frac{\partial r}{\partial x}(X_1, x) = c(x)\, t(X_1, x)\, \tau(X_1, x); \qquad r(X_1, X_1) = 0.$$

$$(4\text{-}18) \quad \frac{\partial t}{\partial x}(X_1, x) = t(X_1, x)\left(c(x)\rho(X_1, x) + b_1(x)\right); \qquad t(X_1, X_1) = 1.$$

$$\frac{\partial \tau}{\partial x}(X_1, x) = \tau(X_1, x)\left(c(x)\rho(X_1, x) + b_2(x)\right); \qquad \tau(X_1, X_1) = 1.$$

...entially, the initial conditions are what one would naturally ...ociate with a two-parameter transformation; the transformation reduces ... the identity transformation when the two parameters are equal. The ...sical model corresponding for the transformation is that illustrated ... Fig. 4-1, i.e., it is a transmission-line imbedded in a homogeneous ...nsmission-line whose characteristics are assumed given and fixed ...oughout the chapter. Therefore, the imbedding process leaves equations ...-18) invariant in the sense that they hold for all values of the parameters ..., X_1. Because the scattering matrix relates the field at two points ... space, it describes global properties, whereas the coefficients, b_1, b_2, and c describe local properties.

Eqs. (4-18) form a Ricatti system because (4-18)(a) is a Ricatti ...ation. The remaining equations are linear but have variable coefficients.*

---

*Taken as a system, however, (4-18) is non-linear because it involves products of unknowns within the system.

[illegible] appear for the scattering [illegible] are also non-linear with qualitatively [illegible] of quadratic non-linearity that symbolizes the Ricatti [illegible] (Chandrasekhar, 1960).

[illegible] transmission-line parameters, $\mathcal{C}$, L, present in the transmission- [illegible] appear in the algebraic formulation. We shall show this [illegible] a, b_1, b_2, and c. In Fig. 4-4 we consider an obstacle [illegible] that c is essentially constant across its thickness. The [illegible] is taken to be $\mathcal{C}(x+0)$, i.e., the limiting value of c [illegible] the left end. We treat this obstacle as a composite of [illegible] obstacles -- an interface obstacle at x and one at x + Δx [illegible] a length, Δx, of transmission-line possessing parameters $\mathcal{C}(x+0)$, L_0. The interface obstacles have scattering coefficients, [illegible] τ_1, ρ_1, and t_2, r_2, τ_2, ρ_2, respectively at x and x+Δx.

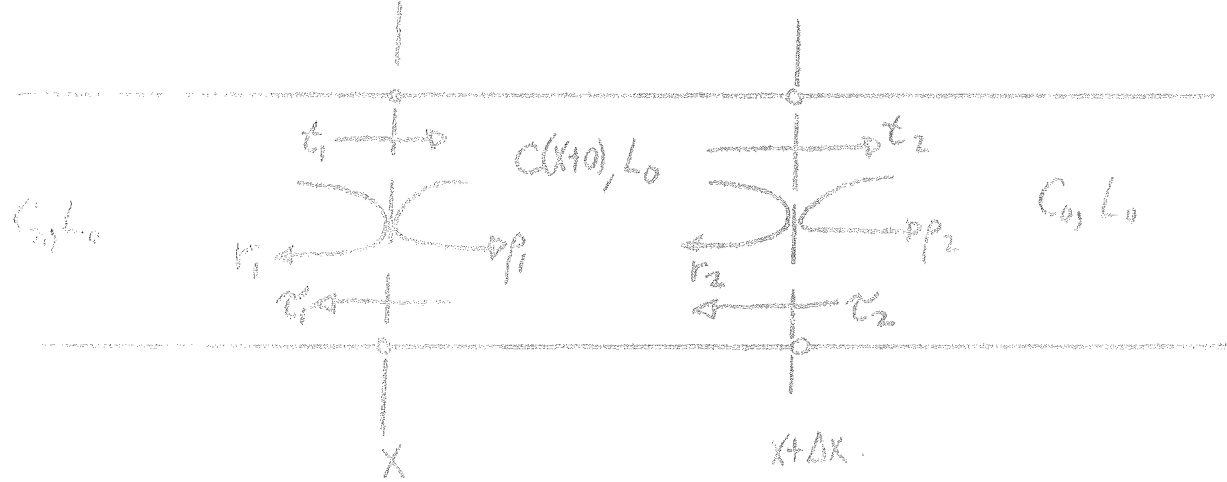


Figure [illegible]-4. A Thin Obstacle with Interface Scattering Coefficients [illegible] across each interface requires [illegible] r_2 [illegible] $1+r_2$, $\tau_1 = 1 + \rho_1$, and $\tau_2 = 1 + \rho_2$.

The scattering matrix of the slab is obtained either from the appropriate triple star-product or from the signal-flow diagram for the three scattering matrices,

$$S_1 = \begin{pmatrix} 1+r_1 & \rho_1 \\ r_1 & 1+\rho_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} e^{-jk\Delta x} & 0 \\ 0 & e^{-jk\Delta x} \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1+r_2 & \rho_2 \\ r_2 & 1+\rho_2 \end{pmatrix},$$

where S_2 is the scattering matrix of a length of transmission-line with propagation constant, $k = \omega\sqrt{L_0 C(x+0)}$. The signal-flow diagram is given in Fig. 4-5. The result for the composite scattering matrix is

$$(4\text{-}19) \qquad S(\Delta x) = \begin{pmatrix} \dfrac{(1+r_1)(1+r_2)e^{-jk\Delta x}}{1-\rho_1 r_2 e^{-j2k\Delta x}} & \rho_2 + \dfrac{\rho_1(1+r_2)(1+\rho_2)e^{-j2k\Delta x}}{1-\rho_1 r_2 e^{-j2k\Delta x}} \\ r_1 + \dfrac{r_2(1+r_1)(1+\rho_1)e^{-j2k\Delta x}}{1-\rho_1 r_2 e^{-j2k\Delta x}} & \dfrac{(1+\rho_1)(1+\rho_2)e^{-j2k\Delta x}}{1-\rho_1 r_2 e^{-j2k\Delta x}} \end{pmatrix},$$

By taking $\rho_1 = -r_1$, $r_2 = -r_1$, $\rho_2 = r_1$, we get $S(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, as required. We also put

$$r_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{1 - Z}{1 + Z},$$

where $Z_1 = \sqrt{\dfrac{L_0}{C(x+0)}}$, $Z_0 = \sqrt{\dfrac{L_0}{C_0}}$, and $Z = \dfrac{Z_0}{Z_1} = \sqrt{\dfrac{C(x+0)}{C_0}}$. When we insert these expressions into (4-19) and then differentiate the result with respect to Δx and set $\Delta x = 0$, we get upon using (4-17),

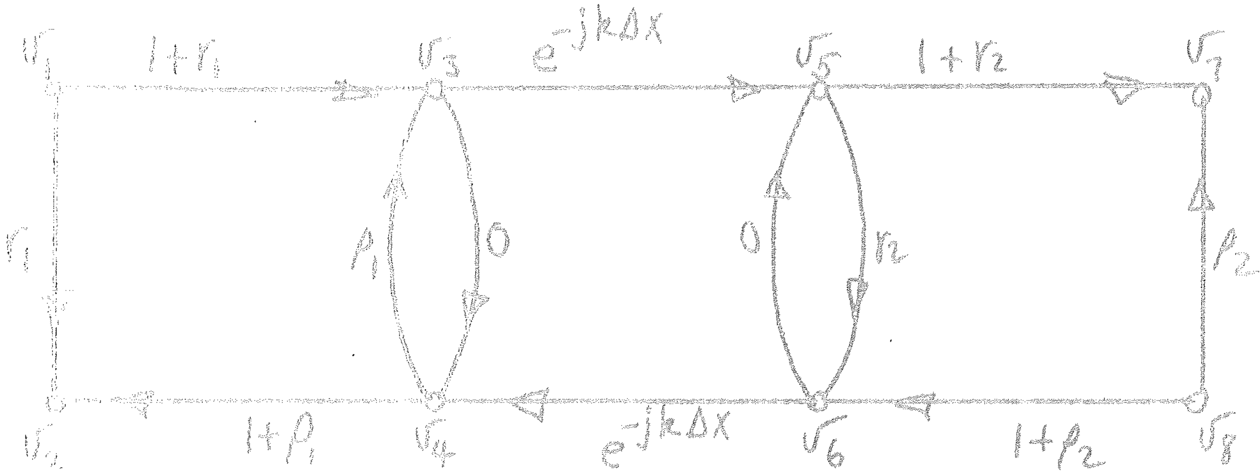


Figure 4-5. The Signal-Flow Graph for Two Interface Obstacles Sandwiched Around a Transmission-Line.

$$\text{(a)}\quad a(x) = c(x) = -\tfrac{1}{2}\, jk(z - z^{-1}) = -\tfrac{1}{2}\, jk_o \frac{c'(x)}{c_o}\,, \tag{4-20}$$

$$\text{(b)}\quad b_1(x) = b_2(x) = -\tfrac{1}{2}\, jk(z + z^{-1}) = -\tfrac{1}{2}\, jk_o\left(\frac{c'(x)}{c_o} + 2\right),$$

where $c'(x) = c(x) - c_o$, $k = k_o Z$, $k_o = 2\pi/\lambda_o$, and λ_o is equal to the wavelength in the reference transmission-line whose parameters are ℓ_o, c_o.

These relations connect the Ricatti system with the wave approach based on the transmission-line equations.

[illegible] (b) these results

$$\frac{d\rho}{dx} = -\frac{1}{2} j k_0 \gamma(x) (\rho + 1)^2 - j 2 k_0 \rho \ , \qquad \rho(0) = 0.$$

(b) $$\frac{dr}{dx} = -\frac{1}{2} j k_0 \gamma(x) t \tau \ , \qquad r(0) = 0.$$

(c) $$\frac{dt}{dx} = t\left[-\frac{1}{2} j k_0 \gamma(x) \cdot \rho - \frac{1}{2} j k_0 \gamma(x) - j k_0\right], \quad t(0) = 1.$$

(d) $$\frac{d\tau}{dx} = \tau\left[-\frac{1}{2} j k_0 \gamma(x) \cdot \rho - \frac{1}{2} j k_0 \gamma(x) - j k_0\right], \quad \tau(0) = 1.$$

where $\gamma(x) = C'(x)/C_0$. We shall consider the left face of the slab to be located at x=0. The initial conditions imply that if the coordinate, x, of the right face of the slab coincides with that of the [illegible], then the slab is of zero thickness and the reflection coefficients vanish, while the transmission coefficients are unity. Note that (4-21) [illegible] and (d) are identical equations, indicating that $t(x) \equiv \tau(x)$. This results because $h_1(x) \equiv h_2(x)$ in (4-20).

The Riccati system, (4-18), satisfied by the elements of the connecting matrix is non-linear. It is interesting to note that the matrix

$$(4\text{-}22) \qquad M = \begin{pmatrix} 1/\tau & \rho/\tau \\ -r/\tau & \dfrac{t\tau - r\rho}{\tau} \end{pmatrix},$$

satisfies a linear differential equation. Before describing this equation, let us define what is meant by the derivative of a matrix.

If

$$(4\text{-}23) \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is an m x n matrix, (m possibly equals n), then the derivative of A is defined by

$$(4\text{-}24) \qquad \frac{dA}{dX} = \begin{bmatrix} \dfrac{da_{11}}{dx} & \dfrac{da_{12}}{dx} & \cdots & \dfrac{da_{1n}}{dx} \\ \vdots & \vdots & & \vdots \\ \dfrac{da_{m1}}{dx} & \dfrac{da_{m2}}{dx} & \cdots & \dfrac{da_{mn}}{dx} \end{bmatrix}.$$

That is, the derivative of a matrix is defined to be the matrix of the derivative of each element.

Let us now consider the derivative of M, defined in (4-22):

$$(4\text{-}25) \qquad \frac{dM}{dX} = \begin{pmatrix} -\dfrac{1}{\tau^2}\dfrac{d\tau}{dx} & \dfrac{1}{\tau}\dfrac{d\rho}{dx} - \dfrac{\rho}{\tau^2}\dfrac{d\tau}{dx} \\ -\dfrac{1}{\tau}\dfrac{dr}{dx} + \dfrac{r}{\tau^2}\dfrac{d\tau}{dx} & \dfrac{dt}{dx} - \dfrac{r}{\tau}\dfrac{d\rho}{dx} - \dfrac{\rho}{\tau}\dfrac{dr}{dx} + \dfrac{r\rho}{\tau^2}\dfrac{d\tau}{dx} \end{pmatrix}$$

… values in (4-25) we obtain

$$(4\text{-}26)\qquad \frac{dM}{dX} = \begin{pmatrix} -\frac{b_2}{\tau} - \frac{c\rho}{\tau} & \frac{a}{\tau} + b_1\frac{\rho}{\tau} \\ b_2\frac{r}{\tau} - c\left(\frac{k\tau - r\rho}{\tau}\right) & -\frac{ar}{\tau} + b_1\left(\frac{k\tau - r\rho}{\tau}\right) \end{pmatrix}.$$

… now, the product of the matrix

$$C = \begin{pmatrix} -b_2 & a \\ -c & b_1 \end{pmatrix}$$

and M

$$MC = \begin{pmatrix} \frac{1}{\tau} & \rho/\tau \\ -\frac{r}{\tau} & \frac{k\tau - r\rho}{\tau} \end{pmatrix} \begin{pmatrix} -b_2 & a \\ -c & b_1 \end{pmatrix}$$

$$(4\text{-}27)\qquad = \begin{pmatrix} -\frac{b_2}{\tau} - c\frac{\rho}{\tau} & \frac{a}{\tau} + b_1\rho/\tau \\ b_2\frac{r}{\tau} - c\left(\frac{k\tau - r\rho}{\tau}\right) & -a\frac{r}{\tau} + b_1\left(\frac{k\tau - r\rho}{\tau}\right) \end{pmatrix}.$$

This is precisely equal to $\frac{dM}{dX}$. Hence, we have the matrix differential equation

$$(4\text{-}28)\ (a)\qquad \frac{dM}{dX} = MC.$$

The initial condition satisfied by M is easily found to be

$$(4\text{-}28)\ (b)\qquad M(X_1, X_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

[illegible] ...al conditions on the scattering [illegible] β, r [illegible] as listed in (4-18).

Now that we have a matrix differential equation, (4-28) (a)-(b), what do we do with it? We can solve it using the following argument. If (4-28) (a) were a scalar differential equation, $\frac{df}{dx} = c(x)f$, with initial condition f(0)=1, then the solution would be $f(x) = e^{\int_0^x c(\xi)d\xi}$, as straightforward differentiation would show. Thus

$$\frac{df}{dx} = \exp\left(\int_0^x c(\xi)d\xi\right)\cdot\frac{d}{dx}\left(\int_0^x c(\xi)d\xi\right) = c(x)\exp\left(\int_0^x c(\xi)d\xi\right) = c(x)\cdot f.$$

Of course if c is a constant, then the solution is $f(x) = e^{cx}$.

Let us assume that the coefficient matrix, C, in (4-28) (a) is independent of x (the special case in which C depends on x will be treated separately). Then the solution of (4-28) (a)-(b) is given by analogy with the scalar case as

(4-29) $$M(x,x_1) = I\, e^{C(x-x_1)}$$

C is a matrix and $(x-x_1)$ is a scalar (i.e., simply a single number), so that $(x-x_1)\cdot C$ is a matrix whose elements are equal to the elements of C, each multiplied by $(x-x_1)$:

(4-30) $$(x-x_1)\cdot C = C\cdot(x-x_1) = \begin{bmatrix} -b_2(x-x_1) & a(x-x_1) \\ -c(x-x_1) & b_1(x-x_1) \end{bmatrix}$$

Hence, (4-29) becomes

(4-31) $$M(x,x_1) = I\cdot\exp\left\{\begin{bmatrix} -b_2(x-x_1) & a(x-x_1) \\ -c(x-x_1) & b_1(x-x_1) \end{bmatrix}\right\}$$

An important question concerning (4-31) arises: how do we define the exponential of a matrix? The answer is given, again, by analogy to the corresponding scalar case.

If the exponential of a scalar, u, is expanded in its Taylor series, there results

$$[illegible] + \frac{a^n}{n!} + \cdots .$$

The corresponding result for the exponential of an n-by-n matrix, A, is

$$\text{(4-33)} \qquad \exp A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots .$$

This series is well defined because powers of a square matrix are well defined (i.e., we know how to multiply a matrix by itself any number of times in order to raise it to any power) and because the series is convergent for any matrix A.

Hence, upon substituting (4-33) into (4-31) there results

$$A(x, x_1) = I + \begin{bmatrix} -b_2 & a \\ -c & b_1 \end{bmatrix}(x - x_1) + \begin{bmatrix} -b_2 & a \\ -c & b_1 \end{bmatrix}^2 \frac{(x-x_1)^2}{2!}$$

$$\text{(4-34)} \qquad + \begin{bmatrix} -b_2 & a \\ -c & b_1 \end{bmatrix}^3 \frac{(x-x_1)^3}{3!} + \cdots + \begin{bmatrix} -b_2 & a \\ -c & b_1 \end{bmatrix}^n \frac{(x-x_1)^n}{n!} + \cdots$$

This, in principle, solves the problem, except that the solution is not in the most convenient form for applications.

Example 4. Determine the wave-transmission matrix for a uniform transmission-line.

Solution. Since the transmission-line is uniform, we might as well take its value of k to be k_0, so that its capacitance-per-unit length, c, equals c_0 and $L = L_0$. Then by (4-20) its coefficient matrix satisfies $a = 0$, $b_1 = b_2 = -jk_0$, and the relevant matrix in (4-34) is diagonal

$$\begin{bmatrix} jk_0 & 0 \\ 0 & -jk_0 \end{bmatrix} .$$

[illegible] obtained, with the result that

$$\begin{pmatrix} jk_0 & 0 \\ 0 & -jk_0 \end{pmatrix}^m = \begin{pmatrix} (jk_0)^m & 0 \\ 0 & (-jk_0)^m \end{pmatrix} .$$

[illegible], (4-34) yields

$$M(X,X_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} jk_0(X-X_1) & 0 \\ 0 & -jk_0(X-X_1) \end{pmatrix} + \begin{pmatrix} (jk_0)^2 \frac{(X-X_1)^2}{2!} & 0 \\ 0 & (-jk_0)^2 \frac{(X-X_1)^2}{2!} \end{pmatrix}$$

$$+ \cdots + \begin{pmatrix} (jk_0)^n \frac{(X-X_1)^n}{n!} & 0 \\ 0 & (-jk_0)^n \frac{(X-X_1)^n}{n!} \end{pmatrix} + \cdots$$

When we add matrices, the elements of the resultant matrix is the sum of the corresponding elements of the summand matrices. With this in mind, the last result for $M(X,X_1)$, can be written

$$M(X,X_1) = \begin{pmatrix} 1 + jk_0(X-X_1) + (jk_0)^2 \frac{(X-X_1)^2}{2!} + \cdots + (jk_0)^n \frac{(X-X_1)^n}{n!} + \cdots , & 0 \\ 0 , & 1 - jk_0(X-X_1) + (-jk_0)^2 \frac{(X-X_1)^2}{2!} + \cdots + \frac{(-jk_0(X-X_1))^n}{n!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{jk_0(X-X_1)} & 0 \\ 0 & e^{-jk_0(X-X_1)} \end{pmatrix} .$$

The result for an arbitrary uniform transmission-line is obtained from this result by simply replacing k_0 by k, where k is the wave number of the line. This corresponds to replacing the reference transmission-line of (4-20) by a new reference line whose wave number is not k_0 but k.

... gives us ... of the wave-transmission matrix. Consider ... line of Figure 4-6.

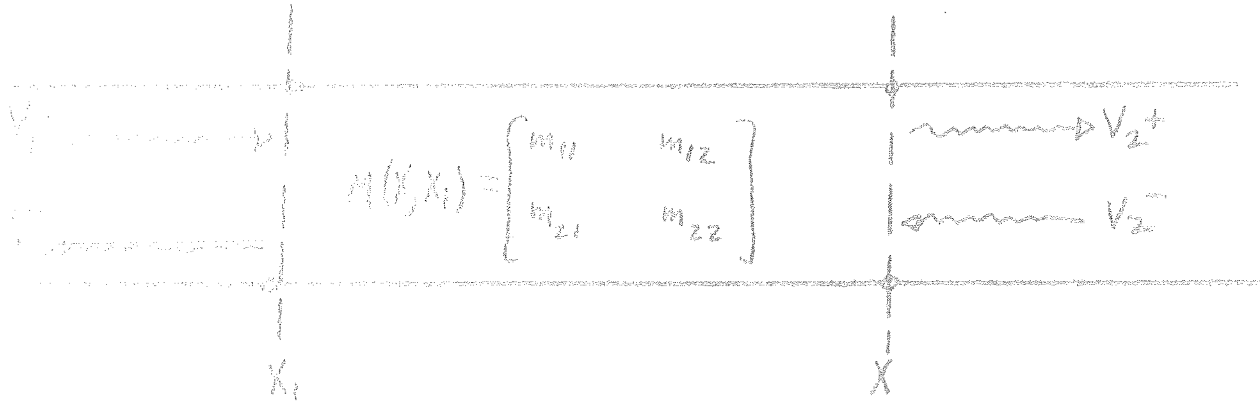


Figure 4-6. Illustrating the meaning of the wave-transmission matrix.

Let us relate the positive and negative going voltage waves at ... those at x_1:

$$V_2^- = m_{11} V_1^- + m_{12} V_1^+$$

$$V_2^+ = m_{21} V_1^- + m_{22} V_1^+ \qquad (4-35)$$

From the solution of the usual transmission-line wave equation, we have

$$V(x) = V_+(x_1) e^{-jk(x-x_1)} + V_-(x_1) e^{jk(x-x_1)}, \qquad (4-36)$$

so that

$$V_2^- = V_1^- e^{jk(x-x_1)}, \quad V_2^+ = V_1^+ e^{-jk(x-x_1)}, \qquad (4-37)$$

which means that the coefficients in (4-35) become

$$m_{11} = e^{jk(x-x_1)}, \quad m_{12} = m_{21} = 0, \quad m_{22} = e^{-jk(x-x_1)}, \qquad (4-38)$$

which is precisely the matrix of the solution of Example 4. Hence, we conclude that the wave-transmission matrix $M(x_1 x_1)$, is the matrix relating left- and right-going wave components at x to those at $x_1 (\leq x)$ in the manner of (4-35).

[illegible] the composite of two systems (such that the "output" of the first is the "input" to the second) is simply the usual matrix product of the wave-transmission matrices of the individual subsystem. Contrast this result with the scattering matrix of the composite of two systems; here we must replace ordinary matrix multiplication by "star-multiplication" as in 4-4.

The Voltage-Current (ABCD) Transmission Matrix

Closely related to the wave-transmission matrix, which relates right- and left-going voltage (or current) waves at either end of an obstacle, is a transmission matrix which relates voltage and current at one end of a two-port system (such as a transmission-line) to the voltage and current at the other end (see Figure 4-7).

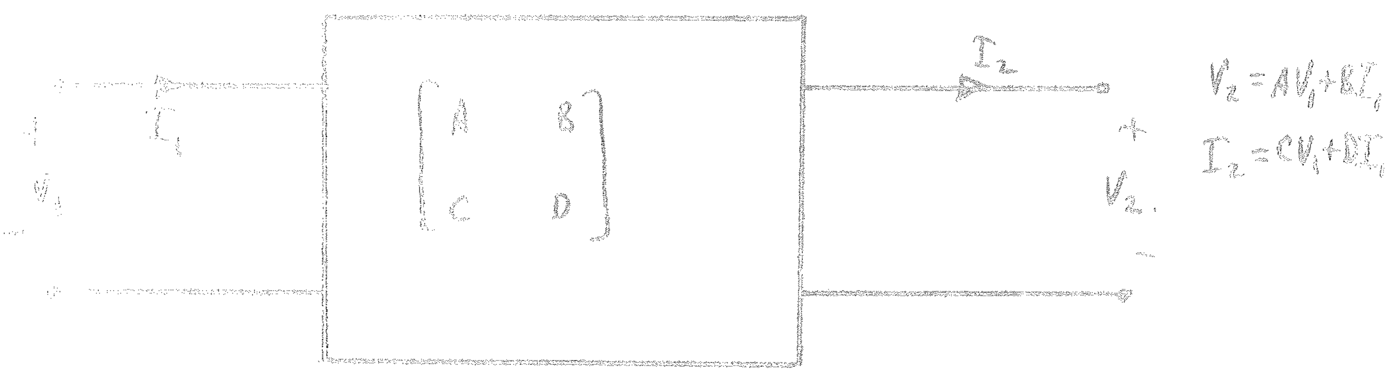


Figure 4-7. Defining the ABCD, voltage-current, matrix of a two-port.

Let us determine the ABCD matrix for a uniform transmission-line. We start with the usual transmission-line equations in the sinusoidal steady state

$$\frac{dV}{dx} = -j\omega L I \;, \qquad \frac{dI}{dx} = -j\omega C V, \tag{4-39}$$

which, in matrix form can be written

$$\frac{d}{dx}\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix} . \tag{4-40}$$

[illegible] form as (4-28), except that the [illegible] matrix M is replaced by the column matrix $\begin{bmatrix} V \\ I \end{bmatrix}$. [illegible] solution of (4-40), thus, is given by a form analogous to (4-29)

$$(4\text{-}41) \qquad \begin{bmatrix} V \\ I \end{bmatrix}(x) = \left\{ \exp \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix} (x - x_1) \right\} \cdot \begin{bmatrix} V \\ I \end{bmatrix}(x_1)$$

[illegible] the matrix exponential is defined by (4-34) with $b_1 = b_2 = 0$, $a = -j\omega L$, [illegible] $-j\omega C$. Hence,

$$(4\text{-}42) \qquad \exp \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix} (x - x_1) = I + \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix} (x - x_1)$$

$$+ \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^2 \frac{(x - x_1)^2}{2!} + \cdots + \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^n \frac{(x - x_1)^n}{n!} + \cdots\cdots .$$

Our object now is to express (4-42) in a closed form, and the result will be our ABCD matrix. In order to arrive at the closed form, we will make use of the Cayley-Hamilton theorem for square matrices, which states that a square matrix A satisfies its own characteristic equation*. Let us discuss this theorem. Consider an nxn matrix A

$$(4\text{-}43) \qquad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} .$$

Using this matrix, form a new matrix, $A - \lambda I$, where λ is a scalar parameter and I is the nxn identity matrix:

$$(4\text{-}44) \qquad A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} .$$

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* See G. R. Cooper and C. D. McGillem, Methods of Signal and System Analysis, Holt-Rinehart and Winston, 1967, p. 254.

[illegible] ...ated with A is the polynomial equation in λ obtained by setting the determinant of (4-44) equal to zero:

$$\det\left[A - \lambda I\right] = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \ldots + b_n = 0, \tag{4-45}$$

where the coefficients $\{b_j\}$ are derived from the elements of A.

The Cayley-Hamilton theorem then states that if the scalar, λ, is replaced by the matrix A, itself, (4-45) is satisfied as a <u>matrix equation</u>

$$A^n + b_1 A^{n-1} + b_2 A^{n-2} + \ldots + b_n I = 0. \tag{4-46}$$

Of course, when we say that the matrix polynomial on the left-hand side of (4-46) vanishes, we mean that each element in the resulting matrix on the left-hand side vanishes.

Let us now apply the preceding developments to the transmission-line problem. For A we take the matrix

$$\begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}(X - X_1),$$

so that the characteristic equation, (4-45), becomes

$$\det\begin{bmatrix} -\lambda & -j\omega L(X-X_1) \\ -j\omega C(X-X_1) & -\lambda \end{bmatrix} = \lambda^2 + \omega^2 LC(X-X_1)^2 = 0. \tag{4-47}$$

The Cayley-Hamilton theorem, therefore, reads

$$\begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^2 (X-X_1)^2 + k^2(X-X_1)^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0, \tag{4-48}$$

where $k^2 = \omega^2 LC$.

From (4-48), therefore, it is easily concluded that [illegible] $= k^2 I$

$$\begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^4 = k^4 I^2 = k^4 I$$

$$\begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^{2n} = (-1)^n k^{2n} I^n = (-1)^n k^{2n} I.$$

[illegible] (b)
$$\begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix} \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^2 = -k^2 \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^5 = \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^3 \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^2 = k^4 \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}$$

$\vdots$

$$\begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}^{(2n+1)} = (-1)^n k^{2n} \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}.$$

$n = 1, 2, \ldots$.

Upon substituting (4-49) into (4-42), we get

$$\exp[A] = I + A - \frac{k^2 (x-x_1)^2}{2!} I - \frac{k^2 (x-x_1)^2}{3!} A + \frac{k^4 (x-x_1)^4}{4!} I \quad (4\text{-}50)$$

$$+ \frac{k^4 (x-x_1)^4}{5!} A - \frac{k^6 (x-x_1)^6}{6!} I - \frac{k^6 (x-x_1)^6}{7!} A + \cdots$$

$$= I \left[1 - \frac{k^2 (x-x_1)^2}{2!} + \frac{k^4 (x-x_1)^4}{4!} - \frac{k^6 (x-x_1)^6}{6!} + \cdots \right]$$

$$+ A \left[1 - \frac{k^2 (x-x_1)^2}{3!} + \frac{k^4 (x-x_1)^4}{5!} - \frac{k^6 (x-x_1)^6}{7!} + \cdots \right].$$

$$\begin{bmatrix} \cdots \\ -j\omega C(x-x_1) & 0 \end{bmatrix}$$

The series multiplying the identity matrix, I, in (4-50) is immediately recognized to be cos $k(x-x_1)$, whereas the series multiplying is $\dfrac{\sin k(x-x_1)}{k(x-x_1)}$. Hence, the ABCD matrix for the uniform, lossless transmission-line is found to be

$$\cos k(x-x_1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\sin k(x-x_1)}{k(x-x_1)}\begin{bmatrix} 0 & -j\omega L(x-x_1) \\ -j\omega C(x-x_1) & 0 \end{bmatrix}.$$

(4-51)

$$= \begin{bmatrix} \cos k(x-x_1) & -j\sqrt{\frac{L}{C}}\,\sin k(x-x_1) \\ -j\sqrt{\frac{C}{L}}\,\sin k(x-x_1) & \cos k(x-x_1) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

An alternative derivation of this result starts by using the equations

$$V(x) = V_+(x_1)\,e^{-jk(x-x_1)} + V_-(x_1)\,e^{jk(x-x_1)}$$

(4-52)

$$I(x) = \frac{V_+(x_1)}{\left(\frac{L}{C}\right)^{1/2}}\,e^{-jk(x-x_1)} - \frac{V_-(x_1)}{\left(\frac{L}{C}\right)^{1/2}}\,e^{jk(x-x_1)},$$

and then solving for $V(x)$, $I(x)$, in terms of $V(x_1)$ and $I(x_1)$. This derivation will be left as an exercise.

Transmission matrices of the type considered here are called state-transition matrices in linear systems and are of considerable use in time-domain, state-variable analysis.

[illegible] Non-Uniform System

[illegible] hold for a uniform system, i.e., one for [illegible] the coefficient matrix, C, is independent of x. If, now, we [illegible] the opposite case of a non-uniform system, we must rederive [illegible] expression for $M(x, x_1)$.

[illegible] object is to solve (4-28) in which C is now variable

$$(4\text{-}53)\ (a)\quad \frac{dM}{dx} = MC(x)\,, \qquad (b)\quad M(x_1, x_1) = I.$$

[illegible] our system, e.g., a transmission-line, extends from x=0 to [illegible], we partition the interval [0, l] into many small sub-intervals, [illegible] each of which the coefficient matrix C(x) may be considered to [illegible] constant. We are said to have generated a <u>piecewise constant</u> <u>approximation</u> to a non-uniform, continuous system.

Let the <u>ith</u> subinterval extend from x_{i-1} to x_i, and let ξ_i [illegible] some point (perhaps the midpoint) within the <u>ith</u> subinterval. [illegible] (4-53) is decomposed into the piecewise uniform system of equations

$$(4\text{-}54)\quad \frac{dM_1}{dx} = M_1 C(\xi_1), \qquad 0 \le x \le x_1$$

$$\frac{dM_2}{dx} = M_2 C(\xi_2), \qquad x_1 \le x \le x_2$$

$$\vdots$$

$$\frac{dM_N}{dx} = M_N C(\xi_N)\,, \qquad x_{N-1} \le x \le l.$$

The solution for the <u>ith</u> equation is given by (4-29)

$$(4\text{-}55)\quad M_i(x_{i-1}, x_i) = I \exp[C(\xi_i)\Delta x_i],$$

where $\Delta x_i \equiv x_i - x_{i-1}$ is the infinitesimal length of the <u>ith</u> subinterval.

is not true that for two matrices, A and B,

exp B = exp (A+B), unless A and B commute, i.e., unless

Because $C(\xi_i)$ does not necessarily commute with $C(\xi_{i-1})$.

not follow our temptations and write for (4-56)

$$M(0,\ell) = I \exp\left[C(\xi_N)\Delta x_N + \cdots + C(\xi_1)\Delta x_1\right]$$

$$= I \exp\left[\int_0^\ell C(\xi)\,d\xi\right],$$

could do if C(x) were a scalar. Only in that case for which $C(x_i) \cdot C(x_j) = C(x_j) \cdot C(x_i)$ for all x_i, x_j does (4-57) hold in the case.

In the limit in which all of the Δ x's go to zero, (4-56) is a product integral. As it stands it is in a form suitable digital computation.

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transmission matrix for the overall interval extending from $x=0$ to the product of (4-55) with i ranging from 1 to N:

$$M(0,\ell) = I \exp\left[C(\xi_N)\Delta x_N\right] \cdot \exp\left[C(\xi_{N-1})\Delta x_{N-1}\right] \cdot \exp\left[C(\xi_{N-2})\Delta x_{N-2}\right] \cdots\cdots \exp\left[C(\xi_1)\Delta x_1\right].$$

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